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Research Article

Ranking of generalized fuzzy numbers using distance measure and similarity measure

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Abstract

In this paper, we first of all define the distance measure entitled generalized Hausdorff distance between two trapezoidal generalized fuzzy numbers (TGFNs) that has been introduced by Chen [10]. Then using a other distance and combining with generalized Hausdorff distance, we define the similarity measure. The basic properties of the above mentioned similarity measure are proved in detail. Finally we rank two generalized fuzzy numbers using distance measure and similarity measure between them.

Keywords : Fuzzy numbers; Generalized Fuzzy numbers; Fuzzy distance measure; Similarity measure.

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1 Introduction

The task of measuring similarity occurs in many disciplines. Because the objects to be compared are often represented by sets, this task is frequently performed by means of measures that compare sets. Such measures are usually called similarity measures. In many applications, fuzzy sets are more suitable than crisp sets for representing the objects concerned. Consequently, there is a need for fuzzy similarity measure, i.e., measures that compare fuzzy sets [7]. Similarity measure between fuzzy sets have gained importance due to the widespread applications in diverse fields like decision making, pattern recognition, machine learning and market prediction, etc. Similarity measure between two fuzzy numbers is related to their commonality, in theories of the recognition, identification, and categorization of objects, where a common assumption is that the greater the commonality between a pair objects, more similar they are. Similarity and distance measure between two fuzzy numbers are closely related concepts. So it is possible to express similarity measure and distance measure between fuzzy numbers by a functional relationship. This

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is one of the oldest and most influential theoretical assumptions that similarity measure is inversely related to distance measure. Therefore the study about the distances between fuzzy numbers are very much significant. Thus the distance between fuzzy sets and as well as fuzzy numbers have gained more attention from researchers [9]. In the most of researches, the authors pointed out to construct the distance measure between normal fuzzy numbers, for instance [1, 2, 8, 14] and also in ranking fuzzy numbers [3, 4, 5].
Now in this paper, the introduced distance measure between two normal fuzzy numbers in [2] is developed to distance measure between two TGFNs. Such that, the confidence level has an important role in this concept. The structure of the paper is as follows: In section 2, some basic definitions and results about TGFNs are brought. In section 3, the new distance between TGFNs is introduced. The new concept of similarity measure is defined and considered in section 4 and it follows by comparison of the proposed method with other methods [11, 12, 13, 15]. Conclusion is drawn in section 5.

2 Preliminaries

In this section basic definitions are brought.

Definition 2.1. *[10] A trapezoidal generalized fuzzy number $\tilde{A}$ as $\tilde{A} = (a, b; \beta, \gamma; w)$, where $0 < w \leq 1$, and a, b, β and γ are non-negative real numbers, w represents the degree of confidence of expert regarding $\tilde{A}$ is a fuzzy subset on the real line R, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions:*

(i) $\mu_{\tilde{A}}$ *is a continuous mapping from R to the closed interval* $[0, 1]$*;*

(ii) $\mu_{\tilde{A}}(x) = 0$*, where* $-\infty < x \leq a - \beta$*;*

(iii) $\mu_{\tilde{A}}(x)$*, is strictly increasing on* $(a - \beta, a)$*;*

(iv) $\mu_{\tilde{A}}(x) = w$*, where* $a \leq x \leq b$*;*

(v) $\mu_{\tilde{A}}(x)$ *is strictly decreasing on* $(b, b + \gamma)$*;*

(vi) $\mu_{\tilde{A}}(x) = 0$*, where* $b + \gamma \leq x < \infty$

If $w = 1$, then the trapezoidal generalized fuzzy number $\tilde{A}$ is called a normal trapezoidal fuzzy number and denoted as $\tilde{A} = (a, b; \beta, \gamma)$. If $a = b$, then $\tilde{A}$ is called a triangular generalized fuzzy number and denoted as $\tilde{A} = (a; \beta, \gamma; w)$. If $\beta = 0$, $\gamma = 0$ and $a = b$ and $w = 1$, then $\tilde{A}$ is called a real number.
The α-cut of the $\tilde{A}$ is presented as an interval $[\tilde{A}]_\alpha = [\tilde{A}^L(\alpha), \tilde{A}^R(\alpha)]$ where $\tilde{A}^L(\alpha) = a + (\frac{\beta}{w})\alpha$ and $\tilde{A}^L(\alpha) = a - (\frac{\gamma}{w})\alpha$ for $0 \leq \alpha \leq w$.
Let F^g be the family of all generalized fuzzy numbers.
If K is the set of compact subsets of R^2, and A and B are two subsets of R^2 then the Hausdorff metric $H : K \times K \to [0, \infty)$ is defined by [14]

$$H(A, B) = \max\{\sup_{b \in B} d_E(b, A), \sup_{a \in A} d_E(a, B)\},$$

where d_E is the usual Euclidean metric for R^2.

