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The Combined Reproducing Kernel Method and Taylor Series for Solving Nonlinear Volterra-Fredholm Integro-Differential Equations

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Abstract In this paper, the numerical scheme of nonlinear Volterra-Fredholm integro-differential equations is proposed in a reproducing kernel Hilbert space (RKHS). The method is constructed based on the reproducing kernel properties in which the initial condition of the problem is satisfied. The nonlinear terms are replaced by its Taylor series. In this technique, the nonlinear Volterra-Fredholm integro-differential equations are converted to nonlinear differential equations. The exact solution is represented in the form of series in the reproducing Hilbert kernel space. The approximation solution is expressed by n-term summation of reproducing kernel functions and it is converge to the exact solution. Some numerical examples are given to show the accuracy of the method.

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1. Introduction

Some of the phenomena in physics, biology, electronics, and other applied sciences persuade to nonlinear Volterra- Fredholm integro-differential equations. Since the

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boundary value problems in terms of integro-differential equations have many practical applications, many authors have been investigated the numerical methods of integro-differential equations, such as application of homotopy analysis method [1], new direct method using operational matrix with block-pulse functions [2], spline collocation method [3], homotopy perturbation method [4], differential transform method with Adomian polynomials [5], modified Laplace Adomian decomposition method [6], a sinc-collocation method [7], Chebyshev finite difference method [8], rationalized Haar function method [9], operational Tau method [10], the combined Laplace transform-Adomian decomposition method [11], modified Adomian decomposition method [12], non-standard finite difference method [13], B-spline interpolation method [14], a new modified homotopy perturbation method [15], and see [16–19].
The theory of reproducing kernel has been successfully applied to nonlinear Volterra integro-differential equations of fractional order [20], fractional differential equations with daley [21], multiple solutions of nonlinear boundary value problems [22], nonlinear delay differential equations of fractional order [23], singularly perturbed boundary value problems with a delay [24], Fredholm integro-differential equations with weakly singularity [25], second-order integro-differential equations of Fredholm type [26], Fredholm integro-differential equations [27], two-point boundary value problems of fourth-order mixed integro-differential equations [28], two-point, second-order periodic boundary value problems for mixed integro-differential equations [29], and see [30–35].
In this paper, an effective numerical method based on the RKM presented for solving nonlinear Volterra-Fredholm integro-differential equations.
Consider the nonlinear Volterra-Fredholm integro-differential equations

$$\sum_{i=0}^{q} a_i(x)u^{(i)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)g(u(t))\mathrm{d}t + \lambda_2 \int_a^b K_2(x,t)h(u(t))\mathrm{d}t, \quad (1)$$

subject to the boundary conditions

$$u^{(i)}(a) = d_i, \qquad i = 0, 1, \cdots q-1,$$

where $a_i(x)(i = 0, 1, \cdots q)$, $f(x)$, $K_1(x,t)$, $K_2(x,t)$, are function having n th ($n \geqslant q$) derivatives on an integral $a \leqslant x, t \leqslant b$ and a, b, λ_1, and λ_2 are constants.

This paper is organized in six sections including the introduction. In the next section, we present construction of the method in the reproducing kernel space. The analytical solution is introduced in Section 3. Implementations of the method is presented in Section 4. We report our numerical findings and demonstrate the stability of the new numerical scheme by considering some examples in Section 5. The last section is a brief conclusion.

2. Construction of the Method

2.1 *Reproducing kernel Spaces*

We construct the closed subspaces ${}^oW_2^m[a,b], (m \geqslant q+1)$ of the reproducing kernel space $W_2^m[a,b]$ by imposing homogeneous boundary conditions on ${}^oW_2^m[a,b]$.
Definition 2.1. ${}^oW_2^m[a,b] = \{u(x)|u^{(m-1)}(x)$ is an absolutely continuous real value function, $u^{(m)}(x) \in L^2[a,b], u^{(i)}(a) = 0, i = 1, 2, \cdots, q-1\}$. The inner product and

