International Journal of
Mathematical Modelling & Computations
Vol. 08, No. 03, Summer 2018, 199- 206

Reproducing Kernel Space Method for Solving Generalized Burgers Equation

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Abstract. In this paper, we present a new method for solving Reproducing Kernel Space (RKS) theory, and iterative algorithm for solving Generalized Burgers Equation (GBE) is presented. The analytical solution is shown in a series in a RKS, and the approximate solution $u(x,t)$ is constructed by truncating the series. The convergence of $u(x,t)$ to the analytical solution is also proved.

Received: 8 June 2018, Revised: 17 September 2018, Accepted: 10 December 2018.

Keywords: Reproducing kernel space; Generalized Burgers equation; Norm space.

AMS Subject Classification: 34A35, 34B60, 35A35.

Index to information contained in this paper

1. Introduction

The reproducing kernel was first used in the early 20th century in Zaremba research on boundary value problems for harmonic functions and two harmonic moduli. In 1907, he was the first person to introduce the corresponding kernel to a class of functions and express the reproducing property. After along interruption, the idea of the reproducing kernel was restored in the thesis of three German mathematicians called Zigo(1921), Bergman(1922) and Bacchner(1922). The theory of the reproducing kernels was arranged by Aarozan in 1948. Bergman and schiffer, with

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the development of the basic idea of Zaremba, in solving the boundary value problems, using their reproducing kernels introduced them as a powerful tool for solving elliptic boundary value problems. RKS-based methods include symbolic methods and numerical methods. In symbolic methods, RKS first appears on the basis of derivative and the boundary conditions governing the problem are obtained. Then, the reproducing space function is obtained as a multiplicative function with symbolic operation. RKS-based method for solving singular two-point boundary value problems[4], nonlinear numerical analysis in RKS [1], Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space [6] and Etc, has been successful. In this paper, we use the RKS method to solve the generalized Burgers equation. Then, we analysis the convergence of the method, and finally calculate the exact answer and error in the last example.

We consider the GBE

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\mu(t)\frac{\partial^2 v}{\partial x^2}+\lambda(t)v(x,t)+f(x,t);\;\; 0<x<1, 0<t<1, \tag{1}$$

with the conditions

$$\{v(x,0)=N(x), 0\leqslant x\leqslant 1,\; v(0,t)=v(1,t), 0\leqslant t\leqslant 1,\; v_x(1,t)=0, 0\leqslant t\leqslant 1\}. \tag{2}$$

The direct problem for equation (1) is to find a function $u(x,t)$ satisfying equation (1) with given $N(x),\mu(t)$ and $f(x,t)$ and the conditions (2). $u(x,t),\lambda(t)$ are unknown functions to be determined. It is required the additional condition

$$\int_0^1 v(x,t)dx=E(t), \tag{3}$$

Where $E(t)$ is the function given.

2. Solution method

We integrate equation (1) with respect to x on $[0,1]$. We get

$$\frac{\partial}{\partial t}\int_0^1 vdx+\frac{v^2(1,t)}{2}-\frac{v^2(0,t)}{2}=\mu(t)(\frac{v(1,t)}{\partial x}-\frac{v(0,t)}{\partial x})+\lambda(t)\int_0^1 v(x,t)+\int_0^1 f(x,t)dx.$$

Hence, with the conditions (2), (3), $\lambda(t)$ to get the following

$$\lambda(t)=\frac{E^{'}(t)+\mu(t)\left(\frac{\partial v(0,t)}{\partial x}\right)-\int_0^1 f(x,t)dx}{E(t)}. \tag{4}$$

