Direct method for solving nonlinear two-dimensional Volterra-Fredholm integro-differential equations by block-pulse functions

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ABSTRACT

In this paper, an effective numerical method is introduced for the treatment of nonlinear two-dimensional Volterra-Fredholm integro-differential equations. Here, we use the so-called two-dimensional block-pulse functions.First, the two-dimensional block-pulse operational matrix of integration and differentiation has been presented. Then, by using this matrices, the nonlinear two-dimensional Volterra-Fredholm integro-differential equation has been reduced to an algebraic system. Some numerical examples are presented to illustrate the effectiveness and accuracy of the method.

Keywords

Nonlinear equations, Two-dimensional Volterra-Fredholm integro-differential equations, Two-dimensional block-pulse functions, Operational matrix

Introduction

An area of increasing scientific interest over the past decades is the study of Volterra-Fredholm integro-differential equation. This equation is encountered in various applications such as physics, mechanics, and applied science [1-4]. A general form of the Volterra-Fredholm integral equation can be written as:

$$U_{xx}+U_{tx}+U_{tt}+U(x,t)=g(x,t)+\int_0^1\int_0^x k(x,t,y,s)\left[U(y,s)\right]^p dyds \quad (1)$$

$$(x,t)\in[0,A_1]\times[0,A_2]$$

with given supplementary conditions, where *U(t, x)* is an unknown function which should be determined; g(t, x) and k(x,t,y,s) are analytical functions, respectively[5]. In this paper, we consider the nonlinear function $[U(y,s)]^p$in the following form

$$F(u(s,y))=u^p(s,y)$$

where p is a positive integer. With regard to the fact that every finite interval can be transformed to [0, 1] by linear map, without loss of generality, we can consider A1 =A2 = 1

As we know, the block-pulse functions (BPFs) presented by Harmuth [6] are a powerful mathematical tool for solving various kinds of integral equations. These functions are a set of orthogonal functions with piecewise constant values which are defined in the time interval [0, T1] as:

$$\phi_i(t,x)=\begin{cases}1, & (i-1)\frac{T_1}{m}\le x\le i\frac{T_1}{m}\\ 0, & otherwise\end{cases} \quad (2)$$

where i = 0, . . . ,m − 1 with m as a positive integer.The solution of Fredholm and Volterra integral equations of the second kind have been approximated using BPFs in [7]. Maleknejad and Mahmoudi in [8] have applied a combination of Taylor and block-pulse functions to solve linear Fredholm integral equation. The BPFs and Lagrange interpolating polynomials have been used to approximate the solution of Volterra's population model by Marzban et al. [9]. Recently, Maleknejad and Mahdiani have applied two dimensional (2D-BPFs) for solving nonlinear mixed Volterra-Fredholm-integral equations [10]. In this paper, we use 2D-BPFs to approximate the solution of Equation (1).

This paper is organized as follows. In section 'Properties of the 2D-BPFs', the definition and some properties of the 2D-BPFs are presented. The 2D-BPFs are applied to solve

Equation 1 in ‘Applying the method’ section. The error analysis of the proposed method has been investigated in section ‘The error analysis’. Some numerical results have been presented in section ‘Numerical results’ to show accuracy and efficiency of the proposed method. Finally, some concluding remarks are given in ‘Conclusion’section.

Properties of the 2D-BPFs

We usually call the block-pulse functions containing two variables as two-dimensional block-pulse functions. An (m_1m_2) set of 2D-BPFs are defined in region $t \in [0, T_1)$ and $x \in [0, T_2)$ as:

$$\phi_{i_1,i_2}(t,x) = \begin{cases} 1, (i_1-1)h_1 \le x \le i_1h_1 \ and\ (i_2-1)h_2 \le y \le i_2h \\ 0, \quad otherwise \end{cases} \quad (3)$$

where $i_1 = 1, 2, \ldots, m_1$ and $i_2 = 1, 2, \ldots, m_2$ with positive integer values for m_1, m_2, and $h1 = T_1/m_1$, $h_2 = T_2/m_2$ .There are some properties for 2D-BPFs, e.g. disjointness,orthogonality, and completeness.

