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Some Common Fixed Point Results for Finite Family of G-Monotone Generalized Quasi- Contraction Mappings

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Abstract. In this paper, we prove some fixed point theorems for finite family of G- monotone generalized quasi contraction mappings in a metric space endowed with a graph. Example is provided to show the effectiveness of our results.

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Index to information contained in this paper

1. Introduction
2. Basic Concept
3. Common Fixed Point Theorems for Family of Mappings
4. Conclusion

1. Introduction

The concepts of metric spaces are introduced in [1]. In 2004, the concept of a partial ordering structure in the notions of metric spaces was introduced in [2]. Later, the authors in [2] proved fixed point theorems for monotone singlevalued mappings in a metric space endowed with partial order. The investigation of a new approach in metric fixed point theory by replacing an order structure with a graph structure in metric spaces was initiated in [3]. Still in [3], they obtained the fixed point theorem for Banach contraction principle on metric spaces endowed with a graph.
Banach contraction result was initiated in 1922 and the unique fixed point of this operator is proved in a complete metric space. The result gained wide range of applications in mathematics and applied mathematics. Due to its applications, numerous researchers extended and generalized it in several areas (see [4], [5], [6], [7], [8]). In particular, the author in [9] generalized the Banach contraction mappings to quasi contraction mappings. Fixed point theorems for quasi contraction mappings in both metric and modular metric spaces with a graph structure are proved in [10]. The work of [10] was extended to G-monotone generalized quasi contraction mappings in the same spaces. The family of contraction mappings was introduced and studied by [12]. The study of existence of common fixed point for finite and infinite family of self mappings has been proved by many authors. For example, [1 3 -16].

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The aim of this paper is to prove some new results on the existence and uniqueness of common fixed points for finite family of self-mappings satisfying certain contractive conditions in a metric space endowed with a graph.

2. Basic Concept

In this section, we review some definitions and motivations that will be needed to prove our results.
In 1971, the following definitions and facts were introduced in [9, 12].
Definition 2.1. Let (X, d) be a metric space and $T: X \to X$be a selfmap. The map$T$is called quasi contraction if there exists $0 \le k < 1$ such that for each $x, y \in X$,

$$d(Tx, Ty) \le k \max\{ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Definition 2.2. Let X be a nonempty set and let $\{T_i\}$ be a family of self mappings on X. A point $x_0 \in X$is called a common fixed point for this family if and only if $T_i(x_0) = x_0$, for each $i \in \mathbb{N}$.

The following theorem was given by [12] for a family of generalized contraction mappings.
Theorem 2.3. Let (X, d) be a complete metric space and let $\{T_i\}_{i\in J}$ be a family of self mappings of X. If there exists fixed $j \in J$such that for each$i \in J$
$d(T_i x, T_j y) \le \lambda \max \left\{d(x, y), d(x, T_i x), d(y, T_j y), \frac{1}{2}\left[d(x, T_j y) + d(y, T_i x)\right]\right\}$, for some $\lambda = \lambda(i) \in (0, 1)$ and all $x, y \in X$, then all $T_i$have a unique common fixed point, which is unique fixed point of each $\{T_i\}$,$i \in J$.

We now give brief description of graph theory. Details of graph theory can also be found in [17].
Let (X, d) be a metric space, $\delta = \delta(X)$ is the diagonal of$X$. Let V be a set and $E \subset V \times V$ be a binary relation on$V$, the ordered pair (V, E) is called a graph$G$. The elements of E are called edges and are denoted by$E(G)$. If the edges are directed then we have a directed graph. Suppose Ghas no parallel edges then the graph can be represented by the ordered pair $(V(G), E(G))$and the metric space is equipped with$G$.
If the direction of the edges is reversed then we have graph G^{-1}. Also we have undirected graph $\overline{G}$, if the direction of the edges is ignore. In other words, we have $V(G^{-1}) = V(\overline{G}) = X$, $E(G^{-1}) = \{(x, y): (y, x) \in E(G)\}$ and $E(\overline{G}) = E(G) \cap E(G^{-1})$.
If $x, y \in X$, then a finite sequence $\{x_i\}_{i=0}^{\infty}$ consisting of $N + 1$vertices is called a path in $G$from x to$y$whenever $x_0 = x, x_N = y$and (x_{i-1}, x_i) is an edge of G for $i = 1, \ldots, N$. The graph G is called connected if there exists a path in G between each two vertices of G.

The following definitions are found in [11].
Definition 2.4. A mapping $T: X \to X$ is called a G -monotone if T preserves edges of$G$, that is, for all $x, y \in X$,$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$.

Definition 2.5. A mapping $T: X \to X$ is called G -contraction if T preserves edges of G that is, for all $x, y \in X$, $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$ and T decreases weight of edges of G in the following way; there exists $\lambda \in (0, 1)$ and for all $x, y \in X$,$(x, y) \in E(G) \Rightarrow d(Tx, Ty) \le \alpha\, d(x, y)$.