Definition 2.2. *The generalized Hausdorff metric d^g_∞ on $F^g \times F^g$ is defined by*

$$d^g_\infty(\tilde{A}_1, \tilde{A}_2) = \sup_{0 \le \alpha < w} \{H([\tilde{A}_1]_\alpha, [\tilde{A}_2]_\alpha)\} + \beta \sup_{w \le \alpha \le w'} |R(\alpha) - L(\alpha)|,$$

where

$$\beta = \begin{cases} 1 & for \quad w' \neq w \\ 0 & w' = w, \end{cases}$$

and

$$R(\alpha) = \begin{cases} \tilde{A}^R_1(\alpha) & for \quad w' = w_1 \\ \tilde{A}^R_2(\alpha) & w' = w_2, \end{cases}$$

and

$$L(\alpha) = \begin{cases} \tilde{A}^L_1(\alpha) & for \quad w' = w_1 \\ \tilde{A}^L_2(\alpha) & w' = w_2. \end{cases}$$

Where $w = \min\{w_1, w_2\}$ and $w' = \max\{w_1, w_2\}$.

Definition 2.3. *[7] A fuzzy comparison measure is a binary relation on $F(X)$, i.e., a fuzzy set in $F(X) \times F(X)$. We call a fuzzy comparison measure s a fuzzy similarity measure if it is reflexive, i.e., $s(\tilde{A}, \tilde{A}) = 1$ for all $\tilde{A} \in F(X)$.*

Hence, we consider reflexivity to be an inherent property of a fuzzy similarity measure. In addition to reflexivity, we also consider the following properties [6]:

$s(\tilde{A}_1, \tilde{A}_2) \in [0, 1]$,

$s(\tilde{A}_1, \tilde{A}_2) = 1 \iff \tilde{A}_1 = \tilde{A}_2$ $(coreflexive)$,

$s(\tilde{A}_1, \tilde{A}_2) = s(\tilde{A}_2, \tilde{A}_1)$ $(symmetric)$,

for all $\tilde{A}_1, \tilde{A}_2 \in F(X)$ and we know $F^g \subset F(X)$.

3 Distance measure for two trapezoidal generalized fuzzy numbers

Let us consider two TGFNs $\tilde{A}_1$ and $\tilde{A}_2$, denoted as $\tilde{A}_1 = (a_1, b_1; \beta_1, \gamma_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2; \beta_2, \gamma_2; w_2)$. The α cut of the $\tilde{A}_1$ and $\tilde{A}_2$ represent following two intervals respectively $[\tilde{A}_1]_\alpha = [\tilde{A}^L_1(\alpha), \tilde{A}^R_1(\alpha)]$ and $[\tilde{A}_2]_\alpha = [\tilde{A}^L_2(\alpha), \tilde{A}^R_2(\alpha)]$ for all $\alpha \in [0, 1]$.
Now the distance between two TGFNs $\tilde{A}_1$ and $\tilde{A}_2$ can be defined by

$$\begin{aligned} d(\tilde{A}_1, \tilde{A}_2) &= | \textstyle\int_0^{\frac{1}{2}} [(1-\alpha)(\tilde{A}^R_1(\alpha) - \tilde{A}^R_2(\alpha)) + \alpha(\tilde{A}^L_1(\alpha) - \tilde{A}^L_2(\alpha))] d\alpha \\ &+ \textstyle\int_{\frac{1}{2}}^{w} [\alpha(\tilde{A}^R_1(\alpha) - \tilde{A}^R_2(\alpha)) + (1-\alpha)(\tilde{A}^L_1(\alpha) - \tilde{A}^L_2(\alpha))] d\alpha| \\ &+ | \textstyle\int_w^{w'} \alpha(R(\alpha) - L(\alpha)) d\alpha| \end{aligned} \tag{3.1}$$

for $w \ge \frac{1}{2}$ and if $w < \frac{1}{2}$ it can be defined by

$$\begin{aligned} d(\tilde{A}_1, \tilde{A}_2) &= | \textstyle\int_0^{w} [(1-\alpha)(\tilde{A}^R_1(\alpha) - \tilde{A}^R_2(\alpha)) + \alpha(\tilde{A}^L_1(\alpha) - \tilde{A}^L_2(\alpha))] d\alpha| \\ &+ | \textstyle\int_w^{w'} \alpha(R(\alpha) - L(\alpha)) d\alpha| \end{aligned} \tag{3.2}$$

Remark 3.1. *For $w = 1$ the result is the same as [2].*

Theorem 3.1. *For fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$, we have*

(i) $d(\tilde{A}_1, \tilde{A}_2) \geq 0$ *and* $d(\tilde{A}_1, \tilde{A}_1) = 0$;

(ii) $d(\tilde{A}_1, \tilde{A}_2) = d(\tilde{A}_2, \tilde{A}_1)$;

(iii) $d(\tilde{A}_1, \tilde{A}_2) \leq d(\tilde{A}_1, \tilde{A}_3) + d(\tilde{A}_3, \tilde{A}_2)$.

