International Journal of
Mathematical Modelling & Computations
Vol. 07, No. 03, Summer 2017, 299- 307

Approximation of a Fuzzy Function by Using Radial Basis Functions Interpolation

R. Firouzdor[a] and M. Amirfakhrian[b,*]

[a] *Young Researcher Elite Club, Central Tehran Branch, Islamic Azad University, Tehran, Iran,*
[b] *Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.*

Abstract. In the present paper, Radial Basis Function interpolations are applied to approximate a fuzzy function $\tilde{f}: \mathbb{R} \to \mathcal{F}(\mathbb{R})$, on a discrete point set $X = \{x_1, x_2, \ldots, x_n\}$, by a fuzzy-valued function $\tilde{S}$. RBFs are based on linear combinations of terms which include a single univariate function. Applying RBF to approximate a fuzzy function, a linear system will be obtained which by defining coefficient vector, target function will be approximated. Finally for showing the efficiency of the method we give some numerical examples.

Received: 03 November 2016, Revised: 30 January 2018, Accepted: 26 August 2018.

Keywords: Radial Basis Function interpolation; Fuzzy function; Fuzzy-valued function; An approximation of a fuzzy function.

Index to information contained in this paper

1. Introduction

Among variety of the numerical methods, Radial Basis Function (RBF) method appears to be one of the best one in literature. RBF approximations are usually finite linear combinations of the translation of a radially symmetric basis function, $\varphi(\| \,.\, \|)$ where $(\| \,.\, \|)$ is the Euclidean norm. The set of RBFs, $\{\phi_i\}_1^m$ is as follows:

$$\phi_i : \mathbb{R}^d \to \mathbb{R}, \quad \phi_i(x) = \phi(\| x - x_i \|)$$

*Corresponding author. Email: amirfakhrian@iauctb.ac.ir

where $\| \, . \, \|$ denote the Euclidean norm and x_i is the center of RBF. Gaussian (GA) $\phi(r) = \exp(-\sigma r^2)$, multiquadric (MQ) $\phi(r) = \sqrt{r^2 + \sigma^2}$ are some well-known functions that generate RBF. More functions are shown in Table 1 in which $r = \|x - x_i\|$. The mentioned functions due to having the parameter c_i have exponentially convergence [6].

Table 1. Well-known functions that generate RBF.

Name of function	Definition
Gaussian (GA)	$\phi(r) = \exp(-\sigma r^2)$
Multiquadric (MQ)	$\phi(r) = (r^2 + \sigma^2)^{\frac{1}{2}}$
Inverse multiquadric (IMQ)	$\phi(r) = (r^2 + \sigma^2)^{-\frac{1}{2}}$
Inverse multiquadric (IQ)	$\phi(r) = (r^2 + \sigma^2)^{-1}$

RBFs are computationally means to approximate functions which are complicated or have many variables, by other simpler functions which are easier to understand and readily evaluated. One of the outstanding advantages of interpolation by RBF, unlike multivariable polynomial interpolation or splines [10], is applicability in scattered data aspect of existence and uniqueness results since there is little restrictions on dimension and also high accuracy or fast convergence to the target function. As another advantage of RBF there are not required to triangulations of the data points, while other numerical methods such as finite elements or multivariate spline methods need triangulations [9, 10]. This requirement cost computationally especially in more than two dimensions. In this paper we consider RBF to approximate the solution of the problem as meshfree approximations. In this study, we consider to approximate fuzzy function by Redial Basis functions.

To approximate the target function $y(x)$, we employ RBF interpolation in distinct grids from a definite domain. To this purpose, a linear component is considered as follows:

$$y(x) \approx \sum_{i=0}^{n} c_i \phi_i(x), \tag{1}$$

where $\phi_i(x)$ can be chosen from one of the basis functions which is mentioned in Table 1, according to type of the target function which is desired to approximate.

In this paper we consider RBF to approximate the fuzzy function and for this purpose we consider Gaussian function, then we would have:

$$\phi_i(x) = e^{-\|x - x_i\|^2}. \tag{2}$$

2. Materials and definitions

In this section, some definitions and features of fuzzy numbers and fuzzy differential equations which will be used throughout the paper, will reviewed.