1. Disjointness

The two-dimensional block-pulse functions are disjoined with each other, i.e.

$$\phi_{i_1,i_2}(t,x)\phi j_1, j_2(t,x) = \begin{cases} \phi_{i_1,i_2}(t,x), \quad i_1 = j_1 \ and\ i_2 = j_2 \\ 0, \quad otherwise \end{cases} \quad (4)$$

2. Orthogonality

The two-dimensional block-pulse functions are orthogonal with each other, i.e.

$$\int_0^{T_1}\int_0^{T_2} \phi_{i_1,i_2}(t,x)\phi j_1, j_2(t,x)dxdt = \begin{cases} h_1h_2, \quad i_1 = j_1 \ and\ i_2 = j_2 \\ 0, \quad otherwise \end{cases} \quad (5)$$

in the region of $t \in [0, T_1)$ and $x \in [0, T_2)$ where $i_1, j_1 = 1, 2, \ldots, m_1$ and $i_2, j_2 = 1, 2, \ldots, m_2$.

3. Completeness

For every $f \in L_2([0,T_1) \times [0,T_2))$ when m_1 and m_2 go to infinity, Parseval identity holds:

$$\int_0^{t_1}\int_0^{t_2} f^2(t,x)dxdt = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} f_{i_1,i_2}^2 \|\phi_{i_1,i_2}(t,x)\|^2, \quad (6)$$

where

$$f_{i1,i2} = \frac{1}{h_1h_2}\int_0^{T_1}\int_0^{T_2} f(t,x)\phi_{i1,i2}(t,x)dxdt. \quad (7)$$

The set of 2D-BPFs may be written as a (m_1m_2) vector ϕ (t,x):

$$\phi(t,x) = [\phi_{1,1}(t,x), \ldots, \phi_{1,m2}(t,x), \ldots, \phi_{m1,1}(t,x), \ldots, \times\phi_{m1,m2}(t,x)]^T, \quad (8)$$

Where $(t,x) \in [0,T_1) \times [0,T_2)$. From the above representation and disjointness property, it follows that

$$\phi(t,x)\phi^T(t,x) = \begin{pmatrix} \phi_{1,1}(t,x) & 0 & \ldots & 0 \\ 0 & \phi_{1,2}(t,x) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \phi_{m1,m2}(t,x) \end{pmatrix} \quad (9)$$

$$\phi^T(t,x)\phi(t,x) = 1, \quad (10)$$

$$\phi(t,x)\phi^T(t,x)V = \tilde{V}\phi(t,x), \quad (11)$$

where V is an m_1m_2 vector and $\tilde{V}$ = diag(V).Moreover, it can be clearly concluded that for every (m_1m_2) × (m_1m_2) matrix A

$$\phi^T(t,x)A\phi(t,x) = \hat{A}^T \phi(t,x) \quad (12)$$

where $\hat{A}$ is an m_1m_2 vector with elements equal to the diagonal entries of matrix A.

2D-BPFs expansion

A function $f \in L_2([0,T_1) \times [0,T_2))$ may be expanded by the 2D-BPFs as:

$$f(t,x) \cong \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2} f_{i1,i2}\phi_{t1,i2}(t,x)$$

$$= F^T\phi(t,x) = \phi^T(t,x)F$$

where F is a (m_1m_2) × 1 vector given by

$$F = [f_{1,1,\ldots,} f_{1,m2,\ldots,} f_{m1,1,\ldots,} f_{m1,m2}]^T \quad (14)$$

and $\phi(t,x)$ is defined in (8).

The block-pulse coefficients $f_{i1,i2}$ are obtained as:

$$f_{i1,i2} = \frac{1}{h_1h_2}\int_{(i_1-1)h_1}^{i_1h_1}\int_{(i_2-1)h_2}^{i_2h_2} f(t,x)dxdt, \quad (15)$$

such that the error between f (t, x) and its block-pulse expansion (13) in the region of

$t \in [0,T_1)$, $y \in [0,T_2)$, i.e,

$$\varepsilon = \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\left(f - \sum_{i_1=1}^{m1}\sum_{i_2=1}^{m2} f_{i1,i2}\phi_{i1,i2}(t,x)\right)^2 dxdt, \quad (16)$$

is minimal.

