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Modified homotopy perturbation method for solving non-linear oscillator's equations

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Abstract

In this paper a new form of the homtopy perturbation method is used for solving oscillator differential equation, which yields the Maclaurin series of the exact solution. Nonlinear vibration problems and differential equation oscillations have crucial importance in all areas of science and engineering. These equations equip a significant mathematical model for dynamical systems. The accuracy of the Solution equation is very important because the analysis component of the system like vibration amplitude control, synchronization dynamics are dependent to the exact solution of oscillation equation.

Keywords : Homotopy perturbation method (HPM); Differential equations; Non-linear oscillator's equation; Laplace transformation.

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1 Introduction

IN recent years, many engineers and scientists in various sciences like Mathematics, Physics, Biology and particularly in branches of engineering like Fluid mechanics, Numerical calculations in Aerospace and Electronics are faced with nonlinear phenomena and many nonlinear problems. Since solving nonlinear problems plays a crucial role in various fields of engineering and science, Scientists are interested in obtaining techniques for solving nonlinear problems and have performed extensive researchers to achieve nonlinear problem solving techniques. As solving nonlinear problems are generally difficult and achieving their exact solutions are hard, various approximate methods have been developed to solve them. Our objectives in this work are, first to present the basic idea of the homotopy perturbation method HPM for solving nonlinear problems; second to demonstrate the basic idea of the new form of homotopy perturbation transform method (NHPTM) for solving differential equations; and then to apply this method to solve oscillator differential equation. The homotopy perturbation method was first proposed by J. Huan He in 1998 and was further improved by him [13, 8, 14]. This method is based on supposition that a small parameter must exist. The applications of the HPM in nonlinear problems have been demonstrated by many researchers, cf. [1, 4, 5, 6, 16]. Perturbation methods are extensively used in engineering and Science to solve nonlinear problems. This method has also been applied to boundary value problems [12], nonlinear wave equations [9], nonlinear oscillators [15],

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heat radiation equations [5], nonlinear heat conduction , convection equations [21] and to the other fields [11, 10]. Recently, some modifications of this Method are presented in order to decrease the volume of calculations [19, 3, 20] and increase the rapid convergence and accuracy of calculation series solution. In this study, we have applied a new form of homotopy perturbation transform method for solving nonlinear differential equations oscillation. This paper is organized as follows. In Section 2, The Structure of the homotopy perturbation method HPM is given. A new form of homotopy perturbation transform method (NHPTM)is described in Section 3. Then in section 4, we applied the application of (NHPTM) for solving the oscillation equation. In section 5, to show the efficiency of the method, we considered two examples and solved them by (NHPTM). Finally, some conclusions are given in Section 6.

2   Structure of HPM for solving nonlinear differential equations

In this section, To illustrate the principal ideas of the homotopy perturbation method we consider the following nonlinear differential equations

$$A(v(t)) = g(r(t)), r(t) \in \Omega \qquad (2.1)$$

with boundary conditions

$$B(\frac{\partial v(t)}{\partial(n)} = 0), r(t) \in \Gamma \qquad (2.2)$$

where A is a general differential operator, B is boundary operator, $g(r(t))$ analytic function, is the boundary of the domainΩ. The operator A can be rewritten as a sum of R and N, where R is a linear operator and N is a nonlinear one. Therefore, Eq. (2.1) may be expressed as

$$R(v(t)) + N(v(t)) = g(r(t)). \qquad (2.3)$$

By the homotopy technique, we construct a homotopy $v(r(t), p) : \Omega \times [0,1] \longrightarrow R$

$$
\begin{aligned}
H(V(t), p) &= (1-p)[R(V(t) - v_0(t)] \\
&\quad + p[A(U(t)) - g(r(t))] = 0, \\
&\quad p \in [0,1], r(t) \in \Omega \qquad (2.4)
\end{aligned}
$$

p is an embedding parameter, v_0 is an initial approximation of Eq. (2.1). Obivously, from Eq. (2.4) We have

$$
\begin{aligned}
H(V(t), 0) &= R(v(t)) - v_0(t) \\
&= 0, \qquad (2.5) \\
H(V(t), 1) &= A(v(t)) - g(r(t)) \\
&= 0, \qquad (2.6)
\end{aligned}
$$