Let $u(x,t)=v(x,t)-N(x)$, equation (1) can be transformation into following equivalent form

$$\frac{\partial u}{\partial t}-\mu(t)\frac{\partial^2 u}{\partial x^2}-\frac{E^{'}(t)+\mu(t)\left(\frac{\partial u(0,t)}{\partial x}+N(0)\right)-\int_0^1 f(x,t)dx}{E(t)}u=F(x,t,u,u_x), \tag{5}$$

with the conditions

$$u(x,0)=0,\quad 0\leqslant x\leqslant 1,$$

$$u(0,t)=u(1,t),\quad 0\leqslant t\leqslant 1,$$

$$u_x(1,t)=0,\quad 0\leqslant t\leqslant 1.$$

Where

$$F(x,t,u,u_x)=-\mu(t)N^{''}(x)+N(x)\frac{E^{'}(t)+\mu(t)\left(\frac{\partial u(0,t)}{\partial x}+N(0)\right)-\int_0^1 f(x,t)dx}{E(t)}$$

$$-uu_x-uN^{'}(x)-N(x)u_x-N(x)N^{'}(x)+f(x,t).$$

Definition 2.1 The RKS $W_2^3[0,1]$ is defined by $W_2^3=\{u\in W^3 \mid u(0)=u(1),u^{'}(1)=0,$ are absolutely continuous real valued functions in $[0,1]$, $u^3\in l^2[0,1]\}$.

[1] With the inner product and norm in $W_2^3[0,1]$

$$\langle u,v\rangle_{W^3}=u(0)v(0)+u^{'}(0)v^{'}(0)+u^{''}(0)v^{''}(0)+\int_0^1 u^3(x)v^3(x)dx,$$

$$u(x),v(x)\in W_2^3[0,1],\qquad \|u\|_{W^3}=\sqrt{\langle u,u\rangle_{W^3}}.$$

The reproducing kernel function $R_y(x)$ in [0, 1]

$$R_y(x)=\{\sum_{i=0}^{6}c_i(y)x^{i-1};x\leqslant y\qquad,\sum_{i=0}^{6}d_i(y)x^{i-1};y<x\}.$$

There for $R_y(x)$ is the solution of the following generalized differential equations

$R_y(0)-R_y(1)=0,$
$R_y^{'}(1)=0,$
$R_y^{(3)}(1)=0,$
$R_y(0)-\frac{\partial^5 R_y(0)}{\partial x^5}+\frac{\partial^5 R_y(1)}{\partial x^5}=0,$
$\frac{\partial R_y(0)}{\partial x}+\frac{\partial^4 R_y(0)}{\partial x^4}=0,$
$\frac{\partial^2 R_y(0)}{\partial x^2}+\frac{\partial^3 R_y(0)}{\partial x^3}=0,$
$\frac{\partial^i R_y(y^-)}{\partial x^i}-\frac{\partial^i R_y(y^-)}{\partial x^i}=0,\qquad i=0,1,2,3,4,$
$\frac{\partial^5 R_y(y^+)}{\partial x^5}-\frac{\partial^5 R_y(y^-)}{\partial x^5}=-1.$

The reproducing kernel functions $R_y(x)$and$R_x(y)$ in $w_2^3[0,1]$

$$R_y(x) = \{1 + \tfrac{1}{120}x^5 - \tfrac{17}{1131}x^5y + \tfrac{11}{2262}x^5y^2 + \tfrac{11}{6786}x^5y^3 + \tfrac{17}{27144}x^5y^4 - \tfrac{7}{16965}x^5y^5$$
$$-\tfrac{3}{3016}x^4y + \tfrac{7}{4524}x^4y^2 + \tfrac{7}{13572}x^4y^3 - \tfrac{23}{13572}x^4y^4 + \tfrac{17}{27144}x^4y^5 - \tfrac{14}{1131}x^3y$$
$$+d\tfrac{43}{1508}x^3y^2 - \tfrac{62}{3393}x^3y^3 + \tfrac{7}{13572}x^3y^4 + \tfrac{11}{6786}x^3y^5 - \tfrac{14}{377}x^2y + \tfrac{129}{1508}x^2y^2 - \tfrac{62}{1131}x^2y^3$$
$$+\tfrac{7}{4524}x^2y^4 + \tfrac{11}{2262}x^2y^5 + \tfrac{9}{377}xy - \tfrac{14}{377}xy^2 - \tfrac{14}{1131}xy^3 + \tfrac{46}{1131}xy^4$$
$$-\tfrac{17}{1131}xy^5, \qquad x \leqslant y;$$