A function of four variables $k(t,s,x,y)$ on $[0,T_1)\times[0,T_2)\times[0,T_3)\times[0,T_4)$ may be approximated with respect to BPFs such as

$$k(t,s,x,y)=\phi^T(t,x)K\phi(s,y) \qquad (17)$$

where $\phi(t,x)$ and $\phi(s,y)$ are 2D-BPF vectors of dimension m_1m_2 and m_3m_4, respectively, and K is a $(m_1m_2)\times(m_3m_4)$ two dimensional block-pulse coefficient matrix.
Also, the positive integer powers of a function u(s, y) may be approximated by 2D-BPFs as:

$$[u(s,y)]^p=[\phi^T(s,y)u]^p=\phi^T(s,y)\Lambda, \quad (18)$$

where Λ is a column vector, the elements of which are pth power of the elements of the vector U.

Operational matrix of integration

The integration of the vector $\phi(t,x)$ defined in (3) may be obtained as:

$$\int_0^1\int_0^x \phi_{i,j}(s,y)dsdy \,\square\, [0,0,...,\frac{h^2}{2},h^2,...,h^2], \qquad (19)$$

in which $h^2/2$, is i th component. Thus

$$\int_0^1\int_0^t \phi(s,y)dsdy \,\square\, P\phi(\chi,t), \quad (20)$$

where P is a (m2)×(m2) matrix and is called operational matrix of double integration and can be denoted by $P=(\frac{h^2}{2})P_2$, where

$$P=\frac{h^2}{2}\underbrace{\begin{bmatrix}1&2&2&\dots&2\\0&1&2&\dots&2\\0&0&1&\dots&2\\0&0&0&\dots&1\end{bmatrix}_{m^2\times m^2}}_{P2} \qquad (21)$$

So the double integral of very function U(x,t) can be approximate by :

$$\int_0^1\int_0^t u(s,y)dsdy \cong \frac{h^2}{2}U^TP_2\phi(\chi,t), \quad (22)$$

By similar method $\int_0^1 \varphi_{ij}(s,t)ds$, in terms of 2D-BPFs as:

$$\int_0^t \phi_{i,j}(s,y)ds \cong [0,0,...,h,0,...,0]^T\phi(0,0), \quad (23)$$

And

$$\int_0^t \phi(s,t)ds \cong hI\phi(0,0). \quad (24)$$

Operationalmatrix of differentiation

We now need to compute the operational matrix of differentiation.For this, let

$$D_T=\{(x,t):a<x<b, 0<t<t\}, where -\infty\le a<b\le\infty,$$

$\partial_p D_T$ be the parabolic boundary of D_T. If a,b are finite

$$\partial_p D=\{x=a,x=b,0\le t\le T\}\bigcup\{a\le x\le b,t=0\},$$

If a,b are infinite $\partial_p D=\{x\in\Re,t=0\}$

And

$$L^{2,1}(D_T)=\{u(x,t):u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2}\in L^2(D_T)\}, \quad (25)$$

The expansion of function U(x,t) over D_t whit respect to $\phi_{i,j}(x,t)$, *i,j* =0,1,…,m-1, can be written as:

$$u(x,t)\cong\sum_{i=0}^{m-1}\sum_{i=0}^{m-1}u_{i,j}\phi_{i,j}(x,t)=U^T\phi=\phi^TU, \quad (26)$$

where

$$U=[u_{0,0},u_{0,1},...,u_{0,m-1},u_{1,0},...,u_{1,m-1},...,u_{m-1,m-1}]^T$$

$$\phi=[\phi_{0,0},\phi_{0,1},...,\phi_{0,m-1},\phi_{1,0},...,\phi_{1,m-1},...,\phi_{m-1,m-1}]^T$$

and

$$\phi_{i,j}(x,t)=\begin{cases}1\ , & \frac{i}{m}\le x<\frac{i+1}{m},\frac{j}{m}\le t<\frac{j+1}{m}\\0\ , & otherwise\end{cases} \qquad (27)$$

$$u_{i,j}=\frac{1}{h^2}\int_{\frac{j}{m}}^{\frac{j+1}{m}}\int_{\frac{i}{m}}^{\frac{i+1}{m}}u(x,t)dxdt. \qquad (28)$$