Definition 2.1 ([12]). A fuzzy number $\tilde{u}$ is completely determined by an ordered pair of functions $\tilde{u} = [\underline{u}(r), \overline{u}(r)]$, $0 \leqslant r \leqslant 1$, satisfy the following requirements:
1. $\underline{u}(r)$ is a bounded, monotonic, increasing (non decreasing) left-continuous function for all $r \in (0, 1]$ and right-continuous for $r = 0$.
2. $\overline{u}(r)$ is a bounded, monotonic, decreasing (non increasing) left-continuous

function for all $r \in (0,1]$ and right-continuous for $r = 0$.
3. For all $r \in (0,1]$ we have $\underline{u}(r) \leqslant \overline{u}(r)$.

For every $\tilde{u} = [\underline{u}(r), \overline{u}(r)]$, $\tilde{v} = [\underline{v}(r), \overline{v}(r)]$ and $k > 0$ addition and multiplication have the following properties:

$$\tilde{u} \oplus \tilde{v} = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)] \tag{3}$$

$$\tilde{u} \ominus \tilde{v} = [\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)] \tag{4}$$

$$k\tilde{u} = \begin{cases} [k\underline{u}(r), k\overline{u}(r)], & k \geqslant 0, \\ [k\overline{u}(r), k\underline{u}(r)], & k < 0, \end{cases} \tag{5}$$

Definition 2.2 The collection of all fuzzy numbers with addition and multiplication as defined by Eqs. $(1) - (3)$ is denoted by $\mathbb{E}^1$. For $0 < r \leqslant 1$ we define the r-cuts of fuzzy number $\tilde{u}$ with $[\tilde{u}]^r = \{x \in \mathbb{R} \mid \mu_{\tilde{u}}(x) \geqslant r\}$ and support of $\tilde{u}$ is defined as $[\tilde{u}]^0 = \overline{\{x \in \mathbb{R} \mid \mu_{\tilde{u}}(x) > 0\}}$.

Definition 2.3 Let $\tilde{u} = (m, n, \alpha, \beta)_{LR}, (\alpha < m \leqslant n < \beta)$, where m, n are two centers (defuzzifiers) and $\alpha > 0, \beta > 0$ are left and right spreads, respectively. $L(\frac{m-x}{\alpha})$ and $R(\frac{x-n}{\beta})$ are non-increasing functions with $L(0) = 1$ and $R(0) = 1$ respectively. $\tilde{u}$ is a L-R fuzzy number if its membership function be as the following form:

$$\mu_{\tilde{u}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & -\infty < x < m, \\ 1 & m \leqslant x \leqslant n, \\ R(\frac{x-n}{\beta}) & n < x < \infty, \\ 0 & otherwise. \end{cases}$$

This definition is very general and covers quite different type of information. For example, fuzzy number $\tilde{u}$ is trapezoidal fuzzy number when $m < n$ and $L(\frac{m-x}{\alpha})$, $R(\frac{x-n}{\beta})$ be linear functions or when $m = n$ and $L(\frac{m-x}{\alpha})$, $R(\frac{x-n}{\beta})$ are linear functions, $\tilde{u}$ denotes triangular fuzzy number and we write $\tilde{u} = (m, \alpha, \beta)$.

Definition 2.4 The Hausdorff distance $D : \mathbb{E}^1 \times \mathbb{E}^1 \to \mathbb{R}_+ \cup \{0\}$ between fuzzy numbers is given by:

$$D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} \{\| [\tilde{u} \ominus_H \tilde{v}]^r \|_* \},$$

where, for an interval [a, b], the norm is

$$\| [a, b] \|_* = \max\{| a |, | b |\}.$$

and $[\tilde{u} \ominus_H \tilde{v}]^r = [\underline{u}(r) - \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$.

It is easy to see that D is a metric in $\mathbb{E}^1$ and has the following properties ([10]).

Lemma 2.5 For $u, v, w, e \in \mathbb{E}^1$ and $k \in \mathbb{R}$, we have the following results
(1) $D(u \oplus w, v \oplus w) = D(u, v)$,
(2) $D(k \odot u, k \odot v) = \mid k \mid D(u, v)$,
(3) $D(u \oplus v, w \oplus e) \leqslant D(u, w) + D(v, e)$.

Definition 2.6 ([3]) Let $x, y \in \mathbb{E}^1$. If there exists $z \in \mathbb{E}^1$ such that $x + y = z$, then z is called the H$-$difference of x, y and it is denoted by $x \ominus y$.

