International Journal of
Mathematical Modelling & Computations
Vol. 09, No. 01, Winter 2019, 61- 69

Heat Transfer in Three-Dimensional Flow along a Porous Plate

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Abstract. An analysis of heat transfer in the flow of a viscous, incompressible fluid along an infinite porous plate is presented when the plate is subjected to a transverse sinusoidal suction. The flow field becomes three-dimensional due to this type of suction velocity. Expressions for the flow and temperature fields are obtained by series expansion method. It is found during the course of discussion that due to more addition of viscous dissipative heat the temperature in the boundary layer increases or decreases accordingly as $E > 0$ or $E < 0$.

Received: 02 December 2018, Revised: 24 January 2019, Accepted: 05 March 2019.

Keywords: Heat transfer; Ecker number; Thermal conductivity; Prandtl number.

AMS Subject Classification: 76S05, 76N20.

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Nomenclature

c_p	specific heat at constant pressure
E	Ecker number
K	thermal conductivity
p'	pressure
p	dimensionless pressure
P_r	Prandtl number
q'	rate of heat transfer
q	dimensionless rate of heat transfer
T'	temperature of fluid
u', v', w'	Velocity components along x', y', z' respectively
u, v, w	dimensionless velocity components along x, y, z respectively
U_0	uniform free stream velocity
$V_0 > 0$	constant mean suction velocity
x', y', z'	co-ordinate system
y, z	dimensionless co-ordinate normal to the main flow direction
τ	dimensionless shear stress in the main flow direction
θ	dimensionless temperature
ε	amplitude of the fluid

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ν	kinematic viscosity
ρ'	density of fluid
λ	Suction parameter
μ	viscosity
ϕ'	dissipation function

1. Introduction

The problem of laminar flow control has become very important in recent years particularly in the field of aeronautical engineering owing to its application to reduce drag and hence to enhance the vehicle power by a substantial amount. Several methods have been developed for the purpose of artificially controlling the boundary layer and detailed account of such methods has been given by Schlichting [5]. The boundary layer suction is one of the effective methods of reducing the drag coefficient which entails large energy losses. By this method the decelerated fluid particles in the boundary layer are removed through the holes or slits in the wall into the interior of the body and thus transition from laminar to turbulent flow may be delayed or prevented which causes increase of drag coefficient. The effects of different arrangements and configurations of the suction holes and slits have been studied extensively by various scholars and development on the subject has been compiled by Lachmann [4] and Dube [2]. Most of the investigators have, however, confirmed themselves to the two-dimensional flow only.

Dube [1] and Gersten et.al [3] studied the flow and heat transfer along a plane porous wall with transverse sinusoidal suction. By assuming such a suction velocity distribution transverse to the potential flow, the flow in the boundary layer becomes three-dimensional. In their study the heat due to viscous dissipation has been neglected. However, there are a number of physical situations where the heat due to viscous dissipation is present in the subsonic flow of an incompressible viscous fluid. Also, in the case of fluid with high Prandtl number the viscous dissipative heat is always present even in slow motion.

Hence, in this paper we propose to investigate the transfer of heat in the flow of a viscous incompressible fluid along an infinite porous plate with transverse sinusoidal suction by which the flow becomes three-dimensional over the surface of the plate in the presence of viscous dissipative heat.

2. Mathematical analysis

We consider the flow of a viscous incompressible fluid along an infinite porous plate with transverse sinusoidal suction. A co-ordinate system is assumed with plate lying horizontally on $x'-z'$ plane. The x'-axis is taken along the plate, being the direction of the flow, and the y'-axis is taken normal to the plate and is directed into the fluid flowing laminarly. Since the plate is considered infinite in x'-direction, so all physical quantities will be independent of x', however, the flow remains three-dimensional because of the variation of suction velocity. Thus, the flow is governed by the following equations:

Continuity equation

$$\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}=0. \tag{1}$$

Momentum equations

$$v'\frac{\partial v'}{\partial y'}+w'\frac{\partial w'}{\partial z'}=\upsilon\left(\frac{\partial^2 u'}{\partial y'^2}+\frac{\partial^2 u'}{\partial z'^2}\right), \tag{2}$$

$$\frac{\partial v'}{\partial y'}+w'\frac{\partial w'}{\partial z'}=\frac{1}{\rho'}\frac{\partial p'}{\partial y'}+\upsilon\left(\frac{\partial^2 v'}{\partial y'^2}+\frac{\partial^2 v'}{\partial z'^2}\right), \tag{3}$$

$$v'\frac{\partial v'}{\partial y'}+w'\frac{\partial w'}{\partial z'}=\frac{1}{\rho'}\frac{\partial p'}{\partial z'}+\upsilon\left(\frac{\partial^2 w'}{\partial y'^2}+\frac{\partial^2 w'}{\partial z'^2}\right). \tag{4}$$