Opreatinal matrix for $\frac{\partial u}{\partial t}$ by 2D-BPFS is approximated as:

$$\frac{\partial u(x,t)}{\partial t}\cong(U_t^d)^T\phi(x,t), \qquad (29)$$

that :

$$U_t^d=\frac{2}{h}(U^T-U_f^T\Delta_1)P_2^{-1}, \qquad (30)$$

Where Δ1 is the following (m2)×(m2) matrix as:

$$\Delta_1=\begin{pmatrix}H_{m\times m}&&&0\\&H_{m\times m}&&\\&&\ddots&\\0&&&H_{m\times m}\end{pmatrix}, \quad (31)$$

With

$$H_{m\times m}=\begin{pmatrix}1&0&0&\dots&0\\0&0&0&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&0\end{pmatrix}, \quad (32)$$

That U_f is initial boundary vector of $\partial_p D_T$
by the same method, operational matrix for $\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}$ are given as follows.

$$\frac{\partial u}{\partial x} \cong (U_x^d)^T \phi(x,t), \qquad (33)$$

$$\frac{\partial^2 u}{\partial x^2} \cong (U_{xx}^d)^T \phi(x,t), \qquad (34)$$

Where

$$U_x^d = \frac{1}{h}(U_{g2}^T \Delta_3 - U_{g1}^T \Delta_2) P_2^{-1}, \qquad (35)$$

$$U_{xx}^d = \frac{1}{h^2}(U_{g2}^T \Delta_3 - U_{g1}^T \Delta_2) P_2^{-1}(\Delta_3 - \Delta_2) P_2^{-1}, \qquad (36)$$

and Δ_2, Δ_3 are the following $m^2 \times m^2$ matrices:

$$\Delta_2 = \begin{pmatrix} i_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad (37)$$

$$\Delta_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & I_{m\times m} \end{pmatrix}, \qquad (38)$$

and U_{g_1}, U_{g_2} are boundary vectors of $\partial_p D_T$.

$$u_{tt} \cong \frac{2}{h^2} U_t^T (I - \Delta_2) P_2^{-1} \phi(\chi, t), \qquad (39)$$

$$u_{tx} \cong \frac{2}{h^2} U_t^T (\Delta_3 - I) P_2^{-1} \phi(x, t), \qquad (40)$$

Applying the method

In this section, we solve the nonlinear two-dimensional Volterra-Fredholm integro-differential equations using 2D-BPFs. As we have shown before, we can write

$$u(t,x) = U^T \Phi(t,x), \qquad (41)$$
$$g(t,x) = G^T \Phi(t,x),$$

$$[u(s,y)]^p = \Phi^T(s,y)\Lambda,$$
$$u_x(t,x) = U_x^T \Phi(t,x),$$
$$u_x(t,x) = U_t^T \Phi(t,x),$$
$$u_{xx}(t,x) = U_{xx}^T \Phi(t,x),$$
$$u_{tt}(t,x) = U_{tt}^T \Phi(t,x),$$
$$u_{tx}(t,x) = U_{tx}^T \Phi(t,x),$$
$$k(t,s,x,y) = \phi^T(t.x),$$

Where the $m_1 m_2$ vectors $U, G, \Lambda, U_x, U_t, U_{xx}, U_{tt}, U_{tx}$ and Matrixare K the BPF coefficients of $u(t,x), g(t,x),$ $[u(s,y)]^p, u_x(t,x), u_t(t,x), u_{xx}(t,x), u_{tt}(t,x), u_{tx}(t,x)$

And

$[u(s,y)]^p, u_x(t,x), u_t(t,x), u_{xx}(t,x), u_{tt}(t,x), u_{tx}(t,x)$

Now, consider the following equation:

$$u_{xx} + x_{tx} + u_{tt} + u(t,x) = g(t,x) + \int_0^1 \int_0^x k(t,s,x,y) \times u^p(s,y) dy ds,$$

$$(t,x) \in [0, A_1] \times [0, A_2].$$

Using the proposed equations in section 'Properties of the 2D-BPFs' to approximate the partial derivatives, we have

$$[\frac{1}{h^2}(U_{g_2}^T \Delta_3 - U_{g1}^T \Delta_2) p_2^{-1}(\Delta_3 - \Delta_2) p_2^{-1}]\phi(\chi,t) +$$
$$\frac{2}{h^2} U_t^T (I - \Delta_2) p_2^{-1} \phi(\chi,t) + \frac{2}{h^2} U_t^T (\Delta_3 - I) p_2^{-1} \phi(\chi,t)$$
$$+\phi^T(\chi,t) U = \phi^T(\chi,t) G + \phi^T(\chi,t) Q U^P$$