Definition 2.6. Let C be a nonempty subset of X. A mapping $T: C \to C$ is called G - monotone quasi contraction if T is G - monotone and there exists $k < 1$ such that for any $x, y \in C, (x, y) \in E(G)$, we have

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Definition [3] 2.7. Suppose that (X, d) is a metric space and G is a directed graph. The triple (X, d, G) is said to have Property (A) if and only if for any sequence $\{x_n\}$ in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ where $n \in N$, we have that $(x_n, x) \in E(G)$.
Definition 2.8 . Let $T: X \to (-\infty, \infty)$ be a function on a topological space X. Then, T is upper semi continuous at the point $x \in X$ if and only if $x_n \to x \Rightarrow \limsup_{n\to\infty} T(x_n) \leq Tx$.
Recall that $\vartheta: 0, \infty) \to 0, \infty)$ is called a comparison function if it is increasing and upper semi- continuous. As a consequence, we also have $\vartheta(t) < t$, for each $t > 0, \vartheta(0) = 0$. For example, $\vartheta(t) = at(\text{where } a \in 0, 1), \vartheta(t) = \frac{t}{1+t}$ and $\vartheta(t) = \ln(1 + t), t \in \mathbb{R}^+$.

Definition 2.9. Let C be a nonempty subset of a metric space X. A mapping $T: C \to C$ is called
(i) G - monotone if T preserves edges of G,
(ii) G - monotone generalized quasi contraction if T is G - monotone and there exists a $\vartheta \in \Phi$ such that for any $x, y \in C, (x, y) \in E(G)$, we have $d(Tx, Ty) \leq \max \{\vartheta(d(x, y)), \vartheta(d(x, Tx)), \vartheta(d(y, Ty)), \vartheta(d(x, Ty)), \vartheta(d(y, Tx))\}$.
Definition [18] 2.10. Let $T: X \to X$ be a selfmap on a metric space. For each $x \in X$ and for any positive whole number n, $O_T(x, n) = \{x, Tx, T^2x, T^3x, \ldots, T^nx\}$ and $O_T(x, \infty) = \{x, Tx, T^2x, T^3x, \ldots\}$.
The set $O_T(x, \infty)$ is called the orbit of T at x and the metric space X is called T - orbitally complete if every Cauchy sequence in $O_T(x, \infty)$ is convergent in X.

3. Common Fixed Point Theorems for Family of Mappings

In this segment, we present the definition of finite G- monotone generalized quasi contraction mappings in a metric space. The existence and uniqueness of the common fixed point of this map is proved in a metric space endowed with a graph. Example is provided to validate our results.
Definition 3.1. Let (X, d) be a metric space endowed with a graph G and $\{T_i\}_{i=1}^m$ be a finite family of self mappings on X. $\{T_i\}_{i=1}^m$ is called ;
(A_1) G - monotone if $\{T_i\}_{i=1}^m$ preserves the edges of G , that is, $(x, y) \in E(G)$ implies $(T_ix, T_iy) \in E(G)$ for all $x, y \in X$;
(A_2) G - monotone generalized quasi contraction if T_i is G - monotone and there exists a $\vartheta \in \Phi$ such that for any $x, y \in X$, with $(x, y) \in E(G)$, we have

$$d(T_ix, T_iy) \leq M_i(x, y), \text{ for all } i \in \{1, 2, \cdots, m\} \tag{1}$$

where

$$M_i(x, y) = \max\left\{\begin{matrix}\vartheta(d(T_{i-1}x, T_{i-1}y)), \vartheta(d(T_{i-1}x, T_ix)), \\ \vartheta(d(T_{i-1}y, T_iy)), \vartheta(d(T_{i-1}x, T_iy)), \vartheta(d(T_{i-1}y, T_ix))\end{matrix}\right\}.$$

Lemma 3.1. Let (X, d) be a metric space and G be a reflexive and transitive digraph on X. Let $\{T_i\}_{i=1}^m$ be a finite family of self mappings on X. Let C be a nonempty subset of X

and T_i: C $\to C$ be a G - monotone generalized quasi contraction mapping. Let $x \in C$ be such that $(x, T_i x) \in E(G)$ and $\delta(x) < \infty$ then for any $n \in \mathbb{N}$, we have

$$\delta(T_{i+n}(x)) \leq \vartheta^n(\delta(x))$$

where $\vartheta \in \Phi$is the comparison function associated with the G - monotone generalized quasi contraction definition of T_i. Moreover, we have

$$d(T_{i+n}x, T_{i+n+m}x) \leq \vartheta^n(\delta(x))$$

for all n,m $\in \mathbb{N}$.

Proof. Let $x \in X$be arbitrary chosen. Since $T_i$is$G$ -monotone and $(T_{i+n}x, T_{i+n+1}x) \in E(G)$ for each $n \in \mathbb{N}$. By the transitivity of G, for each $n \in \mathbb{N}$, we also have $(T_{i+n}x, T_{i+n+m}x) \in E(G)$ for any $m \in Z^+$.

As G is reflexive, $(x, x) \in E(G)$ and by the$G$ - monotonicity of T_i, we obtain $(T_{i+n}x, T_{i+n}x) \in E(G)$ for any $n \in \mathbb{N}$,

and hence (1) holds for m=0. Thus, we obtain

$$(T_{i+n}x, T_{i+n+m}x) \in E(G) \text{ for any n,m } \in \mathbb{N}. \quad (2)$$

Now we show that

$\delta(T_{i+n}(x)) \leq \vartheta^n(\delta(x))$, for each $n \in \mathbb{N}$.