norm in ${}^oW_2^m[a,b]$ are given respectively by

$$\langle u,v\rangle=\sum_{i=0}^{m-1}u^{(i)}(0)v^{(i)}(0)+\int_a^b u^{(m)}(x)v^{(m)}(x)\mathrm{d}x, \tag{2}$$

and

$$\|u\|_m=\sqrt{\langle u,u\rangle_m}, \qquad u,v\in {}^oW_2^m[a,b]. \tag{3}$$

According to [36], ${}^oW_2^m[a,b]$ is a complete reproducing kernel Hilbert space, i.e., for each fixed $y\in {}^oW_2^m[a,b]$, and any $u(x)\in {}^oW_2^m[a,b]$, there exists a function $R_x(y)$ such that $\langle u(y),R_x(y)\rangle=u(x)$, the reproducing kernel $R_x(y)$ can be denoted by

$$R_x(y)=\begin{cases}\sum_{i=1}^{2m}c_i(y)x^{i-1}, & x\leqslant y,\\ \sum_{i=1}^{2m}d_i(y)x^{i-1}, & x>y,\end{cases} \tag{4}$$

where coefficients $c_i(y),d_i(y),\{i=1,2,\cdots,2m\}$, could be obtained by solving the following equations

$$\frac{\partial^i R_y(x)}{\partial x^i}|_{x=y^+}=\frac{\partial^i R_y(x)}{\partial x^i}|_{x=y^-}, \qquad i=0,1,2,\cdots,2m-2, \tag{5}$$

$$(-1)^m(\frac{\partial^{2m-1}R_y(x)}{\partial x^{2m-1}}|_{x=y^+}-\frac{\partial^{2m-1}R_y(x)}{\partial x^{2m-1}}|_{x=y^-})=1, \tag{6}$$

$$\begin{cases}\frac{\partial^i R_y(a)}{\partial x^i}-(-1)^{m-i-1}\frac{\partial^{2m-i-1}R_y(a)}{\partial x^{2m-i-1}}=0, & i=q,q+1,\cdots,m-1,\\ \frac{\partial^{2m-i-1}R_y(b)}{\partial x^{2m-i-1}}=0, \quad i=0,1,\cdots,m-1. \\ \frac{\partial^i R_y(a)}{\partial x^i}=0, \quad i=0,1,\cdots,q-1.\end{cases} \tag{7}$$

2.2 *An Equivalent Transformation of Nonlinear Term (1)*

In this section, for solving Eq. (1) an equivalent transformation of integral parts is proposed. With the Taylor series expansion of $u(t)$ based on expanding about the given point x belonging to the interval $[a,b]$, we have the Taylor series approximation of $u(t)$ in the following form

$$u(t)=u(x)+(t-x)u'(x)+\frac{(t-x)^2}{2!}u''(x)+\cdots+\frac{(t-x)^n}{n!}u^{(n)}(x)+\frac{(t-x)^{n+1}}{(n+1)!}u^{(n+1)}(\eta_{x,t}), \tag{8}$$

where $\eta_{x,t}$ are between x and t. We use the truncated Taylor series of $u(t)$. By substituting relation (8) into Eq. (1), we have

$$\sum_{i=0}^{q} a_i(x)u^{(i)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t) g\left(\sum_{k=0}^{k=n} \frac{u^{(k)}(x)(t-x)^k}{k!}\right) \mathrm{d}t$$
$$+ \lambda_2 \int_a^b K_2(x,t) h\left(\sum_{k=0}^{k=n} \frac{u^{(k)}(x)(t-x)^k}{k!}\right) \mathrm{d}t, \qquad (9)$$

where $u^{(0)}(x) = u(x)$ and $\int_a^x K_1(x,t) g\left(\sum_{k=0}^{k=n} \frac{u^{(k)}(x)(t-x)^k}{k!}\right) \mathrm{d}t$, $\int_a^b K_2(x,t) h\left(\sum_{k=0}^{k=n} \frac{u^{(k)}(x)(t-x)^k}{k!}\right) \mathrm{d}t$ in term of $x, u(x)$ and its derivatives are computable. Hence, Eq. (9) can be written as follows

$$\sum_{i=0}^{q} a_i(x)u^{(i)}(x) = H(x, u(x), u'(x), \cdots, u^{(n)}(x)). \qquad (10)$$