Energy equation

$$\rho' c_p\left(v'\frac{\partial T'}{\partial y'}+w'\frac{\partial T'}{\partial z'}\right)=K\left(\frac{\partial^2 T'}{\partial y'^2}+w'\frac{\partial^2 T'}{\partial z'^2}\right)+\mu\phi', \tag{5}$$

where $\phi'=2\left\{\left(\frac{\partial v'}{\partial y'}\right)^2+\left(\frac{\partial w'}{\partial z'}\right)^2\right\}+\left\{\left(\frac{\partial u'}{\partial y'}\right)^2+\left(\frac{\partial w'}{\partial y'}+\frac{\partial v'}{\partial z'}\right)^2+\left(\frac{\partial u'}{\partial z'}\right)^2\right\}.$

All the physical variables are defined in nomenclature.

The boundary conditions are

$$\left.\begin{array}{l} y'=0: u'=0,\ v'=-v_0\left(1+\varepsilon\cos\frac{\pi v_0 z'}{\upsilon}\right),\ w'=0,\ T'=T'_w, \\ y'\to\infty: \qquad u'=U_0,\ v'=-v_0,\ w'=0,\ p'=p'_\infty,\ T'=T'_\infty \end{array}\right\}. \tag{6}$$

The negative sign in the boundary conditions for v' indicates that the suction is towards the plate. The subscripts w and ∞ denote physical variables at the plate and in the free stream respectively.

On introducing the following non-dimensional quantities:

$$\left.\begin{array}{r} y=\frac{U_0 y'}{\upsilon},\ z=\frac{U_0 z'}{\upsilon},\ u=\frac{u'}{U_0},\ v=\frac{v'}{U_0}, \\ w=\frac{w'}{U_0},\ \lambda=\frac{V_0}{U_0},\ \theta=\frac{(T'-T'_\infty)}{(T'_w-T'_\infty)}, \\ p=\frac{p'}{\rho' U_0^2},\ P_r=\frac{\mu c_p}{K},\ E=\frac{U_0^2}{c_p(T'_w-T'_\infty)}. \end{array}\right\} \tag{7}$$

Equations (1) to (5) reduce to the following non-dimensional form

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{8}$$

$$v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{9}$$

$$v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{10}$$

$$v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z'}z=-\frac{\partial p}{\partial z}+\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right), \tag{11}$$

$$v\frac{\partial \theta}{\partial y}+w\frac{\partial \theta}{\partial z}=\frac{1}{P_r}\left(\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2}\right)+E\phi, \tag{12}$$

where $\phi = 2\left\{\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\}$.

The corresponding boundary conditions become

$$\left.\begin{array}{l} y' = 0 : u = 0,\ v = -\lambda\left(1 + \varepsilon \cos \pi z\right),\ w = 0,\ \theta = 1, \\ y' \to \infty : \quad u = 1,\ v = -\lambda,\ w = 0,\ p = p_\infty,\ \theta = 0 \end{array}\right\}. \tag{13}$$

Assuming the amplitude of the suction velocity ε to be small ($<<1$). We now represent the velocity components, pressure and temperature in the neighbourhood of the plate as follows:

$$\left.\begin{array}{r} u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ... \\ v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + ... \\ w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + ... \\ p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + ... \\ \theta = \theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + ... \end{array}\right\}. \tag{14}$$

Substituting (14) in equations (8) and (12) and comparing the coefficients of like powers ε, and neglecting those of ε^2, we get the following equations as the terms free from ε:

$$v_0' = 0, \tag{15}$$

$$u_0'' - v_0 u_0' = 0, \tag{16}$$

$$v_0'' - v_0 v_0' = p_0', \tag{17}$$

$$w_0'' - v_0 w_0' = 0, \tag{18}$$

$$\theta_0'' - v_0 P_r \theta_0' = -E P_r u_0'^2, \tag{19}$$

where the primes denote differentiation with respect to y. The corresponding boundary conditions are:

$$\left.\begin{array}{r} y = 0 : u_0 = 0,\ v_0 = -\lambda,\ w_0 = 0,\ \theta_0 = 1, \\ y \to \infty : u_0 = 1,\ v_0 = -\lambda,\ w_0 = 0,\ p_0 = p_\infty,\ \theta_0 = 0 \end{array}\right\}. \tag{20}$$

The solutions of equations (15) to (19) under the boundary conditions (20) are

$$u_0 = 1 - \exp(-\lambda y), \tag{21}$$

$$\theta_0 = (1 + E_1)\exp(-\lambda P_r y) - E_1 \exp(-2\lambda y), \tag{22}$$

$$v_0 = -\lambda,\ w_0 = 0,\ p_0 = p_\infty, \tag{23}$$

where $E_1 = \frac{EP_r}{2(2-P)_r}$. This is the solution of a two-dimensional problem in the presence of heat due to viscous dissipation.