The error analysis

Here, we investigate the representation error of a differentiable function f (t, x) when it is represented in a series of 2D-BPFs over the region D=[0,1)×[0,1). For this, we briefly review and use some results from [10,11]. For details, see the mentioned references.We put $m_1=m_2=m$, $h_1=h_2=\frac{1}{m}$

We define the representation error between f(x,t) and its 2D-BPF expansion over every subregion $D_{i1,i2}$ as follows:

$$e_{i_1,i_2}(t,x) = f_{i_1,i_2}\phi_{i_1,i_2}(t,x) - f(t,x) = f_{i_1,i_2} - f(t,\chi), \qquad (42)$$

$$t, x \in D_{i1,i2}$$

Where

$$D_{i1,i2} = \{(t,\chi) : \frac{i_1 - 1}{m} \le \frac{i_1}{m}, \frac{i_2 - 1}{m} \le t < \frac{i_2}{m}\}. \qquad (43)$$

Uaing mean value theorem, it can be shown that

$$\|e_{i1,i2}\|^2 \le \frac{2}{m^4} M^2, \qquad (44)$$

Where $\|f'(t,x)\| \le M$[10.11]. Error between $f(t,x)$ and its 2D-BPF expansion, $f_m(t,x)$, over the region D can be obtained as follows:

$$e(t,x) = f_m(t,x) - f(t,x). \qquad (45)$$

Using Equations 44 and 45, it can be shown that (see [10,11])

$$\|e(t,x)\|^2 \le \frac{2}{m^4} M^2. \qquad (46)$$

Hence, $\|e(t,x)\| = o(\frac{1}{m})$. Similar to the proposed method in [10,11], suppose that $f(t,x)$ is approximated by

$$f_m(t,x) = \sum_{i1=1}^{m}\sum_{i2=1}^{m} f_{i1,i2}\phi_{i1,i2}(t,x)$$

We get $\bar{f}_{i1,i2}$ the approximation of $f_{i1,i2}$ and

$$\bar{f}_m(t,x) = \sum_{i1=1}^{m}\sum_{i2=1}^{m} f_{i1,i2}\phi_{i1,i2}(t,x)\cdot$$

Then, from Equation 38 for $(t,x) \in D_{i1,i2}$, we have

$$\|f_{i1,i2}\Phi_{i1,i2} - f(t,x)\| = \leq \frac{\sqrt{2}M}{m} + \frac{\|\bar{f}_m - f\|\infty}{m}, \qquad (47)$$

There fore, from Equation 47, it can be shown that

$$\lim_{x\to\infty} f_m(t,x) = f(t,x).$$

For an error estimation, reconsider the following nonlinear two- dimensional Volterra-Fredholm integro-diffrential equation

$$u_{xx}(t,x) + x_{tx}(t,x) + u_{tt}(t,x) + u(t,x) = g(t,x) + \int_0^1\int_0^x k(t,s,x,y)\times u^p(s,y)dyds, \qquad (48)$$

$(t,x) \in [0,1]\times[0,1].$

Let $e_m^p(t,x) = u^p(t,x) - u_m^p(t,x)$ be the error function of the approximate solution $u_m(t,x)$ to $u(t,x)$, where $u(t,x)$ is the exact solution of Equation 48. Then, we consider

$$R_m(t,x) + (u_{xx}(t,x) + u_{tx}(t,x) + u_{tt}(t,x) + u(t,x))_m = g(t,x) + \int_0^1\int_0^x k(t,s,x,y)\times u_m^p(s,y)dyds, \qquad (49)$$

