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Numerical Solution of the First-Order Evolution Equations by Radial Basis Function

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Abstract. In this work, we consider the nonlinear first-order evolution equations: $u_t = f(x, t, u, u_x, u_{xx})$ for $0 < t < \infty$, subject to initial condition $u(x, 0) = g(x)$, where u is a function of x and t and f is a known analytic function. The purpose of this paper is to introduce the method of RBF to existing method in solving nonlinear first-order evolution equations and also the method is implemented in four numerical examples. The results reveal that the technique is very effective and simple.

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1. Introduction

Over the last decades several analytical/numerical methods have been developed to solve nonlinear equations. For initial-value problems in ordinary differential equations, some of these technique include perturbation [12], variational [6−−8], decomposition [2] methods, etc [14]. In the recent years, radial basis function collocation has become a useful alternative to finite difference and finite element methods for solving elliptic partial differential equations. RBF collocation methods have been shown numerically [10] and theoretically [5] to be very accurate even for a small number of collocation points. In application finite difference methods often have a low approximation order and consequently can require a large grid and considerable computation to obtain a sufficiently accurate solution. RBF collocation has

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been applied to linear elliptic PDEs in $\mathbb{R}^2$ and $\mathbb{R}^3$ [11], to time dependent problems [9], and to non-linear problems [13]. In this paper we present new numerical results for RBF collocation, and the purpose of this work, is to use RBF method to solve the first-order evolution nonlinear equations. We can present this technique to the other similar problems. The paper is organized as follows. In Section 2, we consider radial basis function and newton's method, respectively. We then introduce first-order evolution equation and consider approximating solution. Some applications to the first-order evolution equation problems are presented in Section 4. A summary of the main conclusions presented in the last section of the paper.

2. Preliminaries radial basis functions and newton's method

Radial Basis Functions (RBFs) are popular for interpolating scattered data since the associated system of linear equations is guaranteed to be invertible under very mild conditions on the location of the data points. For example, the thin-plate spline used in this library only requires that the points are not co-linear. In particular, Radial Basis Functions do not require that the data lie on any sort of regular grid [4]. A radial basis function (RBF) is a function of the form:

$$s(x) = p(x) + \sum_{i=1}^{N} \lambda_i \Phi(x - x_i), \tag{1}$$

where: s is the radial basis function (RBF for short) and p is a low degree polynomial, typically linear or quadratic and the λ_i's are the RBF coefficients and Φ is a real valued function called the basic function and the x_i 's are the RBF centres. The RBF consists of a weighted sum of a radially symmetric basic function ϕ located at the centres x_i and a low degree polynomial p. Given a set of N points x_i and values f_i, the process of finding an interpolating RBF, s, such that, $s(x_i) = f_i, \ \ i = 1, 2, ..., N$ is called fitting. The fitted RBF is defined by the λ_i, the coefficients of the basic function in the summation, together with the coefficients of the polynomial term $p(x)$. [4] For a fixed point $x_j \in \mathbb{R}^d$, a radial basis function is defined:

$$\phi_j(x) = \phi(\| (x - x_j) \|) \tag{2}$$

which is function only dependents on the distance between $x_j \in \mathbb{R}^d$ and the point x_j. This function is radially symmetric near the center x_j.

$$\phi(r) = \exp(-cr^2) \tag{3}$$

$$\phi(r) = \sqrt{r^2 + c^2} \tag{4}$$

However c is a shape parameter which should be considered suitably also the Euclidean distance is considered for the **RBF** have a global support [3].

Newton's method for nonlinear equations: A system of nonlinear equations is expressed in the form $F(X) = 0$, where F is a vector-valued function of the vector variable $X; F : R \longrightarrow R$. Given an estimate $X^{(k)}$ of a solution X^*. Newton's method indicates the estimate $X^{(k+1)}$ by setting the local linear approximation to

F at $X^{(k)}$ to zero and solving for X:

$$J(X^{(k)})H^{(k)} = -F(X^{(k)}), \quad X^{(k+1)} = X^{(k)} + H^{(k)} \quad , \quad k = 0, 1, 2, ... \tag{5}$$

In this calculation, $J = J(X^{(k)})$ is the Jacobian matrix of F at $X^{(k)}$.