For n =1, from (2) and using the monotonicity of ϑ, we have

$$d(T_{i+1}x, T_{i+1+m}x) \leq \max\{\vartheta(d(T_i x, T_{i+m}x)), \vartheta(d(T_i x, T_{i+1}x)), \vartheta(d(T_{i+m}x, T_{i+1+m}x), \vartheta(d(T_i x, T_{i+1+m}x)), \vartheta(d(T_{i+m}x, T_{i+1}x))\} \leq \vartheta(\delta(x)),$$

for each m $\in \mathbb{N}$. This shows that

$$\delta(T_{i+1}(x)) \leq \vartheta(\delta(x)). \quad (3)$$

From (3) and the monotonicity of ϑ, we get

$$\vartheta\left(\delta(T_{i+1}(x))\right) \leq \vartheta^2(\delta(x)). \quad (4)$$

By combining (3) and (4) and the monotonicity of ϑ, we have

$$\begin{aligned}\delta(T_{i+2}(x)) &= \delta\left(T(T_{i+1}(x))\right)\\ &\leq \vartheta(\delta((T_{i+1}(x))\\ &\leq \vartheta^2(\delta(x)).\end{aligned}$$

By induction we conclude that

$$\begin{aligned}\delta(T_{i+n}(x)) &= \delta\left(T(T_{i+n-1}(x))\right)\\ &\leq \vartheta(\delta((T_{i+n-1}(x))\\ &\leq \vartheta^n(\delta(x)),\end{aligned}$$

for each $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$,

$$\delta(T_{i+n}(x)) \leq \vartheta^n(\delta(x)). \quad (5)$$

On the other hand, using (2) and the definition of δ, we obtain

$$\begin{aligned}d(T_{i+n}x, T_{i+n+m}x) &= d(T_{i+n}x, T_m(T_{i+n})x)\\ &\leq \delta(T_{i+n}(x)), \quad (6)\end{aligned}$$

for each n,m $\in \mathbb{N}$. From (5) and (6) we conclude that$d(T_{i+n}x, T_{i+n+m}x) \leq \delta^n(\delta(x))$, for each n,m $\in \mathbb{N}$.

Theorem 3.1. Let (X, d) be a metric space and G be a reflexive and transitive digraph defined on X such that the triple (X, d, G) has Property (A), for any sequence $\{x_n\}_{n \in N}$in X, if $x_n \to$ x and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$for each $n \in \mathbb{N}$. Let C be a closed subset of X and T_i, $i = 1, 2, \cdots$, ma finite family of self mappings on X is a G- monotone generalized quasi contraction mapping with $\vartheta \in \Phi$, the comparison function. For $x \in C$ with $(x, T_i x) \in E(G)$ and $\delta(x) < \infty$. Let T(X) be orbitally complete, then we have the following;

(i) There exists p a common fixed point of T_i, $i = 1, 2, \cdots, m$ such that $\{T_i\}_{i=1}^{m}$converges to p. Moreover, we have $(x, p) \in E(G)$ and $d(T_{i+n}x, p) \le \vartheta^n(\delta(x))$ for each $n \in \mathbb{N}$.

(ii) if $u$is any common fixed point of T_i, $i = 1, 2, \cdots, m$ such that $(x, u) \in E(G)$, then $u = p$.

Proof. We need to prove (i). By Lemma 3.1, we proved that $\{T_i\}_{i=1}^{m}$ is a Cauchy sequence in C. Since X is a complete metric space and C is a closed subset of X , there exists $p \in C$ such that $\{T_i\}_{i=1}^{m}$ converges to p. From (7) we have

$$d(T_{i+n}x, T_{i+n+m}x) \le \vartheta^n(\delta(x)), \tag{7}$$

for anyn,m $\in \mathbb{N}$. Letting $m \to \infty$ in (7) yields

$$d(T_{i+n}x, p) \le \vartheta^n(\delta(x)),$$

for $n \in \mathbb{N}$. Since T_i is G-monotone and $(x, T_{i+n}x) \in E(G)$ we have $d(T_{i+n}x, T_{i+n}x) \in E(G)$for each $n \in \mathbb{N}$ and using Property (A), we concluded that $d(T_{i+n}x, p) \in E(G)$ for each $n \in \mathbb{N}$. In particular, $(x, p) \in E(G)$..

