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Numerical Solution of Nonlinear PDEs by Using Two-Level Iterative Techniques and Radial Basis Functions

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Abstract. Radial basis function method has been used to handle linear and nonlinear equations. The purpose of this paper is to introduce the method of RBF to an existing method in solving nonlinear two-level iterative techniques and also the method is implemented to four numerical examples. The results reveal that the technique is very effective and simple. The main property of the method lies in its flexibility and ability to solve nonlinear equations accurately and conveniently.

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1. Introduction

Over the last decades several analytical/approximate methods have been developed to solve nonlinear equations. For initial-value problems in ordinary differential equations, some of these technique include perturbation [17, 19, 20], variational [10–12], decomposition [2–4] methods, etc [21]. In recent years radial basis function collocation has become a useful alternative to finite difference and finite element methods for solving elliptic partial differential equations. RBF collocation methods have been shown numerically (see for example [14]) and theoretically ([7, 8]) to be very accurate even for a small number of collocation points. In application finite difference methods often have a low approximation order and consequently can

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require a large grid and considerable computation to obtain a sufficiently accurate solution. RBF collocation has been applied to linear elliptic PDEs in $\mathbb{R}^2$ and $\mathbb{R}^3$ [15], to time dependent problems [13], and to non-linear problems [18].

In this paper we present new numerical results for RBF collocation, and the purpose of this work, is to use RBF method to solve two-level iterative techniques nonlinear equations. We can present this technique to the other similar linear problems. Here, we try to solve the above equation by RBF collocation methods.

The paper is organized as follows. In Section 2, we consider radial basis function and newton's method, respectively. We then introduce first-order evolution equation and in next section consider two-level iterative techniques and approximating solution. Some applications to the two-level iterative techniques problems are presented in Section 5. A summary of the main conclusions presented in the last section of the paper.

2. Preliminaries radial basis functions and Newton's method

Radial Basis Functions (RBFs) are popular for interpolating scattered data since the associated system of linear equations is guaranteed to be invertible under very mild conditions on the location of the data points. For example, the thin-plate spline used in this library only requires that the points are not co-linear. In particular, Radial Basis Functions do not require that the data lie on any sort of regular grid.
A radial basis function (RBF) is a function of the form:

$$s(x) = p(x) + \sum_{i=1}^{N} \lambda_i \phi(x - x_i), \tag{1}$$

where: s is the radial basis function (RBF for short) and p is a low degree polynomial, typically linear or quadratic and the λ_i's are the RBF coefficients and ϕ is a real valued function called the basic function and the x_i 's are the RBF centres.

The RBF consists of a weighted sum of a radially symmetric basic function ϕ located at the centres x_i and a low degree polynomial p. Given a set of N points x_i and values f_i, the process of finding an interpolating RBF, s, such that,

$s(x_i) = f_i, \quad i = 1, 2, ..., N$

is called fitting. The fitted RBF is defined by the λ_i, the coefficients of the basic function in the summation, together with the coefficients of the polynomial term $p(x)$. [23]

For a fixed point $x_j \in \mathbb{R}^d$, a radial basis function is defined:

$$\phi_j(x) = \phi(\| (x - x_j) \|) \tag{2}$$

which is function only dependents on the distance between $x_j \in \mathbb{R}^d$ and the point x_j. This function is radially symmetric near the center x_j.

$$\phi(r) = \exp(-cr^2) \tag{3}$$

$$\phi(r) = \sqrt{r^2 + c^2} \tag{4}$$

However c is a shape parameter which should be considered suitably also the Euclidean distance is considered for the **RBF** have a global support. [6]

Newton's method for nonlinear equations: A system of nonlinear equations is expressed in the form $F(X) = 0$, where F is a vector-valued function of the vector variable $X; F : R \longrightarrow R$. Given an estimate $X^{(k)}$ of a solution X^*. Newton's method indicates the estimate $X^{(k+1)}$ by setting the local linear approximation to F at $X^{(k)}$ to zero and solving for X:

$$\begin{array}{l} J(X^{(k)})H^{(k)} = -F(X^{(k)}) \\ X^{(k+1)} = X^{(k)} + H^{(k)} \quad ,k = 0, 1, 2, ... \end{array} \tag{5}$$

In this calculation, $J = J(X^{(k)})$ is the Jacobian matrix of F at $X^{(k)}$.