Proof. We consider only *(iii)* and the other cases are similar. For $w \geq \frac{1}{2}$, since $|\int_w^{w'} \alpha(R(\alpha) - L(\alpha))d\alpha|$ is a positive number then we can add it to right hand side of the definition (3.1), so we obtain

$$\begin{aligned} d(\tilde{A}_1, \tilde{A}_2) & \leq |\int_0^{\frac{1}{2}}[(1-\alpha)(\tilde{A}_1^R(\alpha) - \tilde{A}_3^R(\alpha)) + \alpha(\tilde{A}_1^L(\alpha) - \tilde{A}_3^L(\alpha))]d\alpha \\ & + \int_{\frac{1}{2}}^{w}[\alpha(\tilde{A}_1^R(\alpha) - \tilde{A}_3^R(\alpha)) + (1-\alpha)(\tilde{A}_1^L(\alpha) - \tilde{A}_3^L(\alpha))]d\alpha| \\ & +|\int_w^{w'} \alpha(R(\alpha) - L(\alpha))d\alpha| \\ & +|\int_0^{\frac{1}{2}}[(1-\alpha)(\tilde{A}_3^R(\alpha) - \tilde{A}_2^R(\alpha)) + \alpha(\tilde{A}_3^L(\alpha) - \tilde{A}_2^L(\alpha))]d\alpha \\ & + \int_{\frac{1}{2}}^{w}[\alpha(\tilde{A}_3^R(\alpha) - \tilde{A}_2^R(\alpha)) + (1-\alpha)(\tilde{A}_3^L(\alpha) - \tilde{A}_2^L(\alpha))]d\alpha| \\ & +|\int_w^{w'} \alpha(R(\alpha) - L(\alpha))d\alpha| = d(\tilde{A}_1, \tilde{A}_3) + d(\tilde{A}_3, \tilde{A}_2) \end{aligned}$$

Now if $w < \frac{1}{2}$ the proof is similar. □

3.1 Properties

Lemma 3.1. *Let $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ are TGFNs, then*

(i) $d(\tilde{A}_1 + \tilde{A}_2, \tilde{A}_2 + \tilde{A}_3) = d(\tilde{A}_1, \tilde{A}_3)$,

(ii) $d(\tilde{A}_1 + \tilde{A}_2, \tilde{0}) \leq d(\tilde{A}_1, \tilde{0}) + d(\tilde{A}_2, \tilde{0})$,

(iii) $d(\lambda\tilde{A}_1, \lambda\tilde{A}_2) = |\lambda| d(\tilde{A}_1, \tilde{A}_2), \lambda \in R$.

Proof. **(i)**, For $w \geq \frac{1}{2}$

$$\begin{aligned} d(\tilde{A}_1 + \tilde{A}_2, \tilde{A}_2 + \tilde{A}_3) & = |\int_0^{\frac{1}{2}}[(1-\alpha)(\tilde{A}_1^R(\alpha) + \tilde{A}_2^R(\alpha) - \tilde{A}_2^R(\alpha) - \tilde{A}_3^R(\alpha)) \\ & +\alpha(\tilde{A}_1^L(\alpha) + \tilde{A}_2^L(\alpha) - \tilde{A}_2^L(\alpha) - \tilde{A}_3^L(\alpha))]d\alpha \\ & + \int_{\frac{1}{2}}^{w}[\alpha(\tilde{A}_1^R(\alpha) + \tilde{A}_2^R(\alpha) - \tilde{A}_2^R(\alpha) - \tilde{A}_3^R(\alpha)) \\ & +(1-\alpha)(\tilde{A}_1^L(\alpha) + \tilde{A}_2^L(\alpha) - \tilde{A}_2^L(\alpha) - \tilde{A}_3^L(\alpha))]d\alpha| \\ & +|\int_w^{w'} \alpha(R(\alpha) - L(\alpha))d\alpha| \\ & = d(\tilde{A}_1, \tilde{A}_3) \end{aligned}$$