We need to show that $p$is the common fixed point of T_i for all $i = 1, 2, \cdots, m$. From $d(T_{i+n}x, p) \in E(G)$ the G - monotonicity of T_i, we have $d(T_{i+n}x, T_i p) \in E(G)$ for each $n \in \mathbb{N}$. As $d(T_{i+n}x, T_{i+n}x) \in E(G)$ and G is transitive, we obtain

$$d(T_{i+n}x, T_i p) \in E(G) \text{ for each } n \in \mathbb{N}. \tag{8}$$

Therefore using (8) and the condition that G -monotone generalized quasi- contraction, we have a comparison function ϑ satisfying

$$d(T_{i+n}x, T_i p) \le \max \left\{ \begin{matrix} \vartheta\left(d(T_{i+n-1}x, p)\right), \vartheta\left(d(T_{i+n-1}x, T_{i+n}x)\right), \\ \vartheta(d(p, T_i p), \vartheta\left(d(T_{i+n-1}x, T_i p)\right), \vartheta\left(d(p, T_{i+n}x)\right) \end{matrix} \right\}, \tag{9}$$

for each $n \in \mathbb{Z}^+$. Letting $n \to \infty$ in (9) and using the upper semi-continuity of ϑ, yields, $d(p, T_i p) \le \vartheta(d(p, T_i p))$.

This implies that $d(p, T_i p) = 0$, hence $p, = T_i p$ for all $i = 1, 2, \cdots, m$.

Next we show (ii). Let $u \in C$ be any common fixed point of T_i such that $(x, u) \in E(G)$. Then for each $n \in \mathbb{N}$, and T_i a G -monotone, we have $(T_{i+n}x, u) \in E(G)$. Therefore,

$$\begin{aligned} d(T_{i+n}x, u) &\le \max\{\vartheta\left(d(T_{i+n-1}x, u)\right), \vartheta\left(d(T_{i+n-1}x, T_{i+n}x)\right), \\ &\quad \vartheta(d(u, T_i u), \vartheta\left(d(T_{i+n-1}x, u)\right), \vartheta\left(d(u, T_{i+n}x)\right)\} \\ &= \max\{\vartheta\left(d(T_{i+n-1}x, u)\right), \vartheta\left(d(T_{i+n-1}x, T_{i+n}x)\right), \\ &\quad \vartheta\left(d(T_{i+n-1}x, u)\right), \vartheta\left(d(u, T_{i+n}x)\right)\}. \end{aligned} \tag{10}$$

If $\max\{\vartheta\left(d(T_{i+n-1}x, u)\right), \vartheta\left(d(T_{i+n-1}x, T_{i+n}x)\right), \vartheta\left(d(T_{i+n-1}x, u)\right), \vartheta\left(d(u, T_{i+n}x)\right)\} = \vartheta\left(d(u, T_{i+n}x)\right)$

for some $n \in \mathbb{Z}^+$, then from (10) we have,

$d(T_{i+n}x, u) \le \vartheta\left(d(u, T_{i+n}x)\right)$.

By the property of (ii) of ϑ, we obtain

$d(u, T_{i+n}x) = 0$. This implies $u = T_{i+n}x$.

This shows that the sequence $T_{i+n}x \to u$as $n \to \infty$. By the uniqueness of the limit, we conclude that u= p. Otherwise,

$$\max\{\vartheta\left(d(T_{i+n-1}x, u)\right), \vartheta\left(d(T_{i+n-1}x, T_{i+n}x)\right), \vartheta\left(d(T_{i+n-1}x, u)\right), \vartheta\left(d(u, T_{i+n}x)\right)\} \neq \vartheta\left(d(u, T_{i+n}x)\right)$$

Applying (10) again yields,

$$d(T_{i+n}x, u) \leq \max\{\vartheta(d(T_{i+n-1}x, u)), \vartheta(d(T_{i+n-1}x, T_{i+n}x))\}$$
$$\leq \vartheta(d(T_{i+n-1}x, u)) + \vartheta(d(T_{i+n-1}x, T_{i+n}x)), \quad (11)$$

for all $n \in \mathbb{Z}^+$.
Take the limit superior of (11) and using the upper semi-continuity of ϑ yields

$$d(p, u) \leq \limsup_{n\to\infty} \vartheta(d(T_{i+n-1}x, u)) + \limsup_{n\to\infty} \vartheta(d(T_{i+n-1}x, T_{i+n}x)) \leq \vartheta(d(p, u)) \quad (12)$$

By property (ii) of $\vartheta$and (12) we conclude that$d(u, p) = 0$. Hence $u = p$.

Remarks 3.1. (i) If $m = 1$ in $\{T_i\}_{i=1}^m$ then Theorem 3.1 reduces to Theorem 5 of Hunde et al.[11]. We proved our result for finite family of G -monotone generalized quasi contraction mappings while Hunde et al.[11] proved their result for single map.
(ii) Also if we take $\vartheta(t) = kt$, where $k \in 0, 1)$ and $m = 1$ in $\{T_i\}_{i=1}^m$then Theorem 3.3 is reduced to the result of Alfuraidan [10], (Theorem 3.1).
(iii) If we take $\vartheta(t) = t\text{-}\vartheta(t)$ and quasi contraction is replaced with Reich contraction and our space reduces to metric space then Theorem 3.1 reduces to the result of Lin and Wang ([13], Theorem 2.1).
(iv) The finite family $\{T_i\}_{i=1}^m$of self mappings in Theorem 3.1 is neither commuting nor continuous. These conditions are often assumed when proving common fixed point theorems, see [13].