The terms as the coefficient of ε, with the help of (23) give following equations:

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \tag{24}$$

$$v_1 \frac{\partial u_0}{\partial y} - \lambda \frac{\partial u_1}{\partial z} = \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right), \tag{25}$$

$$-\lambda \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right), \tag{26}$$

$$-\lambda\frac{\partial w_1}{\partial y}=-\frac{\partial p_1}{\partial y}+\left(\frac{\partial^2 w_1}{\partial y^2}+\frac{\partial^2 w_1}{\partial z^2}\right), \tag{27}$$

$$v_1\frac{\partial\theta_0}{\partial y}-\lambda\frac{\partial\theta_1}{\partial y}=\frac{1}{P_r}\left(\frac{\partial^2\theta_1}{\partial y^2}+\frac{\partial^2\theta_1}{\partial z^2}\right)+2E\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y}. \tag{28}$$

The corresponding boundary conditions are:

$$\left.\begin{array}{l} y=0: u_1=0,\ v_1=-\lambda\cos\pi z,\ w_1=0,\ \theta_1=0, \\ y\to\infty: u_1=0,\ v_1=-\lambda,\ w_1=0,\ p_1=0,\ \theta_1=0 \end{array}\right\}. \tag{29}$$

These are the linear partial differential equations which describe the three-dimensional flow. To solve the equations (25) to (28), we assume u_1, v_1, w_1, p_1 and θ_1 of the following form

$$u_1(y,z)=u_{11}(y)\cos\pi z, \tag{30}$$

$$v_1(y,z)=v_{11}(y)\cos\pi z, \tag{31}$$

$$w_1(y,z)=-\frac{1}{\pi}v_{11}'(y)\sin\pi z, \tag{32}$$

$$p_1(y,z)=p_{11}(y)\cos\pi z, \tag{33}$$

$$\theta_1(y,z)=\theta_{11}(y)\cos\pi z, \tag{34}$$

where the expressions for $v_1(y,z)$ and $w_1(y,z)$ have been chosen so that the equation of continuity (24) is satisfied. Substituting (30) to (34) in equations (25) to (28), we obtain the following set of equations:

$$u_{11}''+\lambda u_{11}'-\pi^2 u_{11}=v_{11}u_0', \tag{35}$$

$$v_{11}''+\lambda v_{11}'-\pi^2 v_{11}=p_{11}', \tag{36}$$

$$v_{11}''+\lambda v_{11}''-\pi^2 v_{11}=p_{11}', \tag{37}$$

$$\theta_{11}''+\lambda P_r\theta_{11}'-\pi^2\theta_{11}=P_r v_{11}\theta_0'-2P_r E u_0' v_0', \tag{38}$$

where the primes denote differentiation with respect to y. The corresponding boundary conditions are:

$$\left.\begin{array}{l} y=0: u_{11}=0,\ v_{11}=-\lambda,\ v_{11}'=0,\ \theta_{11}=0, \\ y\to\infty: u_{11}=0,\ v_{11}=0,\ p_{11}=0,\ \theta_{11}=0 \end{array}\right\}. \tag{39}$$

Solving the equations (35) to (38) under the boundary conditions (39) and using equations (30) to (34), we get

$$u_1(y,z)=\frac{\lambda}{n-\pi}\left[\frac{\pi}{2n}\exp(-n-\lambda)y-\frac{n}{\pi}\exp(-\pi-\lambda)y-\left(\frac{\pi}{2n}-\frac{n}{\pi}\right)\exp(-ny)\right]\cos(\pi z), \tag{40}$$

$$v_1(y,z)=\frac{\lambda}{n-\pi}\left[\pi\exp(-ny)-n\exp(-\pi y)\right]\cos(\pi z), \tag{41}$$

$$w_1(y,z)=\frac{\lambda}{n-\pi}\left[\exp(-ny)-\exp(-\pi y)\right]\sin(\pi z), \tag{42}$$