3. First-order evolution equations

Consider the first-order nonlinear evolution equation, [1, 21]

$$u_t = f(x, t, u, u_x, u_{xx}), \quad 0 < t < \infty, \quad u(x, 0) = g(x) \tag{6}$$

Where t denotes time $0 < t < \infty$ and x is the spatial coordinates and f is a nonlinear function of u, u_x, u_{xx} and the subscripts denote partial differentiation. Integrated from equation (6) to yield:

$u(x, t) = g(x) + \int_0^t f(x, p, u, u_x, u_{xx})dp,$

solve iteratively as,

$u^{k+1}(x, t) = g(x) + \int_0^t f(x, p, u^k, u^k_x, u^k_{xx})dp,$

which k shows the k th iteration. The $g(x) + \int_0^t f dp$ is a contractive mapping. The convergence of this equation is ensured by Banach's fixed-point theorem [22].
In the case that equation (6) represents an ordinary differential equation, i.e. $f = f(u, t)$, the Picard-Lindelof theorem indicates that equation (6) has a unique solution if f is continuous with respect to t and Lipschitz continuous with respect to u, u_x, u_{xx} [9, 16].

4. Two-level iterative techniques

Equation (6) can be easily written in differential form by partial differentiation with respect to t. This results in differential form,

$$u_t^{k+1}(x,t) = f(x,t,u^{k+1},u_x^{k+1},u_{xx}^{k+1}) \tag{7}$$

Equation (7) is a special case of the Bellman-Kalaba's quasilinearization method [5], that equation (6) is written as,

$$u_t^{k+1}(x,t) = f(t,x,u^{k+1},u_x^{k+1},u_{xx}^{k+1}), \quad 0 < t < \infty, \quad u^{k+1}(x,0) = g(x) \tag{8}$$

f^{k+1} is quasilinearized with respect to the k th iteration as

$$f(x,t,u^{k+1},u_x^{k+1},u_{xx}^{k+1}) \approx f^k + G^k(u^{k+1}-u^k) + H^k(u_x^{k+1}-u_x^k) + L^k(u_{xx}^{k+1}-u_{xx}^k) \tag{9}$$

which,

$$f^k \equiv f(t,x,u^{k+1},u_x^{k+1},u_{xx}^{k+1}) \tag{10}$$

$$G^k \equiv \frac{\partial f}{\partial u}(x,t,u^{k+1},u_x^{k+1},u_{xx}^{k+1}) \tag{11}$$

$$H^k \equiv \frac{\partial f}{\partial u_x}(x,t,u^{k+1},u_x^{k+1},u_{xx}^{k+1}) \tag{12}$$

$$L^k \equiv \frac{\partial f}{\partial u_{xx}}(x,t,u^{k+1},u_x^{k+1},u_{xx}^{k+1}) \tag{13}$$

substitution of equation (9) into (8) yields the first-order evolution equation,

$$\begin{aligned} &u_t^{k+1} = f^k + G^k(u^{k+1}-u^k) + H^k(u_x^{k+1}-u_x^k) + L^k(u_{xx}^{k+1}-u_{xx}^k), \\ &0 < t < \infty, \quad u^{k+1}(x,0) = g(x) \end{aligned} \tag{14}$$

Consider the approximation solution as:

$$\widetilde{u}(x,t) \equiv \sum_{i=1}^{N} c_i \phi_i(x,t) \tag{15}$$

which c_i's are constants and the ϕ_i's are radial basis functions. where $\phi_i(x,t) = \phi(\|(x-x_i, t-t_i)\|)$ is a radial basis function on $r = \|(x,t)\|$. From equation (15), we have:

$$\begin{cases} \widetilde{u}_t \equiv \sum_{i=1}^{N} c_i \frac{\partial \phi_i}{\partial t}(x,t) \\ \\ \widetilde{u}_x \equiv \sum_{i=1}^{N} c_i \frac{\partial \phi_i}{\partial x}(x,t) \\ \\ \widetilde{u}_{xx} \equiv \sum_{i=1}^{N} c_i \frac{\partial^2 \phi_i}{\partial x^2}(x,t) \end{cases} \tag{16}$$