Corollary 3.1. Let (X, d)be a metric space and$G$be a reflexive and transitive digraph defined on Xsuch that the triple (X, d, G)has Property(A), for any sequence $\{x_n\}_{n\in\mathbb{N}}$in X, if $x_n \to x$and $(x_n, x_{n+1}) \in E(G)$for $n \in N$, then $(x_n, x) \in E(G)$ for each $n \in \mathbb{N}$Let C be a subset of $X$and Ta self mappings on $X$is a G - monotone generalized quasi contraction mapping with $\vartheta \in \Phi$, the comparison function. For $x \in C$ with $(x, Tx) \in E(G)$ and $\delta(x) < \infty$, we have the following;
(i) There exists pa fixed point of Tsuch that $\{T_n\}$ converges to p. Moreover, we have $(x, p) \in E(G)$and $d(T_n x, p) \leq \vartheta^n(\delta(x))$ for each $n \in N$.
(ii) if u is any fixed point of T such that $(x, u) \in E(G)$, then $u = p$.

Corollary 3.2. Let (X, d)be a metric space and$G$be a reflexive and transitive digraph defined on Xsuch that the triple (X, d, G)has Property(A), for any sequence $\{x_n\}_{n\in\mathbb{N}}$in X, if $x_n \to x$and $(x_n, x_{n+1}) \in E(G)$for $n \in N$, then $(x_n, x) \in E(G)$ for each $n \in \mathbb{N}$Let C be a subset of $X$and Ta self mappings on X. If there exists $k \in 0, 1)$such that $T$is aG - monotone quasi contraction mapping then for $x \in C$ with $(x, Tx) \in E(G)$ and $\delta(x) < \infty$, we have the following;
(i) There exists pa fixed point of Tsuch that $\{T_n\}$ converges to p. Moreover, we have $(x, p) \in E(G)$and $d(T_n x, p) \leq \vartheta^n(\delta(x))$ for each $n \in N$.
(ii) if u is any fixed point of T such that $(x, u) \in E(G)$, then $u = p$.

The following results give the unique common fixed point for a pair of finite families of$G$ -monotone generalized quasi contraction mappings in metric spaces equipped with a graph.

Definition 3.2. Let Gbe a directed graph, and let $S, T: X \to X$be two mappings. We say that $S$is T -edge preserving with respect to Gif
$(Tx, Ty) \in E(G) \Rightarrow (Sx, Sy) \in E(G)$.

Definition 3.3. Let (X, d) be a metric space endowed with a directed graph G and $S, T\colon \mathrm{X} \to X$ be two finite families, where $S = \{S_1, \mathrm{S}_2, \cdots, \mathrm{n}\}$ and $T = \{T_1, \mathrm{T}_2, \cdots, \mathrm{T}_m\}$. The pair (S, T) is called G -monotone generalized quasi contraction mapping if
(1) S is T -edge preserving with respect to G;
(2) there exists $\vartheta \in \Phi$ and for all $x, \mathrm{y} \in X$ such that $(\mathrm{Tx}, \mathrm{Ty}) \in E(G)$,

$$d(\mathrm{Sx}, \mathrm{S\,y}) \leq M(x, \mathrm{y}), \tag{13}$$

where

$$M(x, \mathrm{y}) = \max\{\vartheta(d(\mathrm{Sx}, \mathrm{Sy})), \vartheta(d(\mathrm{Sx}, \mathrm{Tx})), \vartheta(d(\mathrm{Sy}, \mathrm{Ty})), \vartheta(d(\mathrm{Sx}, \mathrm{Ty})), \vartheta(d(\mathrm{Sy}, \mathrm{Tx}))\}.$$

We establish the following Lemma needed to prove the next theorem.

Lemma 3.2. Let (X, d) be a metric space and G be a reflexive and transitive digraph on X. Let T and S be two finite families of self mappings on X. Let C be nonempty subset of X and $S, \mathrm{T} : \mathrm{C} \to C$ be G -monotone generalized quasi contraction mapping. Let $x \in C$ be such that $(\mathrm{Sx}, \mathrm{Tx}) \in E(G)$ and $\delta(x) < \infty$ then for any $n \in \mathrm{N}$, we have

$$\delta(\mathrm{Tx}_n) \leq \vartheta^n(\delta(x))$$

where $\vartheta \in \Phi$ is the comparison function associated with the G -monotone generalized quasi contraction definition of T and S. Moreover, we have $d(\mathrm{Sx}_n, \mathrm{Sx}_{n+m}) \leq \vartheta^n(\delta(x))$ for each $n, \mathrm{m} \in N$.