$$p_1(y,z)=\frac{\lambda^2 n}{n-\pi}\exp(-\pi y)\cos(\pi z), \tag{43}$$

$$\theta_1(y,z) = \frac{\lambda P_r}{n-\pi}\begin{bmatrix} \frac{2nN_1}{A}\left(e^{-my} - e^{-(\pi+2\lambda)y}\right) - \frac{2nN_2}{B}\left(e^{-my} - e^{-(\pi+2\lambda)y}\right) \\ +\frac{P_r N_3}{1+P_r}\left(e^{-my} - e^{-(n+\lambda P_r)y}\right) - N_4\left(e^{-my} - e^{-(\pi+\lambda P_r)y}\right) \\ -N_5\left(e^{-my} - e^{-(n+\lambda)y}\right) \end{bmatrix}\cos\pi z. \quad (44)$$

where

$$m = \frac{1}{2}\left[\lambda P_r + \left(\lambda^2 P_r^2 + 4\pi^2\right)^{1/2}\right],\ n = \frac{1}{2}\left[\lambda_r + \left(\lambda^2 + 4\pi^2\right)^{1/2}\right],$$
$$A = 4(\pi+\lambda) + P_r(\pi+2\lambda),\ B = 5n + 4\lambda - P_r(n+2\lambda),$$
$$N_1 = \frac{n(1+E_1)}{\pi},\ N_2 = E_1 + \frac{E(n+\lambda)}{2n},\ N_3 = \frac{\pi(1+E_1)}{n},$$
$$N_4 = \frac{n(1+E_1)}{\pi},\ N_5 = \frac{2nE}{3n+\lambda-P_r(n+\lambda)}\left(\frac{n}{\pi} - \frac{\pi}{2n}\right).$$

Since the expressions for $v_1(y,z)$ and $w_1(y,z)$ are of the same form as obtained in reference (5), so there will not be considered any more. Substituting equations (20), (21), (40) and (41) in equation (14), we can obtain the expressions for the main flow u and the temperature θ.

3. Results and discussions

For the purpose of discussing the effects of various parameters on the flow and temperature fields near the plate, numerical calculations are carried out for different values of λ, P_r and E. In order to be realistic, the values of Prandtl number are chosen 0.71 and 7 approximately, which correspond to air and water respectively at $20^\circ C$. The other value of P_r is taken arbitrarily. The Eckert number would take positive or negative accordingly as the temperature difference $T'_w - T'_\infty >$ or < 0 where T'_w is the wall temperature and T'_∞ is the free stream temperature. Both these cases physically correspond to the addition of heat due to viscous dissipation. The value of Eckert number E are very small for an incompressible fluid. Thus, to be more appropriate from the practice point of view, the values of the Eckert number have been chosen -0.05 and $+0.05$ and that of the suction parameter between 0 and 1.

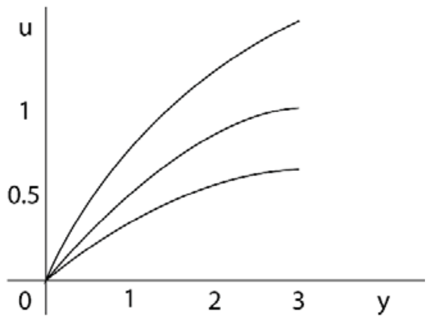
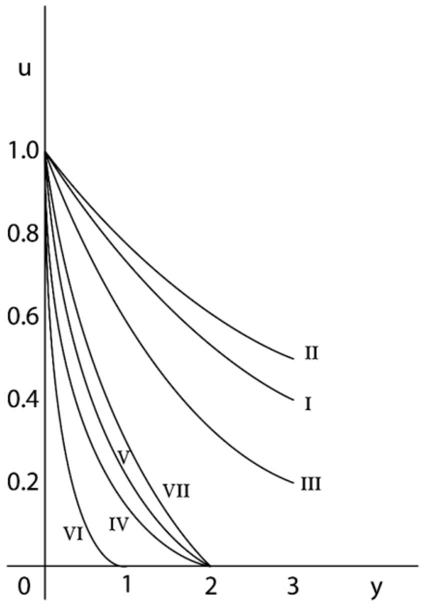


Figure 1. Velocity profile for $\varepsilon = 0.2$, $z = 0$.