(ii), We have

$$
\begin{aligned}
d(\tilde{A}_1+\tilde{A}_2,0) &= |\textstyle\int_0^{\frac{1}{2}}[(1-\alpha)(\tilde{A}_1^R(\alpha)+\tilde{A}_2^R(\alpha))+\alpha(\tilde{A}_1^L(\alpha)+\tilde{A}_2^L(\alpha)]d\alpha \\
&+\textstyle\int_{\frac{1}{2}}^{w}[\alpha(\tilde{A}_1^R(\alpha)+\tilde{A}_2^R(\alpha))+(1-\alpha)(\tilde{A}_1^L(\alpha)+\tilde{A}_2^L(\alpha)]d\alpha| \\
&+|\textstyle\int_w^{w'}\alpha(R(\alpha)-L(\alpha))d\alpha| \\
&\le |\textstyle\int_0^{\frac{1}{2}}[(1-\alpha)\tilde{A}_1^R(\alpha)+\alpha\tilde{A}_1^L(\alpha)]d\alpha\int_{\frac{1}{2}}^{w}[\alpha\tilde{A}_1^R(\alpha)+(1-\alpha)\tilde{A}_1^L(\alpha)]d\alpha| \\
&+|\textstyle\int_w^{w'}\alpha(R(\alpha)-L(\alpha))d\alpha| \\
&+|\textstyle\int_0^{\frac{1}{2}}[(1-\alpha)\tilde{A}_2^R(\alpha)+\alpha\tilde{A}_2^L(\alpha)]d\alpha\int_{\frac{1}{2}}^{w}[\alpha\tilde{A}_2^R(\alpha)+(1-\alpha)\tilde{A}_2^L(\alpha)]d\alpha| \\
&+|\textstyle\int_w^{w'}\alpha(R(\alpha)-L(\alpha))d\alpha| = d(\tilde{A}_1,0)+d(\tilde{A}_2,0)
\end{aligned}
$$

and the proof in case **(iii)** is clear because $\int \lambda d\alpha = \lambda \int d\alpha$.
In case $w < \frac{1}{2}$ the proof is similar. □

Theorem 3.2. *For two generalized fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$, We have*

$$0 \le d(\tilde{A}_1,\tilde{A}_2) \le d_\infty^g(\tilde{A}_1,\tilde{A}_2).$$

Proof. By definition (3.1) we have,

$$d(\tilde{A}_1,\tilde{A}_2) = |\int_0^{\frac{1}{2}}(1-\alpha)(A_1^R(\alpha)-A_2^R(\alpha))d\alpha + \int_0^{\frac{1}{2}}\alpha(A_1^L(\alpha)-A_2^L(\alpha))d\alpha$$

$$\int_{\frac{1}{2}}^{w}\alpha(A_1^R(\alpha)-A_2^R(\alpha))d\alpha + \int_{\frac{1}{2}}^{w}(1-\alpha)(A_1^L(\alpha)-A_2^L(\alpha))]d\alpha| + |\int_w^{w'}\alpha(R(\alpha)-L(\alpha))d\alpha|$$

Also by assumptions $\sup\limits_{0\le\alpha\le w}\{H([\tilde{A}_1]_\alpha,[\tilde{A}_2]_\alpha)\} = M$ and $\sup\limits_{w<\alpha\le w'}|R(\alpha)-L(\alpha)| = M'$, we have $\left|A_1^R(\alpha)-A_2^R(\alpha)\right| \le M$, $\left|A_1^L(\alpha)-A_2^L(\alpha)\right| \le M$, $|R(\alpha)-L(\alpha)| \le M'$, and from the mean value theorem for integrals, we obtain

$$d(\tilde{A}_1,\tilde{A}_2) \le M\int_0^{\frac{1}{2}}(1-\alpha)d\alpha + M\int_0^{\frac{1}{2}}\alpha d\alpha$$

$$+M\int_{\frac{1}{2}}^{w}\alpha d\alpha + M\int_{\frac{1}{2}}^{w}(1-\alpha)d\alpha + M'\int_w^{w'}\alpha d\alpha$$

since $0 \le \alpha \le 1$

$$\le M\int_0^1(1-\alpha)d\alpha + M\int_0^1(\alpha)d\alpha + M'\int_w^{w'}d\alpha = M + (w'-w)M' \le M + M'$$

Therefore

$$d(\tilde{A}_1,\tilde{A}_2) \le d_\infty^g(\tilde{A}_1,\tilde{A}_2).$$

□

4 Similarity measure

Let us consider two TGFNs denoted as $\tilde{A}_1 = (a_1, b_1; \beta_1, \gamma_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2; \beta_2, \gamma_2; w_2)$. Therefore by proposed distance measure the similarity measure between $\tilde{A}_1$ and $\tilde{A}_2$ is defined by

$$s(\tilde{A}_1, \tilde{A}_2) = \begin{cases} 1 - \frac{d(\tilde{A}_1, \tilde{A}_2)}{d^g_\infty(\tilde{A}_1, \tilde{A}_2)}, & d(\tilde{A}_1, \tilde{A}_2) < d^g_\infty(\tilde{A}_1, \tilde{A}_2) \\ 1 - \frac{d(\tilde{A}_1, \tilde{A}_2)}{m}, & d(\tilde{A}_1, \tilde{A}_2) = d^g_\infty(\tilde{A}_1, \tilde{A}_2), \end{cases} \tag{4.3}$$

where $m = \max\{b_1 + \gamma_1, b_2 + \gamma_2\}$.