Proof. Suppose $x_0 \in X$ such that $(\mathrm{Tx}_0, \mathrm{Sx}_0) \in E(G)$. By the assumption that $S(X) \subset T(X)$ and $\mathrm{Sx}_0 \in X$, we can easily construct sequences $\{x_n\}$ in X for which

$$\mathrm{Tx}_{n+1} = \mathrm{Sx}_n, \tag{14}$$

for all $n \in N$. If $\mathrm{Tx}_{n_0} = \mathrm{Sx}_{n_0}$ for some $n_0 \in N$, then x_{n_0} is the coincidence point of the mappings T and S. Thus we assume that for each $n \in N$, $\mathrm{Tx}_{n+1} \neq \mathrm{Sx}_n$ holds.
Since $(\mathrm{Sx}_0, \mathrm{Tx}_0) = (\mathrm{Sx}_0, \mathrm{Sx}_1) \in E(G)$ and S is edge preserving with respect to T, we have $(\mathrm{Sx}_0, \mathrm{Sx}_1) = (\mathrm{Tx}_1, \mathrm{Tx}_2) \in E(G)$. Continuing inductively, we obtain $(\mathrm{Sx}_n, \mathrm{Sx}_{n+1}) \in E(G)$ for each $n \in N$. By the transitivity of G, for each $n \in N$, we also have $(\mathrm{Sx}_n, \mathrm{Sx}_{n+m}) \in E(G)$ for any $m \in Z^+$.

As G is reflexive, $(\mathrm{Sx}_0, \mathrm{Sx}_0) \in E(G)$ and by the G -monotonicity of T , we have $(\mathrm{Sx}_n, \mathrm{Sx}_n) \in E(G)$ for any $n \in N$ and hence (14) holds for $m = 0$. Thus, we have $(\mathrm{Sx}_n, \mathrm{Sx}_{n+m}) \in E(G)$ for any n,m $\in N$.

Now we show that
$\delta(\mathrm{Sx}_n) \leq \vartheta^n(\delta(x))$, for each $n \in N$.
For $m = 1$, from (12) and using the monotonicity of ϑ, we have

$$\begin{aligned} d(\mathrm{Sx}_n, \mathrm{Sx}_{n+1}) &\leq \max\{\vartheta(d(\mathrm{Tx}_n, \mathrm{Tx}_{n+1})), \vartheta(d(\mathrm{Sx}_n, \mathrm{Tx}_n)), \\ &\vartheta(d(\mathrm{Sx}_{n+1}, \mathrm{Tx}_{n+1})), \vartheta(d(\mathrm{Tx}_n, \mathrm{Sx}_{n+1})), \vartheta(d(\mathrm{Sx}_n, \mathrm{Tx}_{n+1}))\} \\ &= \max\{\vartheta(d(\mathrm{Sx}_{n-1}, \mathrm{Sx}_n)), \vartheta(d(\mathrm{Sx}_{n-1}, \mathrm{Sx}_n)), \\ &\vartheta(d(\mathrm{Sx}_n, \mathrm{Sx}_{n+1})), \vartheta(d(\mathrm{Sx}_{n-1}, \mathrm{Sx}_{n+1})), \vartheta(d(\mathrm{Sx}_n, \mathrm{Sx}_n))\} \\ &= \vartheta(\delta(x)). \end{aligned}$$

for each $m \in N$. This shows that

$$\delta(S(x)) \leq \vartheta(\delta(x)). \tag{15}$$

From (14) and the monotonicity of ϑ, we get

$$\vartheta(\delta(\mathrm{Sx})) \leq \vartheta^2(\delta(x)). \tag{16}$$

Combining (15) and (16) and the monotonicity of ϑ we obtain

$$\delta(S^2x) = \delta\big(S(\mathrm{Sx})\big) \le \vartheta\big(\delta(\mathrm{Sx})\big) \le \vartheta^2\big(\delta(x)\big).$$

By induction we conclude that,

$$\begin{aligned}\delta(S^nx) &= \delta\big(S(S^{n-1}x)\big)\\ &\le \vartheta(\delta(S^{n-1}x)\\ &\quad\vdots\\ &\le \vartheta^n\big(\delta(x)\big)\end{aligned}$$

for each $n \in N$.

On the other hand, using (14) and the definition ofδwe obtain

$$d(S^n\mathrm{x}, \mathrm{S}^{n+m}x) = \mathrm{d}(S^n(x), \mathrm{S}^m\big(S^n(x)\big) \le \delta\big(S^n(x)\big), \tag{17}$$

for each n,m $\in N$. From (16) and (17) we conclude that,

$d(S^nx, \mathrm{S}^{n+m}x) \le \vartheta^n\big(\delta(x)\big)$,

for each n,m $\in N$.

Now, we prove the theorem for a pair of finite families of$G$ -monotone generalized quasi contraction in metric spaces endowed with a graph.

Theorem 3.2. Let $\{T_1, \mathrm{T}_2, \cdots, \mathrm{T}_n\}$ and $\{S_1, \mathrm{S}_2, \cdots, \mathrm{S}_m\}$ be two finite families of selfmappings on metric space and G be a reflexive, transitive digraph defined on X such that the triple $(X, \mathrm{d}, \mathrm{G})$has Property (A), $T = \{T_1, \mathrm{T}_2, \cdots, \mathrm{T}_n\}$and $S = \{S_1, \mathrm{S}_2, \cdots, \mathrm{S}_m\}$. Let C be a subset of $X$and $S, \mathrm{T}\colon \mathrm{C} \to C$ finite families of G -monotone generalized quasi contraction mappings with $\vartheta \in \Phi$, the comparison function. For x $\in C$ with $(\mathrm{Sx}, \mathrm{Tx}) \in E(G)$ and $\delta(x) < \infty$, we have that if $S(X) \subset T(X)$and $T(X)$ and $S(X)$ are compatible then $S$and T have a coincidence point. Moreover,

(i) There exists p a common fixed point of T and S such that $\{S_i\}_{i=1}^m$converges to p. Also, we have $(x, \mathrm{p}) \in \mathrm{E}(G)$ and $d(S_{i+n}x, \mathrm{p}) \le \vartheta^n\big(\delta(x)\big)$ for each $n \in N$.