From the embedding parameter p changes from zero to unity, $H(v(t), p)$ change from $R(v(t)) - v_0(t)$ to $A(v(t)) - g(r(t))$. Conforming to He's homotopy perturbation method, we can first use the embedding parameter p as a small parameter and assume that the solution of (2.6) can be given as a power series in p, i.e,

$$V(t) = V_0(t) + pV_1(t) + p^2V_2(t) + \cdots \qquad (2.7)$$

Setting $p = 1$ results in the approximate solution of Eq. (2.1).

$$
\begin{aligned}
v(t) &= \lim_{p \longrightarrow 1} V(t) \\
&= V_0(t) + V_1(t) + V_2(t) + \cdots \qquad (2.8)
\end{aligned}
$$

Finally, we approximate the solution $V(t)$ by

$$v(t) = \sum_{n=0}^{\infty} p^n V_n(t) \qquad (2.9)$$

3   Fundamental idea of NHPTM

In this section we illuminate the theory of the new homotopy perturbation transform method by the using solution procedure of the (NHPTM). We consider the following nonlinear differential equation as

$$D(V(t)) + N(V(t)) = g(t) \qquad (3.10)$$

Where D is the second order linear differential operator, N is the general nonlinear differential operator and $g(t)$ is a known analytic function. with boundary condition

$$v(o) = h(t), v_t(0) = f(t), \qquad (3.11)$$

Taking the Laplace transform (denoted throughout this paper by L)

$$L[D(V(t))] + L[N(V(t))] = L[g(t)] \qquad (3.12)$$

By using the differentiation property of Laplace transform, we have

$$s^2 L[V(t)] - sv(0) - v_t(0) + L[N(V(t))]$$
$$= L[g(t)] \quad (3.13)$$

Next, we obtain

$$s^2 L[V(t)] = sh(t) + f(t) + L[g(t)] - L[N(V(t))] \quad (3.14)$$

we get

$$L[V(t)] = \frac{h(t)}{s} + \frac{f(t)}{s^2} + \frac{1}{s^2} L[g(t)]$$
$$- \frac{1}{s^2} L[N(V(t))] \quad (3.15)$$

Applying Laplace inverse on both sides of Eq.(3.15)

$$V(t) = L^{-1}\left[\frac{h(t)}{s} + \frac{f(t)}{s^2} + \frac{1}{s^2} L[g(t)]\right]$$
$$- L^{-1}\left[\frac{1}{s^2} L[N(V(t))]\right], \quad (3.16)$$

Put $G(t) = L^{-1}\left[\frac{h(t)}{s} + \frac{f(t)}{s^2} + \frac{1}{s^2} L[g(t)]\right]$ and consider the equivalence convex homotopy (2.4) as

$$H(V(t), p) = V(t) - v_0(t) + pv_0(t)$$
$$+ p[N(V(t)) - G(t)] = 0, \quad (3.17)$$

which is equivalent to

$$V(t) = v_0(t) + p[G(t) - v_0(t) - N(V(t))], \quad (3.18)$$

Now, we apply the new form of homotopy perturbation transform method (NHPT) , consider the convex homotopy defined in Eq. (3.16). then we

$$V(t) = v_0(t) + p[G(t) - v_0(t)$$
$$- L^{-1}\left[\frac{1}{s^2} L[N(V(t))]\right]], \quad (3.19)$$

We can assume that the solution of Eq. (3.10)can be written as a power series in p as following

$$V(t) = \sum_{n=0}^{\infty} p^n V_n(t) \quad (3.20)$$

In order to demonstrate the (NHPTM) suppose that the initial approximation of Eq. (2.1) has from

$$v_0(t) = \sum_{n=0}^{\infty} b_n K_n(t) \quad (3.21)$$

Where b_0, b_1, b_2, $\cdots$ are unknown coefficients and $k_0(t)$, $k_1(t)$, $k_2(t)$, $\cdots$ determined function depending on the problem. By substituting Eqs. (3.18) and (3.19) into Eq. (3.17) We get

$$\sum_{n=0}^{\infty} p^n V_n(t) = \sum_{n=0}^{\infty} b_n K_n(t) + p[G(t)$$
$$- \sum_{n=0}^{\infty} b_n K_n(t)$$
$$- L^{-1}\left[\frac{1}{s^2} L[N(\sum_{n=0}^{\infty} p^n V_n(t))]\right]] \quad (3.22)$$