$$R_x(y) = 1 + \tfrac{1}{120}y^5 - \tfrac{17}{1131}xy^5 + \tfrac{11}{2262}x^2y^5 + \tfrac{11}{6786}x^3y^5 + \tfrac{17}{27144}x^4y^5 - \tfrac{7}{16965}x^5y^5$$
$$-\tfrac{3}{3016}xy^4 + \tfrac{7}{4524}x^2y^4 + \tfrac{7}{13572}x^3y^4 - \tfrac{23}{13572}x^4y^4 + \tfrac{17}{27144}x^5y^4 - \tfrac{14}{1131}xy^3$$
$$+d\tfrac{43}{1508}x^2y^3 - \tfrac{62}{3393}x^3y^3 + \tfrac{7}{13572}x^4y^3 + \tfrac{11}{6786}x^5y^3 - \tfrac{14}{377}xy^2 + \tfrac{129}{1508}x^2y^2 - \tfrac{62}{1131}x^3y^2$$
$$+\tfrac{7}{4524}x^4y^2 + \tfrac{11}{2262}x^5y^2 + \tfrac{9}{377}xy - \tfrac{14}{377}x^2y - \tfrac{14}{1131}x^3y + \tfrac{46}{1131}x^4y - \tfrac{17}{1131}x^5y, y < x.$$

Definition 2.2 From refer.[1], The RKS $W^1[0,1] = \{u(t) \mid u(t), u^{'}(t)$ are absolutely continuous, $u^{''}(t) \in L^2[0,1], u(0) = 0\}$.

the inner product and norm are defined by

$$\langle u, v\rangle_{W^1} = u(0)v(0) + u^{'}(0)v^{'}(0) + \int_0^1 u^{''}(t)v^{''}(t)dt,$$

$$\|u\|_{W^1} = \sqrt{\langle u, u\rangle_{W^2}}.$$

The reproducing kernel functions $r_s(t)$and $r_t(s)$ in $[0,1]$

$$r_s(t) = \{st + \frac{1}{2}st^2 - \frac{1}{6}t^3; t \leqslant s\},$$

$$r_t(s) = \{st + \frac{1}{2}ts^2 - \frac{1}{6}s^3; s < t\}.$$

The equation (5) in space $L : W^{(2,3)}(\Omega) \longrightarrow W^{(2,1)}(\Omega), \Omega = [0,1] \times [0,1]$, we have the following

$$Lu = F(x, t, u, u_x), \tag{6}$$

With The conditions

$$u(x, 0) = 0, \quad 0 \leqslant x \leqslant 1,$$

$$u(0, t) = u(1, t), \quad 0 \leqslant t \leqslant 1,$$

$$u_x(1,t) = 0, \quad 0 \leqslant t \leqslant 1,$$

Where $Lu = \dfrac{\partial u}{\partial t} - \mu(t)\dfrac{\partial^2 u}{\partial x^2} - \dfrac{E^{'}(t) + \mu(t)\left(\frac{\partial u(0,t)}{\partial x} + N(0)\right) - \int_0^1 f(x,t)dx}{E(t)}u$, and from refer. [3], it is explain that L is a bounded linear operator.

Now, assume that $\varphi_i(x,t) = K_{(x_i,t_i)}(x,t), \{(x_i,t_i)\}_{i=0}^{\infty}$Is dense in region Ω
Let

$$\psi_i(x,t) = L^*\varphi_i(x,t),$$

Where L^* adjoint operator L, and ψ_iGram-schmith orthonormal sequence.
From the property of the reproducing kernel, we have

$$\langle \psi_i(x,t), \psi_j(x,t)\rangle_{W^{(2,3)}} = \langle L^*\varphi_i(x,t), \psi_j(x,t)\rangle_{W^{(2,3)}}$$

$$= \langle \varphi_i(x,t), L\psi_i(x,t)\rangle_{W^{(2,1)}} = L\psi_j(x,t),$$

And also

$$\langle u(x,t), \psi_i(x,t)\rangle_{W^{(2,3)}} = \langle u(x,t), L*\varphi_i\rangle_{W^{(2,3)}}$$

$$= \langle Lu(x,t), \varphi_i(x,t)\rangle_{W^{(2,1)}} = Lu(x_i,t_i) = F(x_i,t_i,u(x_i,t_i),\partial_x u(x_i,t_i)).$$