3. First-order evolution

Consider the first-order nonlinear evolution equation, [1, 14]

$$u_t = f(x, t, u, u_x, u_{xx}) \quad , \quad 0 < t < \infty \quad , \quad u(x, 0) = g(x) \tag{6}$$

where t denotes time $0 < t < \infty$ and x is the spatial coordinates and f is a nonlinear function of u, u_x, u_{xx} and the subscripts denote partial differentiation. Integrated from equation (6) to yield:

$$u(x, t) = g(x) + \int_0^t f(x, p, u, u_x, u_{xx})dp,$$

solve iteratively as,

$$u^{k+1}(x, t) = g(x) + \int_0^t f(x, p, u^k, u_x^k, u_{xx}^k)dp,$$

which k shows the k th iteration. The $g(x) + \int_0^t f dp$ is a contractive mapping. The convergence of this equation is ensured by Banach's fixed-point theorem [15].

Let (X, d) be a nonempty complete metric space and let $X \longrightarrow X$ be a contraction mapping on X, then is, there exists a nonnegative real number $q < 1$ such that $d(Tx, Ty) \leqslant qd(x, y)$ for all x and y in X.Then the map T admits one and only one fixed point x^* in X such that $Tx^* = x$ which can be determined by starting with an arbitrary x^0 in X and the iterative sequence $x^k = Tx^k - 1, \ \ n = 1, 2, 3, ...,$ converge and its limits is x^* with the following speed of convergence $d(x^*, x^k) \leqslant \frac{q^k}{1-q} d(x^1, x^0)$, and the smallest value of q is sometimes called the Lipschitz condition.

In the case that equation (6) represents an ordinary differential equation, i.e., $f = f(u, t)$, the Picard-Lindelof theorem indicates that equation (6) has a unique solution if f is continuous with respect to t and Lipschitz continuous with respect to u, u_x, u_{xx}.

To illustrate the basic concepts of this technique, consider the approximation solution as:

$$\widetilde{u}(x, t) \equiv \sum_{i=1}^{N} c_i \phi_i(x, t) \tag{7}$$

which c_i's are constants and the ϕ_i's are radial basis functions. where $\phi_i(x, t) = \phi(\|(x - x_i, t - t_i)\|)$ is a radial basis function on $r = \|(x, t)\|$. From equation (7) we have:

$$\widetilde{u}_t \equiv \sum_{i=1}^{N} c_i \frac{\partial \phi_i}{\partial t}(x, t), \quad \widetilde{u}_x \equiv \sum_{i=1}^{N} c_i \frac{\partial \phi_i}{\partial x}(x, t), \quad \widetilde{u}_{xx} \equiv \sum_{i=1}^{N} c_i \frac{\partial^2 \phi_i}{\partial x^2}(x, t) \tag{8}$$

Substitution of the equation (8) into (6) yields,

$$\sum_{i=1}^{N} c_i \frac{\partial \phi_i}{\partial t}(x_j, t_j) - f(x, t, \sum_{i=1}^{N} c_i \phi_i, \sum_{i=1}^{N} c_i \frac{\partial \phi_i}{\partial x}, \sum_{i=1}^{N} c_i \frac{\partial^2 \phi_i}{\partial x^2})(x_j, t_j) \equiv 0 \tag{9}$$

For simplicity, by considering:

$$\begin{array}{ll} \alpha_{ij}(x_j, t_j) = \phi_i(x_j, t_j), & \beta_{ij}(x_j, t_j) = \frac{\partial \phi_i}{\partial t}(x_j, t_j) \\ \gamma_{ij}(x_j, t_j) = \frac{\partial \phi_i}{\partial x}(x_j, t_j), & \lambda_{ij}(x_j, t_j) = \frac{\partial^2 \phi_i}{\partial x^2}(x_j, t_j) \end{array}$$

and substituting them into equation (9), we have

$$\sum_{i=1}^{N} c_i \beta_{ij}(x_j, t_j) - f(x_j, t_j, \sum_{i=1}^{N} c_i \alpha_{ij}, \sum_{i=1}^{N} c_i \gamma_{ij}, \sum_{i=1}^{N} c_i \lambda_{ij})(x_j, t_j) \equiv 0 \tag{10}$$