3. The Analytical Solution

In order to illustrate the analytical of the model problem, we consider that $\mathbb{L} : {}^oW_2^m[a,b] \longrightarrow W[a,b]$ is an invertible bounded linear operator and $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$, assume a countable dense subset $\{x_i\}_{i=1}^{\infty}$ in $[a,b]$ and define equation (1),

$$\begin{cases} \mathbb{L}u(x) = \sum_{i=0}^{q} a_i(x)u^{(i)}(x) = H(x, u(x), u'(x), \cdots, u^{(n)}(x)), \\ u^{(i)}(a) = 0, \qquad i = 0, 1, \cdots, q-1. \end{cases} \qquad (11)$$

Let $\phi_i(x) = R_x(x_i)$ and $\psi_i(x) = L^*\phi_i(x)$. From the property of the reproducing kernel, it holds $\langle u(x), \phi_i(x)\rangle = u(x_i)$.

THEOREM 3.1 *If $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[a,b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of ${}^oW_2^m[a,b]$.*

Proof. *Note that $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[a,b]$. For $u(x) \in {}^oW_2^m[a,b]$, if*

$$\langle u(x), \psi_i(x)\rangle = \langle \mathbb{L}u(x), \phi_i(x)\rangle = u(x_i) = 0, \qquad (i = 1, 2, \cdots),$$

from the density of $\{x_i\}_{i=1}^{\infty}$ and continuity of $u(x)$, then we have $u(x) \equiv 0$.

The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of ${}^oW_2^m[a,b]$ is constructed from $\{\psi_i(x)\}_{i=1}^{\infty}$ by using the Gram-Schmidt algorithm, and then the approximate solution will be obtained by calculating a truncated series based on these functions, such that

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad (\beta_{ii} > 0, \quad i = 1, 2, \cdots), \qquad (12)$$

where β_{ik} are orthogonal coefficients.

THEOREM 3.2 *Let $\{x_i\}_{i=1}^{\infty}$ be dense in the interval $[a,b]$. If the equation (11) has* *a unique solution, then the solution satisfies the form*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\beta_{ik}H(x_k, u(x_k), u'(x_k), \cdots, u_k^{(n)}(x))\bar{\psi}_i(x). \tag{13}$$

Proof. *Let $u(x)$ be the solution of Eq. (11) $u(x)$ is expanded in Fourier series, it has*

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\langle u(x), \bar{\psi}_i(x)\rangle\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), \psi_k(x)\rangle\bar{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), \mathbb{L}^*\varphi_k(x)\rangle\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle \mathbb{L}u(x), \varphi_k(x)\rangle\bar{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle H(x, u(x), u'(x), \cdots, u^{(n)}(x)), \varphi_k(x)\rangle\bar{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}H(x_k, u(x_k), u'(x_k), \cdots, u_k^{(n)}(x))\bar{\psi}_i(x).
\end{aligned}$$

The proof is complete.

The equation (11) is nonlinear, that is $H(x, u(x), u'(x), \cdots, u^{(n)}(x))$ depend on u and its derivatives , then its solution can be obtained by the following iterative method.