3. Convergence of method

Suppose $u(x,t) \in \Omega : [0,1] \times [0,1]$ and also it is a RKS

$$K_{(y,s)}(x,t) = R_y(x)r_s(t),$$

Where $R_y(x), r_s(t)$are the reproducing kernels of $W^{(2,3)}[0,1]$ and $W^{(2,1)}$, respectively. If $u(x,t) \in \Omega$we have

$$u(x,t) = \langle u(y,s), K_{(x,t)}(y,s)\rangle_{W^{(2,3)}},$$

so

$$|u(x,t)| = |\langle u(y,s), K_{(x,t)}(y,s)\rangle|_{W^{(2,3)}} \leqslant \|u(y,s)\|_{W^{(2,3)}}\|K_{(x,t)}(y,s)\|_{W^{(2,3)}},$$

The exist $c_1 > 0$ so that

$$|u(x,t)| \leqslant c_1\|u(y,s)\|_{W^{(2,3)}},$$

similarly,

$$|u_x(x,t)| = |\langle u(y,s), \frac{\partial K}{\partial x}(y,s)\rangle|_{W^{(2,3)}} \leqslant \|u(y,s)\|_{W^{(2,3)}}\|\frac{\partial K}{\partial x}(y,s)\|_{W^{(2,3)}},$$

Then there exists $c_2 > 0, c_3 > 0$ such that

$$|u_x(x,t)| \leqslant c_2 \|u(y,s)\|_{W^{(2,3)}},$$

$$|u_t(x,t)| \leqslant c_3 \|u(y,s)\|_{W^{(2,3)}}.$$

Theorem 3.1 If $u_n(x,t) \longrightarrow u^*(x,t)$, when the $n \to \infty$,and $(x_n,t_n) \longrightarrow (x,t), n \to \infty$, then$\|u_n(x,t)\|_{W^{(2,3)}}$is bounded ,also , if $F(x,t,u,u_x)$ the continuous, we have

$$F(x_n,t_n,u_{n-1},\partial_x u_{n-1}) \longrightarrow F(x,t,u^*,\partial_x u^*), whenthen \to \infty$$

Proof.

$$\begin{aligned}
&|u_{n-1}(x_n,t_n) - u^*(x,t)| = |u_{n-1}(x_n,t_n) - u_{n-1}(x,t) + u_{n-1}(x,t) - u^*(x,t)| \\
&\leqslant |u_{n-1}(x_n,t_n) - u_{n-1}(x,t)| + |u_{n-1}(x,t) - u^*(x,t)| \\
&\leqslant |\partial_x u_{n-1}(x_n,t_n)||x_n - t| + |\partial_t u_{n-1}(x_n,t_n)||t_n - t| + |u_{n-1}(x,t) - u^*(x,t)|,
\end{aligned}$$

similarly,

$$|\partial_x u_{n-1}(x_n,t_n) - \partial_x u^*(x,t)| = |\partial_x u_{n-1}(x_n,t_n) - \partial_x u_{n-1}(x,t) + \partial_x u_{n-1}(x,t) - \partial_x u^*(x,t)|$$

$$\leqslant |\partial_{xx} u_{n-1}(x_n,t_n)||x_n - t| + |\partial_{xt} u_{n-1}(x_n,t_n)||t_n - t)| + |\partial_x u_{n-1}(x,t) - \partial_x u^*(x,t)|.$$