which is a nonlinear system of equations. Furthermore, we assume $N = N_1 + N_2$ that N_1 denotes the number of boundary points and N_2 shows the number of interior points. Suppose that the following sets contain a collocation of scattered nodes in every levels of interpolation

$$\Xi_1 = \{(x_i, t_i) \in \bar{\Omega} \times [0, T_1], \quad i = 1, ..., m\}, \quad T > T_1. \tag{11}$$

$$\Xi_k = \{(x_i, t_i + (k-1)T_1); \ (x_i, t_i) \in \Xi_1, \quad i = 1, ..., m, k = 2, 3, ...\} \tag{12}$$

and the problem has a solution in $\bar{\Omega} \times [(k-1)T_1, kT_1]$.

4. Numerical examples

To achieve the goal of this work, we consider some examples, and we draw them in the another page, you can compare them.

Example 4.1 Consider the following nonlinear partial differential equation:
$u_t = u_{xx}u + 9u^2 + 2u$
with initial conditions: $u(0, t) = 0, \quad u(1, t) = \sin(3)\exp(2t), \quad u(x, 0) = \sin(3x)$
The exact solution is: $u(x, t) = \sin(3x)\exp(2t)$.
By considering $\phi(r) = \sqrt{c^2 + r^2}$, and with taking $c = 2$, $N = 7$, $0 \leqslant x \leqslant 1$, we obtain error graph for this approximation as in Figure 1.

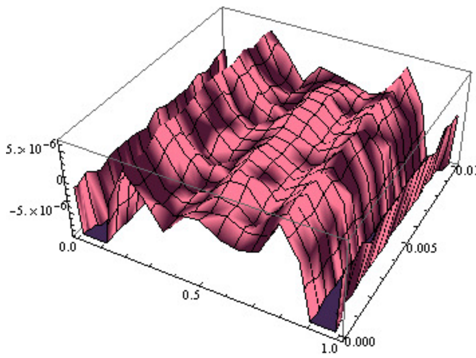
Example 4.2 We assume:
$u_t = u_{xx}u - u_x^2 + \exp(x^2)\cos(t)$
with initial conditions: $u(x, 0) = 0, \quad u(x, 1) = \exp(x^2)\sin(1), \quad u(0, t) = \sin(t)$
The exact solution is: $u(x, t) = \exp(x^2)\sin(t)$.
As stated before, using equation (4), and by taking $c = 3$, $N = 7$, $0 \leqslant x \leqslant 1$, we have Figure 2 for error function.

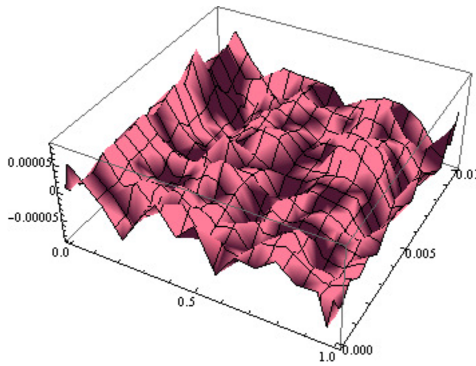
Example 4.3 Consider the following problem:
$u_t = u_{xx}u + u^2 - 3u$
with initial conditions: $u(0, t) = \exp(-3t), \quad u(1, t) = (\sin 1 + \cos 1)\exp(-3t)$,
$u(x, 0) = \sin x + \cos x$
and the exact solution: $u(x, t) = (\sin x + \cos x)\exp(-3t)$.

According to the equation (4) and by considering $c = 3$, $N = 7$, $0 \leqslant x \leqslant 1$, we obtain the error function shown in Figure 3.