Comparing the coefficient of like power of p, the following approximations we have

$$p^0 : V_0(t) = \sum_{n=0}^{\infty} b_n K_n(t),$$
$$p^1 : V_1(t) = G(t) - \sum_{n=0}^{\infty} b_n K_n(t)$$
$$- L^{-1}\left[\frac{1}{s^2} L[N(V_0(t))]\right], \quad (3.23)$$
$$p^2 : V_2(t) = -L^{-1}\left[\frac{1}{s^2} L[N(V_0(t), V_1(t))]\right],$$
$$\vdots$$

Now, we solve these equation so that $V_1(t) = 0$ then the Eq. (3.15) result $V_2(t) = V_3(t) = \ldots = 0$ and finally, the exact solution may be obtained as follows

$$v(t) = V_0(t) = \sum_{n=0}^{\infty} b_n K_n(t) \quad (3.24)$$

4 Application of NHPTM for Oscillation equation

In this section we apply NHPTM for solving oscillation's equation, to this end, first consider a general form of oscillation's equation as follows

$$Dv(x, t) + Rv(x, t) = g(x, t) \quad (4.25)$$

With boundary condition

$$v(x, 0) = u(x), v_t(x, 0) = w(x), \quad (4.26)$$

Where $u(x), w(x) \in C(R)$, D is the second order linear differential operator, R is the linear

differential operator and N is the general nonlinear differential operator and $g(x,t)$ is a known analytical function. After taking the Laplace transform)on Eq. (4.25) we have

$$L[Dv(x,t)] + L[Rv(x,t)] + L[Nv(x,t)]$$
$$= L[g(x,t)], \qquad (4.27)$$

Now we use the differentiation properties of Laplace transform for Eq. (4.27) then we have

$$L[v(x,t)] = \frac{u(x)}{s} + \frac{w(x)}{s^2} + \frac{1}{s^2}L[g(x,t)]$$
$$- \frac{1}{s^2}L[Rv(x,t) + Nv(x,t)], \qquad (4.28)$$

Next we applying the Laplace inverse operators on both sides of Eq. (4.28)

$$v(x,t) = g(x,t) - L^{-1}[\frac{1}{s^2}L[Rv(x,t)$$
$$+ Nv(x,t)]], \qquad (4.29)$$

Now, we apply the homotopy perturbation method, consider the convex homotopy defined in Eq. 3.18). then we have

$$v(x,t) = v_0(x,t) - pv_0(x,t) + p(g(x,t)$$
$$- L^{-1}[\frac{1}{s^2}L[Rv(x,t) + Nv(x,t)]]) \qquad (4.30)$$

By substituting $v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t)$ and $v_o(x,t) = \sum_{n=0}^{\infty} K_n(t)b_n$ where $K_n(t) = t^n$ and the nonlinear term can be decomposed by Adomian's polynomials so that $Nv(x,t) = \sum_{n=0}^{\infty} p^n A_n(v)$ (see [7, 22]) and the Adomian's polynomials A_n is given below

$$A_n(v_0, \cdots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{n=0}^{\infty} p^n v_n(x))],$$
$$n = 0,1,2,3,\ldots \qquad (4.31)$$

Then we have

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \sum_{n=0}^{\infty} t^n b_n(x) - p(\sum_{n=0}^{\infty} t^n b_n(x))$$
$$+ p[g(x,t)$$
$$- L^{-1}[\frac{1}{s^2}L[R\sum_{n=0}^{\infty} p^n v_n(x,t)$$
$$+ \sum_{n=0}^{\infty} p^n A_n(v)]]] \qquad (4.32)$$