4. Implementations of the Method

Here, a method of solving (13) of (11) is given in the reproducing kernel space. Rewrite (13) as

$$u(x) = \sum_{i=1}^{\infty}B_i\bar{\psi}_i(x), \tag{14}$$

where $B_i = \sum_{k=1}^{i}\beta_{ik}H(x_k, u(x_k), u'(x_k), \cdots, u^{(n)}(x_k))$.
Let $x_1 = 0$, from the initial and boundary condition of Eq. (11), it follows that $u^{(i)}(x_1) = 0, (i = 1, 2\cdots n)$, then $H(x_1, u(x_1), u'(x_1), \cdots, u^{(n)}(x_1))$ is known. We put

$$H(x_1, u_0(x_1), u_0'(x_1), \cdots, u_0^{(n)}(x_1)) = H(x_1, u(x_1), u'(x_1), \cdots, u^{(n)}(x_1)),$$

and define the $n-$term approximation to $u(x)$ by

$$u_N(x) = \sum_{i=1}^{N}B_i\bar{\psi}_i(x), \tag{15}$$

where

$$
\begin{aligned}
&B_1 = \beta_{11} H(x_1, u_0(x_1), u_0'(x_1), \cdots, u_0^{(n)}(x_1)),\\
&u_1(x) = B_1\, \bar{\psi}_1(x),\\
&B_2 = \sum_{k=1}^{2} \beta_{2k} H(x_k, u_1(x_k), u_1'(x_k), \cdots, u_1^{(n)}(x_k)),\\
&u_1(x) = B_1\, \bar{\psi}_1(x) + B_2\, \bar{\psi}_2(x),\\
&\qquad\quad \vdots\\
&B_N = \sum_{k=1}^{N} \beta_{Nk} H(x_k, u_{N-1}(x_k), u_{N-1}'(x_k), \cdots, u_{N-1}^{(n)}(x_k)),\\
&u_N(x) = \sum_{i=1}^{N} B_i \bar{\psi}_i(x).
\end{aligned}
$$

Next, the convergence of $u_N(x)$ will be proved.

4.1 *Convergence of Method*

THEOREM 4.1 *Suppose $\|u_N(x)\|_{W_2^m}$ is bounded in (15), if $\{x_i\}_{i=1}^{\infty}$ is dense in $[a, b]$, then the N-term approximate solution $u_N(x)$ converges to the exact solution $u(x)$ of Eq. (11) and the exact solution is expressed as*

$$u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x), \tag{16}$$

where B_i is given by (14).

Proof. *The convergence of $u_N(x)$ will be proved. From (15), one obtains*

$$u_N(x) = u_{N-1}(x) + B_N \bar{\psi}_N(x). \tag{17}$$

From the orthogonality of $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\|u_N(x)\|^2_{W_2^m} = \|u_{N-1}(x)\|^2_{W_2^m} + \|B_N\|^2.$$

The sequence $\|u_N(x)\|_{W_2^m}$ is monotone increasing. Due to $\|u_N(x)\|_{W_2^m}$ being bounded, $\{\|u_N(x)\|_{W_2^m}\}$ is convergent as soon as $N \longrightarrow \infty$. Then there is a constant c such that

$$\sum_{i=1}^{\infty} B_i^2 = c. \tag{18}$$

It implies that

$$B_i = \sum_{k=1}^{i} \beta_{ik} H(x_k, u_{i-1}(x_k), u'_{i-1}(x_k), \cdots, u_{i-1}^{(n)}(x_k)),$$

let $m > N$, *in view of* $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{N+1} - u_N)$, *it follows that*

$$\begin{aligned}\|(u_m - u_N)\|^2_{W_2^m} &= \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{N+1} - u_N\|^2_{W_2^m} \\ &= \|u_m - u_{m-1}\|^2_{W_2^m} + \|u_{m-1} - u_{m-2}\|^2_{W_2^m} + \cdots + \|u_{N+1} - u_N\|^2_{W_2^m} \\ &= \sum_{i=N+1}^{m} (B_i)^2 \longrightarrow 0, (N \longrightarrow \infty). \qquad (19)\end{aligned}$$

Considering the completeness of ${}^oW_2^m[a,b]$, *it has*

$$u_N(x) \overset{\|\cdot\|_{W_2^m}}{\longrightarrow} u(x), \quad (N \longrightarrow \infty).$$

It is proved that $u(x)$ *is the solution of Eq. (11).*
Hence,

$$u(x) = \sum_{i=1}^{\infty} B_i \bar{\psi}_i(x).$$

The proof is complete.