The continuous $F(x,t,u,u_x)$ we have

$$F(x_n,t_n,u_{n-1}(x_n,t_n), \partial_x u_{n-1}(x_n,t_n) \longrightarrow F(x,t,u^*,\partial_x u^*),$$

when the $n \to \infty$.$\square$

Theorem 3.2 If $\{(x_i,t_i)\}_{i=0}^{\infty}$ is dense Ω then the analytic solution (6) as the following them

$$u_n(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_i,t_i,u_n(x_i,t_i), \partial_{x_i} u_n(x_i,t_i)) \psi_i^*(x,t),$$

where β_{ik} are orthogonal coefficients, and $\psi_i^*(x,t) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x,t)$.

Proof. consider that $\{\psi_i^*(x,t)\}_{i=1}^{\infty}$ on $W^{(2,3)}[0,1]$ is complete. we have

$$u_n(x,t) = \sum_{i=1}^{\infty} \langle u_n(x,t), \psi_i^*(x,t) \rangle_{W^{(2,3)}} . \psi_i^*(x,t)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u_n(x,t), \psi_k(x,t) \rangle_{W^{(2,3)}} . \psi_i^*(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u_n(x,t), L^*\varphi_k(x,t)\rangle_{W^{(2,3)}}.\psi_i^*(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu_n(x,t), \varphi_k(x,t)\rangle_{W^{(2,1)}}.\psi_i^*(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}Lu_n(x_i,t_i).\psi_i^*(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_i,t_i,u_n(x_i,t_i,\partial_{x_i}u_n(x_i,t_i))\psi_i^*(x,t).\square$$

4. Example

The consider GBE

$$u_t - e^{-t}(-t^3 - 3t^2 + t + 3)u_{xx} - \lambda(t)u = -uu_x + f(x,t)$$

With The conditions

$$u(x,0) = 0,$$

$$u(0,t) = u(1,t),$$

$$u_x(1,t) = 0.$$

Where

$$f(x,t) = -2\pi t^2 sin(2\pi x) - \pi t^2 sin(4\pi x) + 4\pi^2 te^{-t}(-t^3 - 3t^2 + t + 3)cos(2\pi x),$$

and

$$\lambda(t) = \frac{1}{t}.$$

the exact solution

$$u(x,t) = t(1 + cos(2\pi x)).$$

We have the method an approximation solution in the following

Table 1. Approximation solutions.

(x,t)	Exact solution	Approximation solution	Error
$(0.1, 0.1)$	0.19999398715	0.19999457764	0.00000059049
$(0.16, 0.16)$	0.31997537177	0.31997478296	0.00000058881
$(0.22, 0.22)$	0.43993597767	0.43993584567	0.000000132
$(0.28, 0.28)$	0.55986801504	0.55986802467	0.00000000963
$(0.34, 0.34)$	0.67976369609	0.67976368561	0.00000001048
$(0.4, 0.4)$	0.79961523566	0.79961524749	0.00000001183
$(0.46, 0.46)$	0.91936287521	0.91936286936	0.00000000585
$(0.52, 0.52)$	1.03915476627	1.03915478851	0.00000002224
$(0.58, 0.58)$	1.15882720517	1.15882720895	0.00000000378
$(0.64, 0.64)$	1.27842439946	1.27842439752	0.00000000194
$(0.7, 0.7)$	1.39793858551	1.39793858577	$2.6E-10$
$(0.76, 0.76)$	1.51736200571	1.51736200423	0.00000000148
$(0.82, 0.82)$	1.63668690899	1.63668690752	0.00000000147
$(0.88, 0.88)$	1.75590555138	1.75590555278	0.0000000014
$(0.94, 0.94)$	1.87446578395	1.87446578682	0.00000000287

5. Conclusion

In this paper we proposed a numerical method, based on the RKS method for solving the generalized Burgers equation. The RKS method for solving the GBE is a suitable method that shows the high accuracy as in the example, our proposed method the error gets smaller.

Acknowledgment

We would like to thank Ali Ebadian for many helpful suggestions.

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