Substitution of equation (16) into (14) yields,

$$\sum_{i=1}^{N} c_i\left(\frac{\partial \phi_i}{\partial t} - G^k \phi_i - H^k \frac{\partial \phi_i}{\partial x} - L^k \frac{\partial^2 \phi_i}{\partial x^2}\right)(x_j, t_j) \equiv (f^k - G^k \widetilde{u} - H^k \widetilde{u}_x - L^k \widetilde{u}_{xx}) \tag{17}$$

By considering,

$$\begin{cases} \alpha_{ij}(x_j, t_j) = \phi_i(x_j, t_j) \\ \beta_{ij}(x_j, t_j) = \frac{\partial \phi_i}{\partial t}(x_j, t_j) \\ \gamma_{ij}(x_j, t_j) = \frac{\partial \phi_i}{\partial x}(x_j, t_j) \\ \lambda_{ij}(x_j, t_j) = \frac{\partial^2 \phi_i}{\partial x^2}(x_j, t_j) \end{cases}$$

and substituting them into equation (17), we have:

$$\begin{aligned} &\sum_{i=1}^{N} c_i[\beta_{i,j}(x_j, t_j) - G^k \alpha_{i,j}(x_j, t_j) - H^k \gamma_{i,j}(x_j, t_j) - L^k \lambda_{i,j}(x_j, t_j)] \\ &-[f^k - G^k \widetilde{u} - H^k \widetilde{u}_x - L^k \widetilde{u}_{xx}] \equiv 0 \end{aligned} \tag{18}$$

That is a nonlinear system of equations.

In order to solve , we consider $N = N_1 + N_2$, consequently, N_1 denotes the number of boundary points and N_2 shows the number of interior points.
Suppose that the following sets contain a collocation of scattered nodes in every levels of interpolation

$$\Xi_1 = \{(x_i, t_i) \in \bar{\Omega} \times [0, T_1], \quad i = 1, ..., m\}, \quad T > T_1. \tag{19}$$

$$\Xi_k = \{(x_i, t_i + (k-1)T_1);\ (x_i, t_i) \in \Xi_1, \quad i = 1, ..., m, k = 2, 3, ...\} \tag{20}$$

and the problem has a solution in $\bar{\Omega} \times [(k-1)T_1, kT_1]$.

5. Numerical examples

To illustrate the effectiveness of the present method, several test examples are consider in this section, the accuracy of the technique is assessed by comparison with the exact solutions. We obtain good approximation which is shown in Figure

1 and the error graph for this approximation is shown in Figure 2.

Example 5.1 Consider the following nonlinear partial differential equation:

$$u_t = u_{xx}u + 9u^2 + 2u \tag{21}$$

initial conditions are,

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = \sin(3)\exp(2t) \\ u(x,0) = \sin(3x) \end{cases} \tag{22}$$

The exact solution is:

$$u(x,t) = \sin(3x)\exp(2t) \tag{23}$$

We set ϕ as:

$$\phi(r) = \exp\left(-cr^2\right) \tag{24}$$

By considering only the first step of Newton method for nonlinear system and by taking $c = 2$, $N = 10$, $0 \leqslant x \leqslant 1$, we have the error function which is shown in Figure 1:

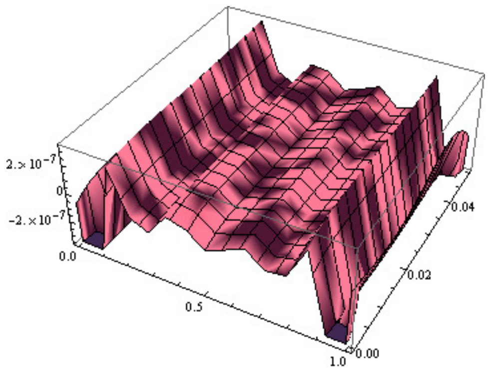

Figure 1. Error function.