THEOREM 4.2 *If* $u_N(x) \overset{\|\cdot\|_{W_2^m}}{\longrightarrow} u(x)$ *and* $x_N \longrightarrow y\,(N \longrightarrow \infty)$, *then*

$$H(x_N, u_N(x_N), u'_N(x_N), \cdots, u_N^{(n)}(x_N)) \longrightarrow H(y, u(y), u'(y), \cdots, u^{(n)}(y))\,(N \longrightarrow \infty). \qquad (20)$$

Proof. *We will prove* $u_N^{(i)}(x_N) \longrightarrow u^{(i)}(y), (N \longrightarrow \infty)$ *and* $i = 0, 1, 2, \cdots, n$. *Observing that*

$$\begin{aligned}|u_N^{(i)}(x_N) - u^{(i)}(y)| &= |u_N^{(i)}(x_N) - u_N^{(i)}(y) + u_N^{(i)}(y) - u^{(i)}(y)| \\ &\leqslant |u_N^{(i)}(x_N) - u_N^{(i)}(y)| + |u_N^{(i)}(y) - u^{(i)}(y)|.\end{aligned}$$

It follows that

$$\begin{aligned}|u_N^{(i)}(x_N) - u_N^{(i)}(y)| &= |\langle u_N(x), \frac{\partial^i}{\partial y^i}(R_{x_N}(x) - R_y(x))\rangle| \\ &\leqslant \|u_N(x)\|_{W_2^m} \|\frac{\partial^i}{\partial y^i}(R_{x_N}(x) - R_y(x))\|_{W_2^m}.\end{aligned}$$

From the convergence of $u_N(x)$, *there exist constants* $N_1 \in \mathbb{N}$ *and* $M \in \mathbb{R}$, *such* *that*

$$\|u_N^{(i)}(x)\|_{W_2^m} \leqslant M\|u(x)\|_{W_2^m},\ for\ N \geqslant N_1\ and\, i = 0, 1, 2, \cdots, n.$$

Since

$$\|R_{x_N}(x) - R_y(x)\|_{W_2^m} \longrightarrow 0 \quad (N \longrightarrow \infty).$$

It follows that $|u_N^{(i)}(x_N) - u^{(i)}(y)| \longrightarrow 0$ *as soon as* $x_N \longrightarrow y$ *from* $\|u_N^{(i)}(x)\|_{W_2^m} \leqslant M\|u(x)\|_{W_2^m}$.
Hence, as soon as $x_N \longrightarrow y$ *it shows that*

$$u_N^{(i)}(x_N) \longrightarrow u^{(i)}(y) \quad (N \longrightarrow \infty).$$

It follows that

$$H(x_N, u_N(x_N), u_N'(x_N), \cdots, u_N^{(n)}(x_N)) \longrightarrow H(y, u(y), u'(y), \cdots, u^{(n)}(y))\ (N \longrightarrow \infty). \tag{21}$$

Consequently, the method mentioned is convergent.

5. Applications and Numerical Results

To test the accuracy of the present method, some examples with exact solutions are given. In these examples we take $N = 10$, where N is the number of terms of the Fourier series of the unknown function $u(x)$. Parameter n is the number of terms of the Taylor series and we choose $m > n$ for solving these examples. Results obtained by the method are compared with the exact solution of each example and are found to be in a good agreement. The approximate solution $u_N(x)$ is calculated by (15). The examples are computed using Mathematica 8.0.
Example 5.1. Consider the nonlinear Volterra-Fredholm integro-differential equation [17]:

$$x^2u''(x)+2u'(x) = 2-\frac{5}{6}x+\frac{1}{2}xe^{-x^2}+\int_0^x txe^{-u^2(x)}\mathrm{d}t+\int_0^1 xu^2(t)\mathrm{d}t, \qquad 0 \leqslant x \leqslant 1,$$

with initial conditions $u(0) = 0$, $u'(0) = 1$. The exact solution is $u(x) = x$.
Let $n = 2$ and applying the reproducing kernel method. The comparison between the exact solution and the approximate solution and the absolute errors in spaces $W^5[0,1], W^6[0,1]$ are graphically shown in figure 1, respectively. The absolute errors between $u(x)$ and $u_{10}(x)$ in spaces $W^5[0,1], W^6[0,1]$ are shown in Table 5. This is an indication of stability on the reproducing Kernel. However, by increasing m, the behavior improves.