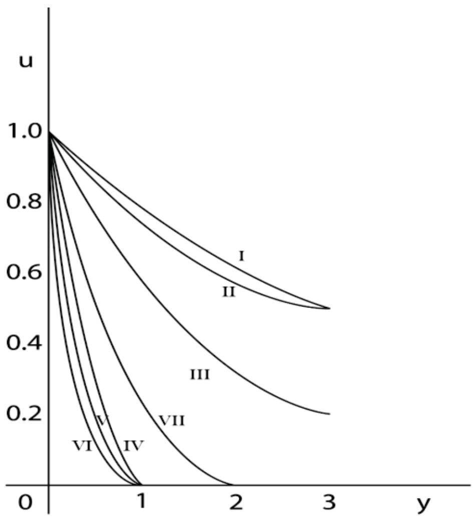
The main flow velocity profiles are shown in Figure 1 and it is observed that the velocity increases with the increase in the suction parameter λ .

The temperature profile for the cases $T'_w - T'_\infty > 0$ and $T'_w - T'_\infty < 0$, i.e., for Eckert number positive and negative are shown in Figures 2 and 3 respectively. From Figure 2, we observe that the temperature in the boundary layer increases due to more addition of viscous dissipative heat but decreases with increasing λ in the case of air $(P_r = 0.71)$ and water $(P_r = 7)$ both. Also, there is a drop in the temperature as Prandtl number increases from 5 to 7. Similarly Figure 3 shows that the boundary layer temperature falls owing to greater viscous dissipative heat or greater suction parameter for both air and water. In this case also the temperature decreases with increasing Prandtl number.



	λ	P_r	E
I	0.5	0.71	0.01
II	0.5	0.71	0.05
III	1.0	0.71	0.01
IV	0.5	7.0	0.01
V	0.5	7.0	0.05
VI	1.0	7.0	0.01
VII	0.5	5.0	0.01

Figure 2. Temperature profile for $\varepsilon = 0.2$ and $z = 0$.



	λ	P_r	E
I	0.5	0.71	-0.01
II	0.5	0.71	-0.05
III	1.0	0.71	-0.01
IV	0.5	7.0	-0.01
V	0.5	7.0	-0.05
VI	1.0	7.0	-0.01
VII	0.5	5.0	-0.01

Figure 3. Temperature profile for $\varepsilon = 0.2$ and $z = 0$.

The main flow velocity being known, the shear stress in the main flow direction is obtained as follows

$$\tau = \frac{\tau'}{\rho' U_0'} = \left(\frac{\partial u}{\partial y}\right)_{y=0} = -\lambda + \frac{\varepsilon\lambda}{n-\lambda}\left[n\left(\frac{\pi}{2n} - \frac{n}{\pi}\right) + \frac{n(\pi+\lambda)}{\pi} - \frac{\pi(n+\lambda)}{2n}\right]\cos\pi z. \quad (45)$$

The values of the shear stress are given in Table 1 and it is clear from these values that the shear stress increases with the increasing suction of the boundary layer.

Table 1. Shear stress τ for $\varepsilon = 0.2$ and $z = 0$.

λ	0.2	0.5	1.0
τ	1.6014	2.1363	6.5979

From the temperature filed, we can now calculate the expression for the rate of heat transfer as

$$q = \frac{q' v}{U_0 K\left(T_w' - T_\infty'\right)} = -\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = \lambda P_r\left(1+E_1\right) - 2\lambda E_1 + \frac{E\lambda P_r}{n-\pi}\left[\begin{array}{c}\frac{2nN_1}{A}(m-\pi-2\lambda) - \frac{2nN_2}{B}(m-n-2\lambda) + \frac{P_r N_3}{1+P_r}\left(m-n-\lambda P_r\right) \\ -N_4\left(m-\pi-\lambda P_r\right) - N_5(m-n-\lambda)\end{array}\right]\cos\pi z \quad (46)$$

The numerical values of the rate of heat transfer q are given in Table 2. It is found that due to greater viscous dissipative heat the rate of heat transfer increases in air irrespective of the fact whether the Eckert number is positive or negative.

Table 2. Rate of heat transfer q : $\varepsilon = 0.2$ and $z = 0$.

λ	$E \setminus Pr$	0.71	5	7
0.5	0.01	0.35891	2.71028	3.87323
0.5	0.05	0.37921	2.46804	3.79688
1.0	0.01	0.72808	5.30583	8.04565
0.5	-0.01	0.36252	2.69583	3.91140
0.5	-0.05	0.36975	2.79087	3.98775
1.0	-0.01	0.73542	5.70470	8.13078

In the case of large Prandtl number, owing to more addition of viscous dissipative heat, the rate of heat transfer decreases or increases accordingly as $E > 0$ or < 0. It is also evident that an increase in the suction parameter give rise to an increase in the rate of transfer.

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