(ii) if u is any common fixed point of$T$and Ssuch that $(x, \mathrm{u}) \in E(G)$, then $u = \mathrm{p}$.

Proof. We need to prove that S and T have coincidence point in X. By Lemma 3.2, we proved that $\{S^nx\}$ is a Cauchy sequence in C. Since X is a complete metric space and C is a closed subset of X, there is a $p = \mathrm{Tx}$ such that;

$\lim_{n\to\infty} \mathrm{Sx}_n = \lim_{n\to\infty} \mathrm{Tx}_n$=p.

The compatibility of S and T give

$$\lim_{n\to\infty} d(\mathrm{STx}_n, \mathrm{TSx}_n) = 0. \tag{18}$$

Using the triangle inequality,

$$d(\mathrm{Sp}, \mathrm{Tp}) \le \mathrm{d}(\mathrm{Sp}, \mathrm{STx}_n) + \mathrm{d}(\mathrm{STx}_n, \mathrm{TSx}_n) + \mathrm{d}(\mathrm{TSx}_n, \mathrm{Tp}).$$

Letting n $\to \infty$ in (18) yields

$d(\mathrm{Sp}, \mathrm{Tp}) \le d(\mathrm{Sp}, \mathrm{Sp}) + 0 + \mathrm{d}(\mathrm{Tp}, \mathrm{Tp})$.

This gives $d(\mathrm{Sp}, \mathrm{Tp}) = 0$, which implies $\mathrm{Sp} = \mathrm{Tp}$. So p is the coincidence point of $S$and T.

Next we show that p is the common fixed point of $S$and T. From (17) we have,

$$d(\mathrm{Sx}_n, \mathrm{Sx}_{n+m}) \le \vartheta^n\big(\delta(x)\big), \tag{19}$$

for any n, m $\in N$. Letting m $\to \infty$ in (19) yields,

$$d(\mathrm{Sx}_n, \mathrm{p}) \le \vartheta^n\big(\delta(x)\big), \tag{20}$$

for n $\in N$. Since S is G -monotone and $(\mathrm{Sp}, \mathrm{Sp}) \in E(G)$, it implies $(\mathrm{Sx}_n, \mathrm{Sx}_{n+m}) \in E(G)$ for n $\in N$. Using Property(A), we conclude that $(\mathrm{Sx}_n, \mathrm{Sp}) \in \mathrm{E}(G)$ for each n $\in N$. With the G - monotonicity of S, we have $(\mathrm{Sx}_{n+1}, \mathrm{Sp}) \in \mathrm{E}(G)$ for each n $\in N$. As $(\mathrm{Sx}_n, \mathrm{Sx}_{n+1}) \in E(G)$ and G is transitive, we have,

$$(\mathrm{Sx}_n, \mathrm{Sp}) \in \mathrm{E}(G) \text{ for each n } \in N. \qquad (21)$$

Thus using (21) and the condition that S and T are G -monotone generalized quasi contraction , we have a comparison function satisfying

$$d(\mathrm{Sx}_n, \mathrm{Sp}) \leq \max\{\vartheta\left(d(\mathrm{Sx}_{n-1}, \mathrm{p})\right), \vartheta\left(d(\mathrm{Sx}_{n-1}, \mathrm{Sx}_n)\right), \vartheta\left(d(\mathrm{Sp}, \mathrm{p})\right), \vartheta\left(d(\mathrm{Sp}, \mathrm{Sx}_{n-1})\right), \vartheta\left(d(\mathrm{Sx}_n, \mathrm{p})\right)\}, \qquad (22)$$

for each $n \in \mathbb{Z}^+$. Letting n $\to \infty$ in (22) and using the upper semi-continuity of ϑyields,
$d(p, \mathrm{Sp}) \leq \vartheta\left(d(p, \mathrm{Sp})\right)$.

This means that $d(p, \mathrm{Sp}) = 0$ which implies $p = \mathrm{Sp} = \mathrm{Tp}$.
Next, assume $u$to be a different common fixed point of $S$and Tsuch that $(x, \mathrm{u}) \in E(G)$. Then for each$n \in N$,S is G -monotone and $(\mathrm{Sx}_n, \mathrm{u}) \in E(G)$for each $n \in \mathbb{Z}^+$. Therefore

$$d(\mathrm{Sx}_n, \mathrm{u}) \leq \max\{\vartheta\left(d(\mathrm{Sx}_{n-1}, \mathrm{u})\right), \vartheta\left(d(\mathrm{Sx}_{n-1}, \mathrm{Sx}_n)\right), \vartheta\left(d(\mathrm{Su}, \mathrm{u})\right), \vartheta\left(d(\mathrm{u}, \mathrm{Sx}_{n-1})\right), \vartheta\left(d(\mathrm{Sx}_n, \mathrm{u})\right)\}.$$

Using the convergence of Sx_nwe have

$$d(p, \mathrm{u}) \leq \vartheta\left(d(p, \mathrm{u})\right). \qquad (23)$$

By the Property (ii) of $\vartheta$and (23) we conclude that $d(p, \mathrm{u}) = 0$. Hence $\mathrm{p} = \mathrm{u}$. Uniqueness proved.