Example 5.2. Let us now study the nonlinear Volterra-Fredholm integro-differential equation [1]:

$$u''(x)+xu'(x) = e^x(2+x^2+3x)-(0.5892858)x+\int_0^x u^2(t)\mathrm{d}t+\int_0^{0.5} xt(1+u(t))^2\mathrm{d}t, \qquad 0 \leqslant x \leqslant 0.5,$$

with initial conditions $u(0) = 0$, $u'(0) = 1$. The exact solution is $u(x) = xe^x$. Let $n = 3$ and applying the reproducing kernel method. The comparison between the exact solution and the approximate solution and the absolute errors in spaces

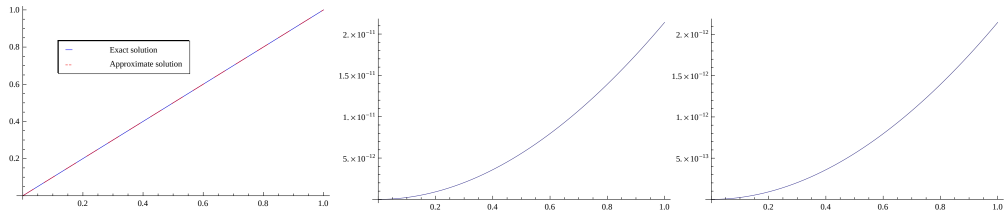

Figure 1. The figures of the approximate solution, the absolute errors in W^5 and W^6, respectively left to right.

Table 1
Numerical results of Ex. 5.1.

Node	$\lvert u_{10}(x) - u(x)\rvert_{W^5}$	$\lvert u_{10}(x) - u(x)\rvert_{W^6}$
0	0	0
0.1	2.29525E-13	2.29677E-14
0.2	9.11105E-13	9.11216E-14
0.3	2.03459E-12	2.03448E-13
0.4	3.59002E-12	3.58991E-13
0.5	5.56755E-12	5.56888E-13
0.6	7.95675E-12	7.96030E-13
0.7	1.07471E-11	1.07558E-12
0.8	1.39275E-11	1.39422E-12
0.9	1.74869E-11	1.75171E-12
1	2.14146E-11	2.14717E-12

$W^5[0, 0.5], W^6[0, 0.5]$ are graphically shown in figure 1, respectively. The absolute errors between $u(x)$ and $u_{10}(x)$ in spaces $W^5[0, 0.5], W^6[0, 0.5]$ are shown in Table 5. This is an indication of stability on the reproducing Kernel. However, by increasing m, the behavior improves. The numerical results compared with [1] are given in Table 5.

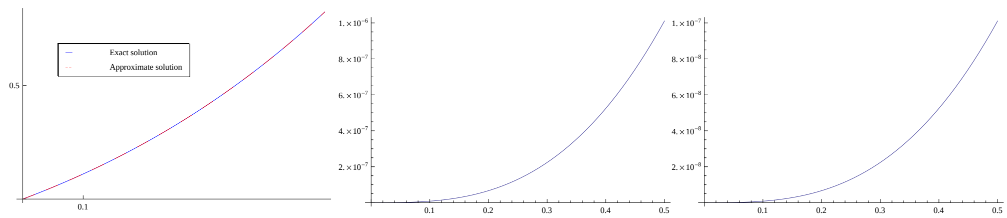

Figure 2. The figures of the approximate solution, the absolute errors in W^5 and W^6, respectively left to right.

Table 2
Numerical results of Ex. 5.2.