Comparing coefficients of terms with identical powers of p leads to

$$p^0 : v_0(x,t) = \sum_{n=0}^{\infty} t^n b_n,$$

$$p^1 : v_1(x,t) = v_0(x,t) = \sum_{n=0}^{\infty} t^n b_n + g(x,t)$$
$$- L^{-1}[\frac{1}{s^2}L[Rv_o(x,t) + A_0(v)]], \qquad (4.33)$$

$$p^2 : v_2(x,t) = -L^{-1}[\frac{1}{s^2}L[Rv_1(x,t) + A_1(v)]]),$$

$$p^3 : v_3(x,t) = -L^{-1}[\frac{1}{s^2}L[Rv_2(x,t) + A_2(v)]]),$$
$$\vdots$$

Now, we solve these equations in such a way that $v_1(x,t) = 0$ then Eq. (4.25) result in

$$v_2(x,t) = v_3(x,t) = \cdots = 0 \qquad (4.34)$$

Therefore, the exact solution may be obtained as following

$$v_0(x,t) = \sum_{n=0}^{\infty} t^n b_n \qquad (4.35)$$

5 Numerical Examples

In this section To illustrate the efficiency of the method, we consider two test problems.

Example 5.1 *Consider the Van der Pols oscillator problem [18, 2, 17]*

$$\frac{d^2v}{dt^2} + \frac{dv}{dt} + v + v^2\frac{dv}{dt} = 2\cos(t) - \cos(t)^3, \quad (5.36)$$

With the initial condition $v(0) = 0$ and $v'(0) = 1$ and the exact solution $v(t) = \sin(t)$. By applying the Laplace transform and initial condition, we have

$$v(s) = \frac{1}{s^2 + s} + \frac{1}{s^2 + s}L[2\cos(t) - \cos(t)^3$$
$$- v - v^2 v'] \qquad (5.37)$$

Apply the inverse Laplace transform on both side Eq. (5.37) we obtain

$$v(t) = 1 - \exp(-t) + L^{-1}[\frac{1}{s^2 + s}L[2\cos(t)$$
$$- \cos(t)^3 - v - v^2 v']] \qquad (5.38)$$

Consider the convex homotopy (3.18) as

$$
\begin{aligned}
v(t) &= v_0(t) - pv_0(t) - p(\exp(-t) - 1) \\
&+ pL^{-1}[\frac{1}{s^2 + s}L[2\cos(t) - \cos(t)^3 \\
&- v - v^2 v']]] = 0
\end{aligned}
\tag{5.39}
$$

Considering the Maclaurin series of the exponential term and trigonometric expressions in Eq. (5.39)

$$
(1 - \exp(-t)) = t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24} + \frac{t^5}{120} + \cdots,
\tag{5.40}
$$

$$
2\cos(t) - \cos(t)^3 = 1 + \frac{t}{2} - \frac{19t^4}{24} + \cdots, \tag{5.41}
$$

Substituting them to $v(t) = \sum_{n=0}^{\infty} p^n v_n(t)$, Eq. (3.21), Eq. (4.31), Eq. (5.40) and Eq. (5.41) into Eq. (5.39)

$$
\begin{aligned}
\sum_{n=0}^{\infty} p^n v_n(t) &= \sum_{n=0}^{\infty} t^n b_n - p(\sum_{n=0}^{\infty} t^n b_n) \\
&+ (t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24} + \frac{t^5}{120}) \\
&+ p[L^{-1}[\frac{1}{s^2 + s}L[(1 + \frac{t}{2} - \frac{19t^4}{24}) \\
&- \sum_{n=0}^{\infty} p^n v_n(t) - \sum_{n=0}^{\infty} p^n A_n]]]
\end{aligned}
\tag{5.42}
$$

By comparing the coefficient of like powers of p, we have

$$
p^0 : v_0(t) = \sum_{n=0}^{\infty} t^n b_n
$$

$$
\begin{aligned}
p^1 : v_1(t) =& -\sum_{n=0}^{\infty} t^n b_n + (t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24} + \frac{t^5}{120}) \\
&- L^{-1}[\frac{1}{s^2 + s}L[(1 + \frac{t}{2} - \frac{19t^4}{24}) \\
&- \sum_{n=0}^{\infty} t^n a_n - A_0]]
\end{aligned}
\tag{5.43}
$$

$$\vdots$$

Now, we solve these equations in such a way that $v_1(t) = 0$, then Eq. (5.46). $v_2(t) = v_3(t) = \cdots =$

0. Solving the above equation for $v_1(t) = o$ leads to the result

$$
\begin{aligned}
v_1(t) =& -b_0 t^0 + (-b_1 + 1)t + t^2(\frac{b}{2} - \frac{b_0^2 b_1}{2} - b_2) \\
&+ t^3(-b_3 - \frac{1}{6}) + \cdots = 0
\end{aligned}
\tag{5.44}
$$