It is clear $m \neq 0$ because in definition 2.1 b and γ are non-negative real numbers and $m = 0$ is only for zero singleton fuzzy number. Also from theorem 3.2 that, if $d^g_\infty(\tilde{A}_1, \tilde{A}_2) = 0$ then $d(\tilde{A}_1, \tilde{A}_2) = 0$ so $d(\tilde{A}_1, \tilde{A}_2) = d^g_\infty(\tilde{A}_1, \tilde{A}_2)$ and we have $s(\tilde{A}_1, \tilde{A}_2) = 1 - \frac{d(\tilde{A}_1, \tilde{A}_2)}{m}$. We can see comparison of the proposed similarity measure with the existing similarity measures in table 2 for the eighteen pairs of sets in Fig. 1.

In table 2, for set 2, Lee's and Chen's methods have the same similarity number, 1, But there aren't similar graphically. This fact is clear in our introduced method. Also for set 8, our method shows that two fuzzy numbers haven't any similarity, 0.1785, this is obvious in figure 1. But in the other methods the similarity measures are at least 0.4028.

In existing methods if the normal fuzzy numbers are symmetric then they are similar even they are separate graphically, but it doesn't appear in our method, for instance table 2, set 11,12,13.

Now we prove that our method satisfies three main properties of a similarity measure.

Property 4.1. $s(\tilde{A}_1, \tilde{A}_2) \in [0, 1]$.

Proof. By theorem 3.2 we have $0 \leq d(\tilde{A}_1, \tilde{A}_2) \leq d^g_\infty(\tilde{A}_1, \tilde{A}_2)$. Also we know that $\tilde{A}^L_1(\alpha)$, $\tilde{A}^R_1(\alpha)$, $\tilde{A}^L_2(\alpha)$ and $\tilde{A}^R_2(\alpha)$ are positive then $|\tilde{A}^R_1(\alpha) - \tilde{A}^R_2(\alpha)| \leq \tilde{A}^R_1(\alpha)$ or $\tilde{A}^R_2(\alpha)$. Therefore $|\tilde{A}^R_1(\alpha) - \tilde{A}^R_2(\alpha)| \leq b_1 + \gamma_1$ or $b_2 + \gamma_2$ and so $|\tilde{A}^R_1(\alpha) - \tilde{A}^R_2(\alpha)| \leq m$. The proof is similar to show $|\tilde{A}^L_1(\alpha) - \tilde{A}^L_2(\alpha)| \leq m$ then $d^g_\infty(\tilde{A}_1, \tilde{A}_2) \leq m$.

So $\frac{d(\tilde{A}_1, \tilde{A}_2)}{d^g_\infty(\tilde{A}_1, \tilde{A}_2)} \leq 1$ and $\frac{d(\tilde{A}_1, \tilde{A}_2)}{m} \leq 1$ then $0 \leq s(\tilde{A}_1, \tilde{A}_2) \leq 1$. □

Property 4.2. $s(\tilde{A}_1, \tilde{A}_2) = 1 \iff \tilde{A}_1 = \tilde{A}_2$.

Proof. From theorem 3.2 if $\tilde{A}_1 = \tilde{A}_2$, then $d(\tilde{A}_1, \tilde{A}_2) = 0$ and then $d^g_\infty(\tilde{A}_1, \tilde{A}_2) = 0$. So $s(\tilde{A}_1, \tilde{A}_2) = 1 - \frac{0}{m} = 1$.

Conversely $s(\tilde{A}_1, \tilde{A}_2) = 1$, then $\frac{d(\tilde{A}_1, \tilde{A}_2)}{m} = 0$. Therefore $d(\tilde{A}_1, \tilde{A}_2) = 0$ and $\tilde{A}_1 = \tilde{A}_2$. □

Property 4.3. $s(\tilde{A}_1, \tilde{A}_2) = s(\tilde{A}_2, \tilde{A}_1)$.

Proof. It is obvious because $d(\tilde{A}_1, \tilde{A}_2)$ and $d^g_\infty(\tilde{A}_1, \tilde{A}_2)$ are symmetric. □

5 Ranking

Let us consider two TGFNs denoted as $\tilde{A}_1 = (a_1, b_1; \beta_1, \gamma_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2; \beta_2, \gamma_2; w_2)$. We rank them using the distance measure and the similarity measure that represent following:

Table 1: Calculated results of 18 pairs of TGFNs

	$\tilde{A}_1$	$\tilde{A}_2$	$d(\tilde{A}_1, \tilde{A}_2)$	$d^g_\infty(\tilde{A}_1, \tilde{A}_2)$
set 1	(0.2,0.3;0.1,0.1;1)	(0.25;0.15,0.15;1)	0.0125	0.05
set 2	(0.2,0.3;0.1,0.1;1)	(0.2,0.3;0.1,0.1;$\frac{4}{5}$)	0.0241	0.1650
set 3	(0.3;0.0,0.0;1)	(0.3;0.0,0.0;1)	0	0
set 4	(0.2;0.0,0.0;1)	(0.3;0.0,0.0;1)	0.1	0.1
set 5	(0.25;0.25,0.25;$\frac{1}{3}$)	(0.6;0.1,0.1;$\frac{2}{7}$)	0.08	0.5714
set 6	(0.2,0.3;0.1,0.1;$\frac{4}{7}$)	(0.55;$\frac{2\sqrt{3}}{9}$,$\frac{2\sqrt{3}}{9}$;$\frac{2}{3}$)	0.3680	0.6449
set 7	(0.2,0.3;0.1,0.1;1)	(0.2,0.3;0.1,0.1;1)	0	0
set 8	(0.2,0.3;0.1,0.1;1)	(0.55;0.15,0.15;1)	0.2875	0.35
set 9	(0.2,0.3;0.1,0.1;1)	(0.5,0.6;0.1,0.1;1)	0.3	0.3
set 10	(0.2;0.1,0.1;1)	(0.3;0,0;1)	0.075	0.2
set 11	(0.2;0.1,0.1;1)	(0.3;0.1,0.1;1)	0.1	0.1
set 12	(0.4;0.3,0.3;1)	(0.4;0.1,0.1;1)	0.05	0.2
set 13	(0.4;0,0.4;1)	(0.4;0.1,0.1;1)	0.125	0.3
set 15	(0.3,0.4;0,0;1)	(0.6,0.7;0,0;1)	0.3	0.3
set 16	(0.5,0.6;0.1,0.1;$\frac{4}{7}$)	(0.2,0.3;0.1,0.1;$\frac{4}{7}$)	0.1714	0.3
set 17	(0.225;0.225,0.225;0.225$\sqrt{3}$)	(0.675;0.0.225,0.225;0.225$\sqrt{3}$)	0.1754	0.45
set 18	(0.225;0.225,0.225;0.225$\sqrt{3}$)	(0.675;0,0;0.15$\sqrt{3}$)	0.893	0.8250

Table 2: Comparison of the proposed similarity measure with the existing similarity measures for 18 sets in table 1.

Set	Lee's method	Chen's method	Chen et al. method	Yong et al. method	proposed method
1	0.9617	0.975	0.8375	0.7954	0.812
2	1	1	0.8	0.8000	0.8539
3	*	1	1	1	1
4	0	0.9	0.9	0.81	0.6666
5	0.75	0.65	0.106	*	0.8599
6	0.6407	0.7	0.49	0.7004	0.4293
7	1	1	1	1	1
8	0.5	0.7	0.42	0.4028	0.1785
9	0.5	0.7	0.49	0.4931	0.5714
10	0.5	0.9	0.54	0.5754	0.625
11	0.6667	0.9	0.81	0.8112	0.75
12	0.8333	0.9	0.9	0.8854	0.75
13	0.75	0.9	0.9	0.6914	0.5
14	0.8	0.9	0.78	0.7744	0.5833
15	0.25	0.7	0.49	0.7781	0.5714
16	0.5	0.7	0.49	0.4931	0.4286
17	0.5	0.55	0.3025	0.3090	0.6102
18	0.3333	0.55	0.3025	0.2870	0.8917

5.1 Ranking of TGFNs using the distance measure

Here we consider the distance is defined in section 3 without the Absolute symbol. Because we need to know the sign of distance measure between two TGFNs. So we have:

$$\begin{aligned} d(\tilde{A}_1,\tilde{A}_2) &= \int_0^{\frac{1}{2}}[(1-\alpha)(\tilde{A}_1^R(\alpha)-\tilde{A}_2^R(\alpha))+\alpha(\tilde{A}_1^L(\alpha)-\tilde{A}_2^L(\alpha))]d\alpha \\ &+\int_{\frac{1}{2}}^{w}[\alpha(\tilde{A}_1^R(\alpha)-\tilde{A}_2^R(\alpha))+(1-\alpha)(\tilde{A}_1^L(\alpha)-\tilde{A}_2^L(\alpha))]d\alpha \\ &+\int_w^{w'}\alpha(R(\alpha)-L(\alpha))d\alpha \end{aligned} \tag{5.4}$$

for $w \geq \frac{1}{2}$ and if $w < \frac{1}{2}$ it can be defined by

$$\begin{aligned} d(\tilde{A}_1,\tilde{A}_2) &= \int_0^{w}[(1-\alpha)(\tilde{A}_1^R(\alpha)-\tilde{A}_2^R(\alpha))+\alpha(\tilde{A}_1^L(\alpha)-\tilde{A}_2^L(\alpha))]d\alpha \\ &+\int_w^{w'}\alpha(R(\alpha)-L(\alpha))d\alpha \end{aligned} \tag{5.5}$$

Be careful for determine the sign of distance measure between two TGFNs Correctly, we must consider:

$$R(\alpha)=\begin{cases} \tilde{A}_1^R(\alpha) & for \quad w'=w_1 \\ -\tilde{A}_2^R(\alpha) & w'=w_2, \end{cases}$$

and

$$L(\alpha)=\begin{cases} \tilde{A}_1^L(\alpha) & for \quad w'=w_1 \\ -\tilde{A}_2^L(\alpha) & w'=w_2. \end{cases}$$