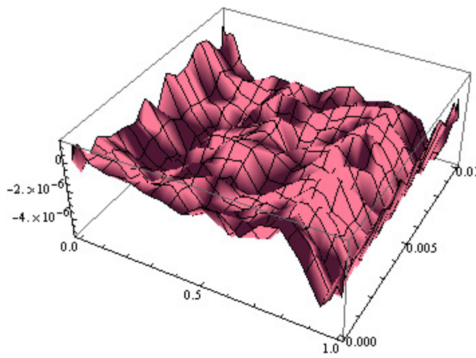
Example 4.4 Consider the problem:
$u_t = \frac{t}{u}$
with the initial conditions: $u(0,t) = \sqrt{t^2+1}$, $\quad u(1,t) = \sqrt{t^2+2}$,
$u(x,0) = \sqrt{x^2+1}$
The exact value is: $u(x,t) = \sqrt{x^2+t^2+1}$.
Using the method and the equation (4), by considering $c = 3$, $N = 7$,
$0 \leqslant x \leqslant 1$, We have the error function which is shown in Figure 4.

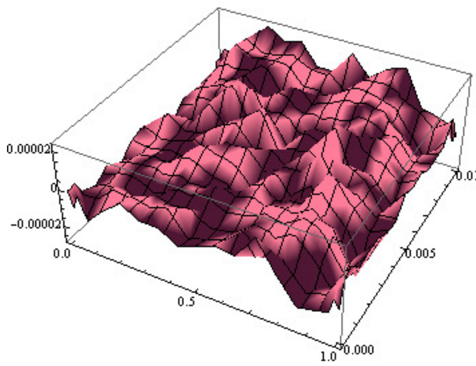
The error functions are shown in four figures.





Left: Figure 1: Error function in $u_t = u_{xx}u + 9u^2 + 2u$
Right: Figure 2: Error function in $u_t = u_{xx}u - u_x^2 + \exp(x^2)\cos(t)$.





Left: Figure 3: Error function in $u_t = u_{xx}u + u^2 - 3u$
Right: Figure 4: Error function in $u_t = \frac{t}{u}$.

5. Conclusions

Radial basis function method has been known as a powerful tool for solving many equations. In this article, we have presented a general framework of the **RBF** method for first-order equations. The present work shows the validity and great potential of **RBF** technique for solving linear and nonlinear equations. All of examples show that the results of **RBF** are in excellent agreement with those obtained by other methods. In this type of problems, if we take care in selection of approximation radial basis functions and their shape parameter, we can obtain more accurate solution with less error.

References

[1] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions, Dover, NewYork, (1965).
[2] G. Adomian, Stochastic systems, Academic Press Inc., New York, (1983).
[3] M. D. Buhmann, Radial basis function, Cambridge monographs on Applied and Computational Mathmatics, (2004).
[4] Fast RBF toolbox matlab manual, version 1.4, 4th August, (2004).
[5] C. Franke and R. Schaback, Convergence order estimates of meshless collocation methods using radial basis functions, Advances in Computational Mathematics, **8** (1998), 381−399.
[6] Ji-H. He, A new approach to nonlinear partial differential equations, Commun. Nonlinear Sci. Numer. Simul, **2** (1997) 230−235.
[7] Ji-H. He, Variational iteration method - a kind of non-linear analytical technique: some examples, Int. J. Non-Linear Mech, **34** (1999) 699−708.
[8] Ji-H. He, Some asymptotic methods for strongly nonlinear equations, Int. J. Modern Phys, **20** (2006) 1141−1191.
[9] Y. C. Hon, A RBFs method for solving options pricing model, Proceedings of Advances in Scientific Computing and Modeling, Alicante, Spain, (1998) 210−230.
[10] E. J. Kansa, Multiquadrics - a scattered data approximation scheme with applications to computational fluid-dynamics-II, Computers and Mathematics with Applications, **19** (1990), 147−161.
[11] E. J. Kansa and Y. C. Hon, Circumventing the ill-conditioining problem with multiquadric radial basis functions: applications to elliptic partial differential equations, Computers and Mathematics with Applications, **39** (2000), 123−137.
[12] J. Kevorkian, and J. D. Cole, Multiple scale and singular perturbation methods, Springer-Varlag, New York, (1996).
[13] C. T. Mouat and R. K. Beatson, RBF collocation, Department of mathematics and statistics, university of canterbury, Private Bag 4800, Christchurch, NewZealand, (2002).
[14] J. I. Ramos, On The Variational iteration method and other iterative techniques for nonlinear differential equation, Appl. Math. Comput, **199** (2008) 39−69.
[15] M. Reed and B. Simon, Methods of modern mathematical physics, I: Functional Analysis, Academemic Press, New York, (1980).