3. Description of the method

The method applied in n dimensional Euclidean space. To this purpose, consider n distinct points as $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$, in this space at which the function to be approximated is known and real scalers $\{\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n\} \in \mathcal{F}(\mathbb{R})$ which are given values at the points. We desire to construct a continuous function $\tilde{S} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ so that $\tilde{S}(x_j) = \tilde{y}_j$ for $j = 1, 2, \ldots, n$. Radial basis function method is based on continuous function such as $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and a norm $\parallel . \parallel$ in $\mathbb{R}^n$, then $\tilde{S}$ can be as the following form:

$$\tilde{S}(x) = \sum_{j=1}^{n} \tilde{c}_j \phi(\parallel x - x_j \parallel) \tag{6}$$

where $\tilde{c}_j$ are scalar parameters which should be chosen so that s approximates $\tilde{y}$ in point x_j for $j = 1, 2 \ldots, n$. Then funtions $\gamma \to \phi(\parallel \gamma - x_j \parallel)$ translates s into a vector space. According to interpolation conditions a linear system will be defined as $\Psi\tilde{C} = \tilde{Y}$, where $\Psi \in \mathbb{R}^{n \times n}$ is called a distance matrix or interpolation matrix, and given by

$$\Psi_{ij} = \phi(\parallel x_i - x_j \parallel) \tag{7}$$

and also $\tilde{C} = (\tilde{c}_1, \ldots, \tilde{c}_n)^T$ and $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_n)^T$

The interpolation matrix is non-singular since it is a positive definite matrix, so we have the unique existence of the coefficients $\tilde{c}_j$.

4. Fuzzy interpolation

In this section we used RBF functions for fuzzy interpolation. We proof that this method is better than when we used polynomial function of degree at most n.
A fuzzy interpolation is a function $\tilde{S}$ from $\mathbb{R}$ to $\mathcal{F}(\mathbb{R})$ such that $\tilde{S}(x) = \sum_{i=0}^{n} \tilde{a}_i \phi_i(x)$. Denote by $\tilde{\prod}^{\phi}$ the set of all fuzzy function $\tilde{S}(x) = \sum_{i=0}^{n} \tilde{a}_i \phi_i(X)$ on $\mathcal{F}(\mathbb{R})$. A fuzzy interpolation function on $\mathcal{F}(\mathbb{R})$ can be put in the following parametric form:

$$\tilde{S}(x) = (\underline{S}_x(r), \overline{S}_x(r)) \tag{8}$$

where

$$\begin{cases} \underline{S}_x(r) = \\ \sum\limits_{\phi_i(x)\geqslant 0} \underline{a_i}(r)\phi_i(x) + \sum\limits_{\phi_i(x)<0} \overline{a_i}(r)\phi_i(x) \\ \\ \overline{S}_x(r) = \\ \sum\limits_{\phi_i(x)\geqslant 0} \overline{a_i}(r)\phi_i(x) + \sum\limits_{\phi_i(x)<0} \underline{a_i}(r)\phi_i(x) \end{cases} \tag{9}$$

for $0 \leqslant r \leqslant 0$.
But we know $\phi_i(x) \geqslant 0$ for all $x \in \mathbb{R}$. So we have

$$\begin{cases} \underline{S}_x(r) = \sum_{i=1}^{n} \underline{a_i}(r)\phi_i(x) \\ \\ \overline{S}_x(r) = \sum_{i=1}^{n} \overline{a_i}(r)\phi_i(x) \end{cases} \tag{10}$$

for $0 \leqslant r \leqslant 1$.

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $0 \leqslant r \leqslant 1$ for (10) so we have

$$\big(\underline{v}_{x_1}(r), \ldots, \underline{v}_{x_n}(r), \overline{v}_{x_1}(r), \ldots, \overline{v}_{x_n}(r)\big)^T$$

$$=$$

$$\begin{pmatrix} \phi_1(x_1) & \ldots & \phi_n(x_1) & 0 & \ldots & 0 \\ \phi_1(x_2) & \ldots & \phi_n(x_2) & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_n) & \ldots & \phi_n(x_n) & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \phi_1(x_1) & \ldots & \phi_n(x_1) \\ 0 & \ldots & 0 & \phi_1(x_2) & \ldots & \phi_n(x_2) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \phi_1(x_n) & \ldots & \phi_n(x_n) \end{pmatrix} \tag{11}$$

$$\times$$

$$\big(\underline{a_1}(r), \ldots, \underline{a_n}(r), \overline{a_1}(r), \ldots, \overline{a_n}(r)\big)^T$$