Eliminating $v_1(t) = 0$ lets the coefficients an for take the $n = 0, 1, 2, \cdots$ we obtain

$$
b_0 = 0, b_1 = 1, b_2 = -\frac{1}{6}, b_3 = 0, \cdots \tag{5.45}
$$

This implies that

$$
\begin{aligned}
v(t) &= v_0(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \cdots \\
&= \sin(t)
\end{aligned}
\tag{5.46}
$$

Example 5.2 *Consider the nonlinear oscillator differential equation [18, 2, 17]*

$$
\frac{d^2v}{dt^2} - v - v^2 - (\frac{dv}{dt})^2 - 1 = 0 \tag{5.47}
$$

With the initial condition $v(0) = 2$ and $v'(0) = 1$ and the exact solution $v(t) = 1 + \cos(t)$ the exact solution. In a similar way, we apply the Laplace transform and the on both sides of Eq. (5.47) then we have

$$
v(s) = \frac{2}{s} + \frac{1}{s^3} + \frac{1}{s^2}L[v - v^2 - (\frac{dv}{dt})^2] \tag{5.48}
$$

Next by applying the Laplace inverse operators on both sides of Eq. (5.48) then we obtain

$$
v(t) = 2 + \frac{t^2}{2} + L^{-1}[\frac{1}{s^2}L[v - v^2 - (\frac{du}{dt})^2]] \tag{5.49}
$$

Consider the convex homotopy (3.18) as

$$
\begin{aligned}
v(t) =& v_0(t) - pv_0(t) - p[v(t) - 2 - \frac{t^2}{2} \\
&- L^{-1}[\frac{1}{s^2}L[v - v^2 - (\frac{dv}{dt})^2]]]
\end{aligned}
\tag{5.50}
$$

Substituting $v(t) = \sum_{n=0}^{\infty} p^n v_n(t)$ and Eqs. (3.21),(4.31) in Eq. (5.50)

$$\sum_{n=0}^{\infty} v_n(t)p^n = \sum_{n=0}^{\infty} b_n t^n - p(\sum_{n=0}^{\infty} b_n t^n)$$
$$-p[\sum_{n=0}^{\infty} v_n(t)p^n - 2 - \frac{t^2}{2}$$
$$-L^{-1}[\frac{1}{s^2}L[\sum_{n=0}^{\infty} v_n(t)p^n$$
$$-\sum_{n=0}^{\infty} p^n A_{1,n} - \sum_{n=0}^{\infty} p^n A_{2,n}]]]$$
$$(5.51)$$

Comparing coefficients of terms with identical powers of p leads to

$$p^0 : v_0(t) = \sum_{n=0}^{\infty} t^n b_n \qquad (5.52)$$

$$p^1 : v_1(t) = -\sum_{n=0}^{\infty} t^n b_n - [2 - \frac{t^2}{2}$$
$$-L^{-1}[\frac{1}{s^2}L[v_0 - A_{1,0} - A_{2,0}]]]$$
$$\vdots$$

Now, we solve these equations in such a way that $v_1(t) = 0$, then Eq. (5.52) results in

$$v_2(t) = v_3(t) = \cdots = 0.$$

$$v_1(t) = t^0(2 - b_0) + tb_1 + t^2(\frac{1}{2} - b_2)$$
$$+t^3(-b_3) + \cdots \qquad (5.53)$$

Eliminating $v_1(t)$ lets the coefficients an for $n = 1, 2, \cdots$ take the following values by substituting

$$b_0 = 2, b_1 = 0, b_2 = -\frac{1}{2}, b_3 = 0, \cdots \qquad (5.54)$$

This implies that

$$v(t) = v_o(t) = 2 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t}{720} - \cdots$$
$$= 1 + \cos(t) \qquad (5.55)$$

6   Conclusion

In this study, we have presented the (NHPTM) to solve non-linear oscillator differential equations. The (NHPTM) yields the Maclaurin series of the true solution.The obtained results indicate that the method is very efficient and simple and leads to the exact solution of non-linear Oscillator differential equation.

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