Example 3.1. Let $X = [0, 1]$ be a metric space with the distance $d(x, \mathrm{y}) = |\mathrm{x} - \mathrm{y}|$, for $\mathrm{x}, \mathrm{y} \in X$. Let $C = 0, 1) \subset [0, 1]$ which is closed. Define a map $T_i\colon \mathrm{C} \to C$by
\

$$T_i(x) = \begin{cases} 1 & \text{if } 0<\mathrm{x} \leq 1 \\ \dfrac{2}{3} + \dfrac{x}{1 + \mathrm{x}} & \text{if } \mathrm{x} = 0 \end{cases}$$

Consider a graph Gon $X$with $E(G) = \mathrm{X} \times X$, then G is connected, reflexive and transitive digraph. To show that $T_i$is a G -monotone generalized quasi contraction mapping, we consider a function , $\vartheta\colon 0, \infty) \to 0, \infty)$, by
$\vartheta(t) = \frac{t}{1+\mathrm{t}}$. We observe that ϑ is a comparison function. Let $x, \mathrm{y} \in C$, without loss of generality we assume $\mathrm{x} > \mathrm{y}$ (since $d$is symmetric then $\mathrm{x} < \mathrm{y}$ holds by the same argument).
There are three possible cases:
(i) Let $x, \mathrm{y} \in 0, 1$. Then

$$d(T_i x, \mathrm{T}_i y) = |T_i\mathrm{x} - T_i y| = 0$$

$$= \left|\frac{x}{1+x} - 1 + 1 - \frac{y}{1+\mathrm{y}}\right|$$
$$= \left|\frac{x}{1+x} - \frac{y}{1+\mathrm{y}}\right|$$
$$= \frac{\mathrm{x} - \mathrm{y}}{1 + \mathrm{x} + \mathrm{y} + \mathrm{xy}}$$
$$\leq \frac{\mathrm{x} - \mathrm{y}}{1 + \mathrm{x} + \mathrm{y}}$$
$$\leq \frac{\mathrm{x} - \mathrm{y}}{1 + \mathrm{x} - \mathrm{y}}$$
$$\leq \vartheta(\mathrm{x} - \mathrm{y})$$

$$= \vartheta(d(x, y))$$
$$\leq \max\left\{\begin{array}{l}\vartheta(d(x, y)), \vartheta(d(x, Tx)), \vartheta(d(y, Ty)),\\ \vartheta(d(x, Ty)), \vartheta(d(y, Tx))\end{array}\right\}.$$

(ii) Let $x, y \in [0,1)$, $y = 0$. Then

$$d(T_i x, T_i y) = |T_i x - T_i y|$$
$$= \left|1 - \left(\frac{y}{1+y} + \frac{2}{3}\right)\right|$$
$$\leq \left|\frac{1}{3} - \frac{y}{1+y}\right|$$
$$\leq \left|\frac{1}{3} - \frac{y}{1+y} + \frac{1}{3} + \frac{x}{1+x}\right|$$
$$= \left|\frac{x}{1+x} - \frac{y}{1+y}\right|$$
$$\leq \frac{x-y}{1+y+x}$$
$$\leq \frac{x-y}{1+x-y}$$
$$= \vartheta(d(x, y))$$
$$\leq \max\left\{\begin{array}{l}\vartheta(d(x, y)), \vartheta(d(x, Tx)), \vartheta(d(y, Ty)),\\ \vartheta(d(x, Ty)), \vartheta(d(y, Tx))\end{array}\right\}.$$

(iii) Let $x = y = 0$. Then

$$d(T_i x, T_i y) = |T_i x - T_i y|$$
$$= \left|\left(\frac{x}{1+x} + \frac{2}{3}\right) - \left(\frac{y}{1+y} + \frac{2}{3}\right)\right|$$
$$= \left|\frac{x}{1+x} - \frac{y}{1+y}\right|$$
$$\leq \frac{x-y}{1+x-y}$$
$$= \vartheta(d(x, y))$$
$$\leq \max\left\{\begin{array}{l}\vartheta(d(x, y)), \vartheta(d(x, Tx)), \vartheta(d(y, Ty)),\\ \vartheta(d(x, Ty)), \vartheta(d(y, Tx))\end{array}\right\}$$

Therefore T_i is a G -monotone generalized quasi contraction mapping. The unique common fixed point of $T_i$is 1 for each $i \in N$

5. Conclusion

This research defines a class of finite family of$G$ - monotone generalized quasi contraction mappings in a metric space. The existence of common fixed point for finite family of these maps is proved in a metric space equipped with a graph. The existence and uniqueness of the common fixed point of this map can be established in other abstract spaces.

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