Therefore we have two systems of equations $\Psi\underline{C} = \underline{Y}$ and $\Psi\overline{C} = \overline{Y}$, where

$$\Psi = \begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) & \ldots & \phi_n(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \ldots & \phi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_n) & \phi_2(x_n) & \ldots & \phi_n(x_n) \end{pmatrix} \tag{12}$$

is a nonsingular matrix. So instead of solving a system of $2n \times 2n$ we solve two systems equations of $n \times n$ with the same matrix coefficients.

5. Numerical examples

In this section we present some numerical examples.

Example 5.1 Consider the following tabular function with $n = 3$ points:

x	$\tilde{f}(x)$
3	$(r+2, 4-r)$
1	$(r, 2-r)$
2	$(r+1, 3-r)$

Figure 1 shows the value of the fuzzy function number in x_1 and its RBF approximation. Figure 2 shows the errors of this approximation in x_1. Figure 3 shows RBF approximation over an interval $x \in [0, 6]$ and $r \in [0, 1]$.

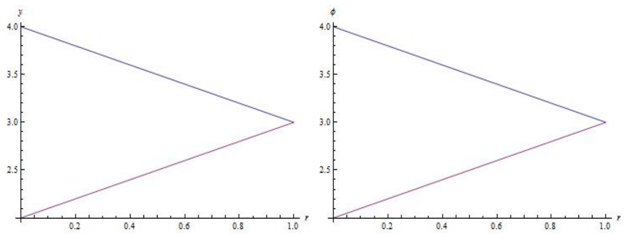

Figure 1. Fuzzy number for x_1 in tabular function and RBF approximation.

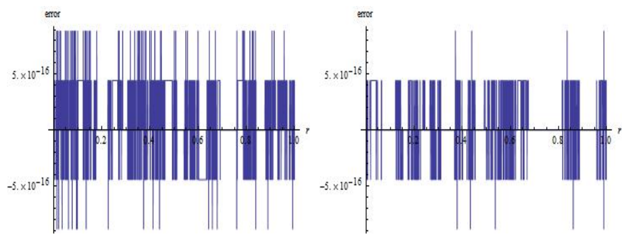

Figure 2. Under and over error of Fuzzy number for x_1 in tabular function and RBF approximation.

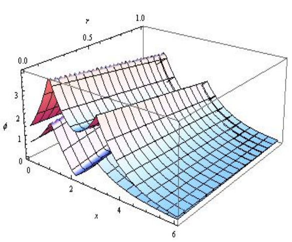

Figure 3. RBF approximation for interval of x and $r \in [0, 1]$.

Example 5.2 Consider the following tabular function with $n = 6$ points:

x	$\tilde{f}(x)$
0	$(1 + r, 3 - r)$
1	$(2 + r, 4 - r)$
2	$(r, 2 - r)$
3	$(3 + r, 5 - r)$
4	$(4 + r, 6 - r)$
5	$(3 + r, 5 - r)$

Figure 4 shows the value of the fuzzy function number in x_3 and its RBF approximation. Figure 5 shows the errors of this approximation in x_3. Figure 6 shows RBF approximation over an interval $x \in [0, 6]$ and $r \in [0, 1]$.

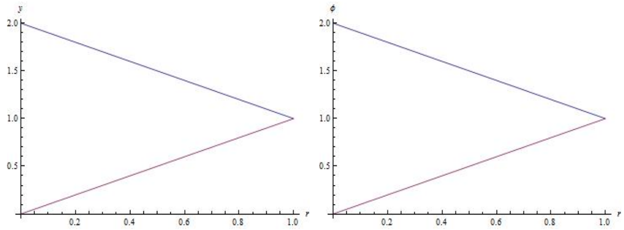

Figure 4. Fuzzy number for x_3 in tabular function and RBF approximation.

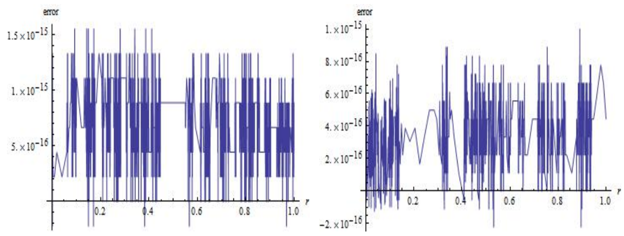

Figure 5. Under and over error of Fuzzy number for x_3 in tabular function and RBF approximation.