A decision maker can rank a pair of fuzzy numbers, u and v, using $D(u,v)$ based on the following rules:

(i) If $d(\tilde{A}_1,\tilde{A}_2) > 0$, then $\tilde{A}_1 \succ \tilde{A}_2$;

(ii) If $d(\tilde{A}_1,\tilde{A}_2) = 0$, then $\tilde{A}_1 \approx \tilde{A}_2$;

(iii) If $d(\tilde{A}_1,\tilde{A}_2) < 0$, then $\tilde{A}_1 \prec \tilde{A}_2$.

We can see calculation results of the proposed method in table 3 for the eight pairs sets and we use an example shown in [12] to present comparison the proposed method to the exiting methods in table 4.

5.2 Ranking of TGFNs using the similarity measure

Ranking fuzzy numbers by using the similarity measure between them in the second class ranking is possible. In this section, we consider the single zero as a reference. We determine the similarity measure between generalized fuzzy numbers with zero, Then we rank them by results. So for two TGFNs $\tilde{A}_1$ and $\tilde{A}_2$ we used the following similarity measures:

$$s(\tilde{A}_1,0)=1-\frac{d(\tilde{A}_1,0)}{m-n} \quad , \quad s(\tilde{A}_2,0)=1-\frac{d(\tilde{A}_2,0)}{m-n} \tag{5.6}$$

where $m=\max\{b_1+\gamma_1,b_2+\gamma_2\}$ and $n=\max\{a_1-\beta_1,a_2-\beta_2\}$.
It is clear $m,n \neq 0$ because in definition 2.1 b and γ are non-negative real numbers and

Table 3: The calculation results for 8 pair TGFNs

	$\tilde{A}_1$	$\tilde{A}_2$	$s(\tilde{A}_1, \tilde{A}_2)$	result
set 1	(0.2,0.3;0.1,0.1;1)	(0.25;0.15,0.15;1)	0.0125	$\tilde{A}_1 \succ \tilde{A}_2$
set 2	(0.3;0.0,0.0;1)	(0.3;0.0,0.0;1)	0	$\tilde{A}_1 \approx \tilde{A}_2$
set 3	(0.2;0.0,0.0;1)	(0.3;0.0,0.0;1)	-0.1	$\tilde{A}_1 \prec \tilde{A}_2$
set 4	(0.2,0.3;0.1,0.1;1)	(0.2,0.3;0.1,0.1;1)	0	$\tilde{A}_1 \approx \tilde{A}_2$
set 5	(0.2,0.3;0.1,0.1;1)	(0.55;0.15,0.15;1)	-0.2875	$\tilde{A}_1 \prec \tilde{A}_2$
set 6	(0.2,0.3;0.1,0.1;1)	(0.5,0.6;0.1,0.1;1)	-0.3	$\tilde{A}_1 \prec \tilde{A}_2$
set 7	(0.2;0.1,0.1;1)	(0.3;0.1,0.1;1)	-0.1	$\tilde{A}_1 \prec \tilde{A}_2$
set 8	(0.3,0.4;0,0;1)	(0.6,0.7;0,0;1)	-0.3	$\tilde{A}_1 \prec \tilde{A}_2$

Table 4: comparison of the proposed method with the existing method for 6 sets

$\widetilde{A}$	$\widetilde{B}$	Chen-Wang	chen	Chen-Chen	Yong et al.	proposed method
(0.3;0.1;0.1)	(1;0;0;1)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$
(-0.3;0.2,0.2;1)	(0.3;0.2,0.2;1)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$
(-0.01;0,0;1)	(0.01;0,0;0.8)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$
(0.3;0.1,0.1;1)	(1;0,0;1)	$\tilde{A} \prec \tilde{B}$	$*$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$
(0.3;0.2,0.2,;1)	(0.3;0.1;0.1;1)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$
(0.2,0.3;0.1;0.1;1)	(0.2,0.3;0.1,0.1;1)	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \approx \tilde{B}$