Where $R_m(t,x)$ is the perturbation function that depends on $u_m(t,x)$, $(u_{xx}(t,x))_m$, $(u_{tx}(t,x))_m$ and $(u_{tt}(t,x))_m$. It can be obtained by substituting $u_m(t,x), (u_{xx}(t,x))_m, (utx(t,x))_m$ *and* $(u_{tt}(t,x))_m$ into equation 48 as

$$R_m(t,x) = g(t,x) + \int_0^1\int_0^x k(t,s,x,y)\times u_m^p(s,y)dyds - (u_{xx}(t,x) + u_{tx}(t,x) + u_{tt}(t,x) + u(t,x))_m$$

Subtracting (49) from (48) gives

$$\int_0^1\int_0^x k(t,s,x,y)\, e_m^p(s,y)dyds = -R_m(t,x) - (e_{xx}(t,x) + e_{tx}(t,x) + e_u(t,x) + e(t,x))_m, \qquad (50)$$

Finally, the proposed method in this paper can be applied to approximate $e_m(t, x)$ in Equation 50.

Numerical results

In this section, three examples are given to show the accuracy of the proposed method. For the all examples,we consider the supplementary conditions from the exact solution. The absolute error is computed for m = m_1 = m_2 terms of 2D-BPF series in all examples. Allcomputations are implemented in MATLAB software ona personal computer.

Example1. For the example, consider the following equation [1]:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^2 u(x,t)}{\partial t^2} + \int_0^1\int_0^x x^2 tu(s,y)dsdy = g(x,t)$$

$, x, t \in [0,1],$

Table 1 Absolute errors for example 1

m = 64	**m = 32**	**$\vert e_{6,6}(x, y)\vert$ [1]**	**$\vert e_{5,5}(x,y)\vert$ [1]**	**(x, t)**
2.124 ×10−8	2.238 ×10−8	1.666 ×10−7	1.666 ×10−7	(0.01, 0.01)
7.742 ×10−8	7.980 ×10−8	1.333 ×10−6	1.333 ×10−6	(0.02, 0.02)
4.714 ×10−6	6.809 ×10−6	1.666 ×10−4	1.670 ×10−4	(0.1, 0.1)
2.230 ×10−4	2.334 ×10−4	1.332 ×10−3	1.347 ×10−3	(0.2, 0.2)

Where

$$g(x,t) = x\ esp(t) - \frac{1}{2}x^2 t + \frac{1}{2}x^4 t\ \exp(t),$$

With subject to the initial conditions

$$u(0,t) = 0, \quad \frac{\partial u(x,t)}{\partial x} = \exp(t),$$

The exact solution of this problem is $u(t,x) = x\ \exp(t).$

The numerical results of problem is shown in Table 1.

Example 2. In this example, we consider a two dimensional nonlinear Volterra-Fredholm integro-diffrential equation as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + u(x,t) - \int_0^1 \int_0^x (y + \cos z) u^2(y,z) dy dz = g(x,t)$$
$$,\quad x,t \in [0,1],$$

Table 2 Absolute errors for example 2

m = 64	m = 32	m = 16	(x, t)
$7.378 \times 10{-}8$	$8.903 \times 10{-}8$	$5.136 \times 10{-}7$	(0.01, 0.01)
$8.703 \times 10{-}8$	$2.918 \times 10{-}7$	$1.307 \times 10{-}6$	(0.02, 0.02)
$1.714 \times 10{-}6$	$1.809 \times 10{-}6$	$5.563 \times 10{-}5$	(0.1, 0.1)
$1.230 \times 10{-}4$	$1.334 \times 10{-}4$	$2.973 \times 10{-}3$	(0.2, 0.2)

Where

$$g(x,t) = \frac{1}{8}x^4 \sin t \cos t - \frac{1}{8}x^4 t - \frac{1}{9}x^3 \sin^3 t.$$

With supplementary conditions,

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = x.$$

The exact solution of this problem is u(x, t) = x sin t. In Table 2, the numerical results are presented.

Conclusion

In this paper, we have successfully approximated the solution of the form (1) of nonlinear Volterra-Fredholm integrodifferential equations. To this end, we have used some orthogonal functions called block-pulse functions. Moreover, the error of the proposed method is analyzed. For more investigation, some examples have been presented. As the numerical results showed, the proposed method is an effective method to solve the Volterra-Fredholm integrodifferential equations.

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