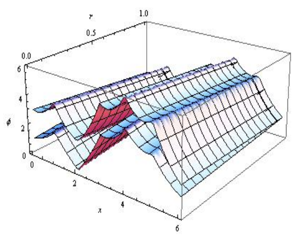

Figure 6. RBF approximation for interval of x and $r \in [0, 1]$.

Example 5.3 In this example we consider $n = 10$ points and used continuous fuzzy function $\tilde{k}\sin(x)$ in interval $[0, \frac{\pi}{5}]$. There $\tilde{k} = (r, 2 - r)$ and $x_i = i \times \frac{\pi}{5n}$ for $i = 1, 2, \ldots, n$.
Figure 7 shows the fuzzy number for $x_3 = \frac{\pi}{25}$ in tabular function and RBF approximation of x_3. Figure 8 shows the error of fuzzy number for $x_3 = \frac{\pi}{25}$ and RBF approximation for interval of x fuzzy function by $r \in [0, 1]$. Figure 9 shows the error fuzzy number for $x = \frac{\pi}{20}$ that isn't in tabular function points and 3D error for $x \in [0, \frac{\pi}{5}]$ and $r \in [0, 1]$.

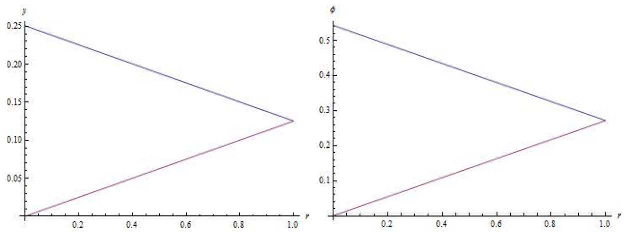

Figure 7. Fuzzy number for $x_3 = \frac{\pi}{25}$ in tabular function and RBF approximation.

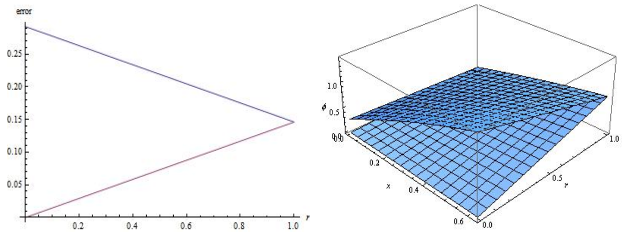

Figure 8. Error of fuzzy number for $x_3 = \frac{\pi}{25}$ and RBF approximation for interval of x and $r \in [0, 1]$.

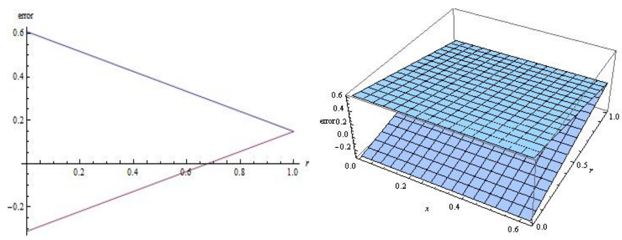

Figure 9. Error fuzzy number for $x = \frac{\pi}{20}$ that isn't in tabular function points and 3D error for $x \in [0, \frac{\pi}{5}]$ and $r \in [0, 1]$.

Example 5.4 In this example we let $n = 10$ points and used continuous fuzzy function $\tilde{k}\exp(x)$ in interval $[0, 0.1]$. There $\tilde{k} = (r, 2 - r)$ and $x_i = i \times \frac{0.1}{n}$ for $i = 1, 2, \ldots, n$.
Figure 10 shows the fuzzy number for $x_5 = 0.04$ in tabular function and RBF approximation of x_5. Figure 11 shows the error of fuzzy number for $x_5 = 0.04$ and RBF approximation for interval of x fuzzy function by $r \in [0, 1]$. Figure 12 shows the error fuzzy number for $x = 0.055$ that isn't in tabular function points and 3D error for $x \in [0, \frac{\pi}{5}]$ and $r \in [0, 1]$.

Finally we show the maximum errors of examples in Table 2.

Table 2. Maximum errors of examples.