Node	$\lvert u_{10}(x) - u(x)\rvert_{W^5}$	$\lvert u_{10}(x) - u(x)\rvert_{W^6}$	Errors, HAM [1]
0	0	0	—
0.05	1.04211E-9	1.04855E-10	0.00071584
0.1	8.36636E-9	8.38119E-10	0.00127473
0.15	2.82122E-8	2.82352E-9	0.00163066
0.2	6.67232E-8	6.67544E-9	0.00196136
0.25	1.29904E-7	1.29944E-8	0.00217893
0.3	2.23578E-7	2.23626E-8	0.00235548
0.35	3.53345E-7	3.53400E-8	0.00268905
0.4	5.24540E-7	5.24602E-8	0.00297645
0.45	7.42203E-7	7.42273E-8	0.00314973
0.5	1.01104E-6	1.01112E-7	0.00347981

Example 5.3. Consider the following Volterra-Fredholm integro-differential equation [4, 19]:

$$u''(x) - xu'(x) + xu(x) = f(x) + \int_{-1}^{x} (x-2t)u^2(t)\mathrm{d}t + \int_{-1}^{1} xtu(t)\mathrm{d}t, \qquad -1 \leqslant x \leqslant 1,$$

where $f(x) = \frac{2}{25}x^6 - \frac{1}{3}x^4 + x^3 - 2x^2 - \frac{23}{15}x + \frac{5}{3}$, with condition $u(0) = -1$, $u'(0) = 0$ and exact solution $u(x) = x^2 - 1$.
Let $n = 2$ and applying the reproducing kernel method. The comparison between the exact solution and the approximate solution and the absolute errors in spaces $W^5[-1,1], W^6[-1,1]$ are graphically shown in figure 3, respectively. The absolute errors between $u(x)$ and $u_{10}(x)$ in spaces $W^5[-1,1], W^6[-1,1]$ are shown in Table 5. This is an indication of stability on the reproducing Kernel. However, by increasing m, the behavior improves.

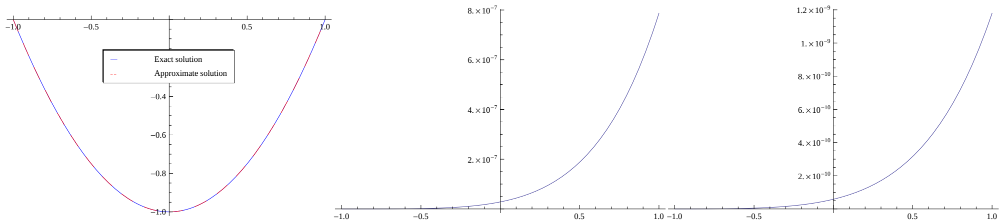

Figure 3. The figures of the approximate solution, the absolute errors in W^5 and W^6, respectively left to right.

Table 3
Numerical results of Ex. 5.3.

Node	$\lvert u_{10}(x) - u(x)\rvert_{W^5}$	$\lvert u_{10}(x) - u(x)\rvert_{W^6}$
-1	0	0
-0.8	7.68430E-11	3.24518E-13
-0.6	3.57387E-10	2.33868E-12
-0.4	1.67649E-9	8.81384E-12
-0.2	6.20583E-9	2.47495E-11
0.0	1.81650E-8	5.82997E-11
0.2	4.47561E-8	1.21964E-10
0.4	9.73603E-8	2.34128E-10
0.6	1.93123E-7	4.21058E-10
0.8	3.57009E-7	7.19472E-10
1	6.24578E-7	1.17988E-9

6. Concluding Remarks

In this study, a computationally attractive method for solving nonlinear Volterra-Fredholm integro-differential equations is presented. Using the definition reproducing kernel space, Taylor series and the properties of proposed method, the initial problem is converted a nonlinear differential equation. By solving the nonlinear differential equations, numerical solutions are obtained. Some examples are solved in two different space W_2^m. However, to obtain better results, use of the larger parameter m is recommended.

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