Example 5.2 We assume another nonlinear example,

$$u_t = u_{xx}u - u_x^2 + \exp(x^2)\cos(t) \tag{25}$$

subject to initial conditions,

$$\begin{cases} u(x,0) = 0 \\ u(x,1) = \exp(x^2)\sin(1) \\ u(0,t) = \sin(t) \end{cases} \tag{26}$$

The exact solution is,

$$u(x,t) = \exp(x^2)\sin(t) \tag{27}$$

As stated before using equation (24) and by taking $c = 2$, $N = 10$, $0 \leqslant x \leqslant 1$,we have the error function which is shown in Figure 2:

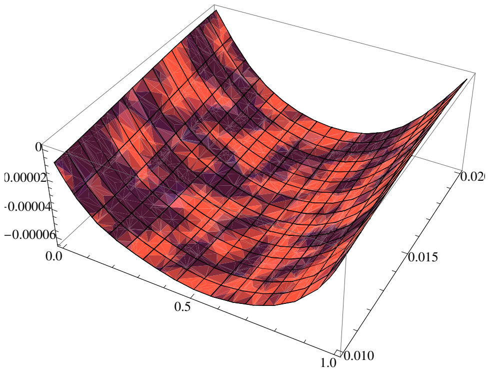

Figure 2. Error function.

Example 5.3 Consider the following problem:

$$u_t = u_{xx}u + u^2 - 3u \tag{28}$$

with the initial conditions:

$$\begin{cases} u(0,t) = \exp(-3t) \\ u(1,t) = (\sin 1 + \cos 1)\exp(-3t) \\ u(x,0) = \sin x + \cos x \end{cases} \tag{29}$$

The exact solution is,

$$u(x,t) = (\sin x + \cos x)\exp(-3t) \tag{30}$$

According to the equation (24) and considering $c = 2$, $N = 10$, $0 \leqslant x \leqslant 1$, we have the error function which is shown in Figure 3:

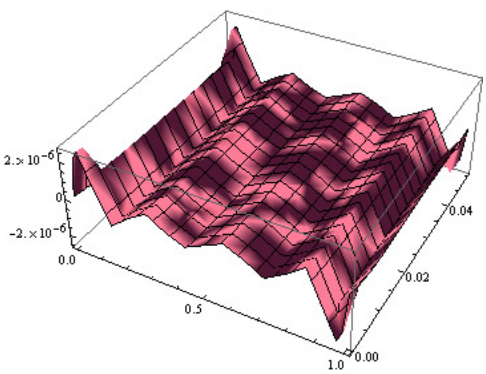

Figure 3. Error function.

Example 5.4 Let this problem:

$$u_t = u_x u + u_{xx}^2 + 2\exp(-x)\cos(2t) \tag{31}$$

initial conditions are:

$$\begin{cases} u(0,t) = \sin(2t) \\ u(1,t) = \exp(-1)\sin(2t) \\ u(x,0) = 0 \end{cases} \tag{32}$$

with the exact solution,

$$u(x,t) = \exp(-x)\sin(2t) \tag{33}$$

According to the equation (24) and by taking $c = 2$, $N = 10$, $0 \leqslant x \leqslant 1$, we have the error function which is shown in Figure 4:

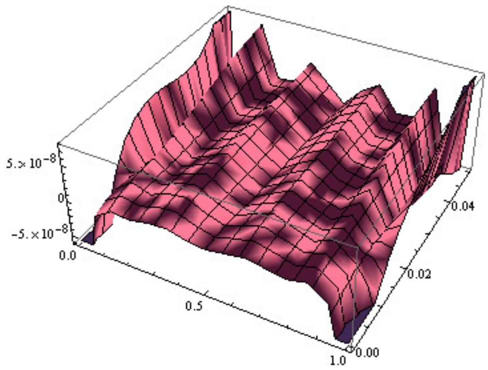

Figure 4. Error function.

A very good agreement between the results from the exact solution and the approximation was observed, which confirms the validity of the present method.

6. Conclusions

Radial basis function method has been known as a powerful tool for solving many equations. In this article, we have presented a general framework of the **RBF** method for two- level iterative equation. The present work shows the validity and great potential of **RBF** technique for solving linear and nonlinear equations. All of examples show that the results of **RBF** are in excellent agreement with those obtained by other methods. In this type of problems, if we take care in selection of approximation radial basis functions and their shape parameter, we can obtain more accurate solution with less error.

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