Example	Maximum error in table points	Maximum error in interval approximation
Example 1	8.88178×10^{-16}	——
Example 2	4.44089×10^{-16}	——
Example 3	0.292323	0.549779
Example 4	0.0471799	0.183151

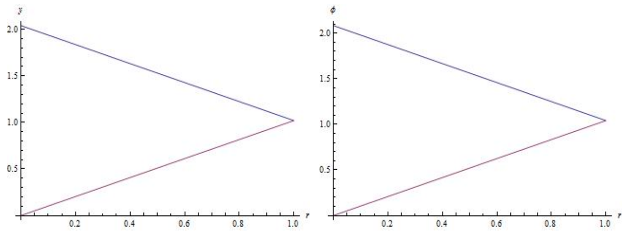

Figure 10. Fuzzy number for $x_5 = 0.04$ in tabular function and RBF approximation.

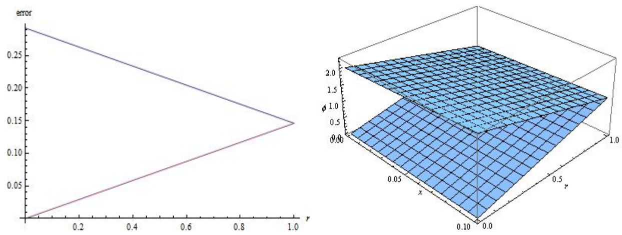

Figure 11. Error of fuzzy number for $x_5 = 0.01$ and RBF approximation for interval of x and $r \in [0, 1]$.

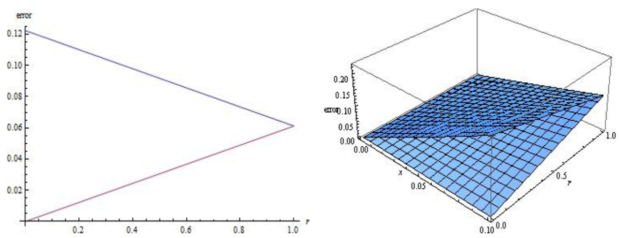

Figure 12. Error fuzzy number for $x = 0.055$ that isn't in tabular function points and 3D error for $x \in [0, \frac{\pi}{5}]$ and $r \in [0, 1]$.

6. Conclusion

In this paper we presented a method to approximate a fuzzy function by using Radial Basis Functions interpolation. The results obtained from this method seems to be acceptable and useful for fuzzy applications.

References

[1] M. Amirfakhrian, Some Approximation Method in Fuzzy Logic, Lambert Academic Publishing, (2012).
[2] M. Amirfakhrian, R. Firouzdor and Z. Mollaramezani, Zero approximation method for functional nonlinear integral equations, Advances in Applied Physics & Materials Science Congress, Antalya, Turkey, MAY 12-15, (2011).
[3] B. Bede, I. Rudas and A. Bencsik, First order linear fuzzy differential equations under generalized differentiablity, Information Science, **177** (2006) 3627-3635.
[4] C. de. Boor, Multivariate piecewise polynomials, Acta Numerica, **2** (1993) 65-109.
[5] M. Buhmann, Radial basis function, Scholarpedia, **5 (5)** (2010) 9837.
[6] Y. Chalco-Cano and H. Roman-Flores, On new solutions of fuzzy differential equations, Chaos, Solitons and Fractals, **38 (1)** (2008) 112-119.
[7] R. Firouzdor, A. Heidarnejad, Z. Mollaramezani, Numerical solution of functional nonlinear integral equations by using B-splines, Journal of Linear and Topological Algebra, **1 (1)** (2012) 45-53.
[8] J. Stoer and R. Bulirsch, Introduction to the Numerical Analysis, Springer-Verlag, (2002).
[9] M. J. Lai and L. L. Schumaker, Spline functions on triangulations Cambridge University Press, Cambridge, (2007).
[10] M. L. Puri and D. Ralescu, Fuzzy random variables, J. Math. Anal. Appl, **114** (1986) 409-422.
[11] M. T. Rashed, An Expansion Method To Treat Integral Equations, Applied Mathematics and Computation, **135** (2003) 73-79.
[12] L. Stefanini, L. Sorini and M. L. Guerra, Parametric representation of fuzzy numbers and application to fuzzy calculus, Fuzzy Sets and Systems, **157** (2006) 2423-2455.