Table 5: Calculated results of 8 pairs of TGFNs

	$\tilde{A}_1$	$\tilde{A}_2$	$s(\tilde{A}_1, 0)$	$s(\tilde{A}_2, 0)$	results
set 1	(0.2,0.3;0.1,0.1;1)	(0.25;0.15,0.15;1)	0.25	0.28	$\tilde{A}_1 \succ \tilde{A}_2$
set 3	(0.3;0.0,0.0;1)	(0.3;0.0,0.0;1)	0.76	0.76	$\tilde{A}_1 \approx \tilde{A}_2$
set 4	(0.2;0.0,0.0;1)	(0.3;0.0,0.0;1)	0.83	0.76	$\tilde{A}_1 \prec \tilde{A}_2$
set 7	(0.2,0.3;0.1,0.1;1)	(0.2,0.3;0.1,0.1;1)	0.25	0.25	$\tilde{A}_1 \approx \tilde{A}_2$
set 8	(0.2,0.3;0.1,0.1;1)	(0.55;0.15,0.15;1)	0.25	0.16	$\tilde{A}_1 \prec \tilde{A}_2$
set 9	(0.2,0.3;0.1,0.1;1)	(0.5,0.6;0.1,0.1;1)	0.25	0.14	$\tilde{A}_1 \prec \tilde{A}_2$
set 11	(0.2;0.1,0.1;1)	(0.3;0.1,0.1;1)	0.25	0.18	$\tilde{A}_1 \prec \tilde{A}_2$
set 15	(0.3,0.4;0,0;1)	(0.6,0.7;0,0;1)	0.06	0.03	$\tilde{A}_1 \prec \tilde{A}_2$

$m = 0$ is only for zero singleton fuzzy number. For the singles numbers we have $m = n = 0$. So we can define the similarity measure by $s(\tilde{A}, \tilde{0}) = 1 - \frac{d(\tilde{A},\tilde{0})}{m+1}$. A decision maker can rank a pair of fuzzy numbers, u and v, using $D(u, v)$ based on the following rules:

(i) If $s(\tilde{A}_1, 0) < s(\tilde{A}_2, 0)$, then $\tilde{A}_1 \succ \tilde{A}_2$;

(ii) If $s(\tilde{A}_1, 0) = s(\tilde{A}_2, 0)$, then $\tilde{A}_1 \approx \tilde{A}_2$;

(iii) If $s(\tilde{A}_1, 0) > s(\tilde{A}_2, 0)$, then $\tilde{A}_1 \prec \tilde{A}_2$.

We can see results of ranking in table 5 for the eight pairs of sets .

6 Conclusion

In this paper a new similarity measure between TGFNs by using generalized distance measure was introduced. The comparison between TGFNs showed that, our introduced measure is useful and better than the others and has the same similarity. Finally we ranked two TGFNs using the distance measure and the similarity measure between them. For future research, one can use fuzzy distance measure between two trapezoidal generalized fuzzy numbers and fuzzy similarity measure between two TGFNs.

References

[1] S. Abbasbandy, T. Hajjari, *A new approach for ranking of trapezoidal fuzzy numbers*, Comput. Math. Appl. 57 (2009) 413-419.

[2] S. Abbasbandy, S. Hajighasemi, *A fuzzy distance between two fuzzy numbers*, IPUM 2010, Part II, CCIS 81 (2010) 376-382.

[3] T. Allahviranloo, S. Abbasbandy, S. Saneifard, *A method for ranking of fuzzy numbers using new weighted distance*, Comput. Math. Appl. 16 (2011) 359-369.

[4] T. Allahviranloo, S. Abbasbandy, S. Saneifard, *An approximation approach for ranking of fuzzy numbers with weighted interval-value*, Comput. Math. Appl. 16 (2011) 588-597.

[5] R. Ezzati, T. Allahviranloo, S. Khezerloo, M. Khezerloo, *An approach for ranking of fuzzy numbers*, Expert Systems with Applications 39 (2012) 690-695.

[6] K. Bosteels, E. Kerre, *A biparametric family of cardinality-based fuzzy similarity measures*, New Math. Natural Comp. to appear.

[7] K. Bosteels, E. Kerre, *A triparametric family of cardinality-based fuzzy similarity measures*, Fuzzy sets and systems 158 (2007)2466-2479.

[8] C. Chakraborty, D. Chakraborty, *A theoretical development on a fuzzy distance measure for fuzzy numbers*, Math. Comput. Modeling 43 (2006) 254-261.

[9] G. Chakraborty, D. Chakraborty, *A new approach to fuzzy distance measure and similarity measure between two generalized fuzzy numbers*, Applied Soft Computing 10 (2010) 90-99.

[10] SH. Chen, *Ranking generalized fuzzy numbers with graded mean integration*, Proceedings of the 8th International fuzzy systems Association World Congress 2 (1999) 899-902.

[11] SM. Chen, *New methods for sobjective mental workload assessment and fuzzy risk analysis*, Cybrenetics and Systems 27 (1996) 449-472 .

[12] SJ. Chen, SM. Chen, *Fuzzy risk analysis based on similarity measures of generelized fuzzy numbers*, IEEE Transactions on fuzzy systems 114 (2003) 1-9.

[13] HS. Lee, *An optimal aggregation method for fuzzy opinions of group decision*, Proceedings of IEEE International conference on Sustems, Man, and Cybernetics 3 (1999) 314-319.

[14] W. Voxman, *Some remarks on distances between fuzzy numbers*, Fuzzy Sets and Systems 100 (1998) 353-365.

[15] D. Yong, SH. Wenkang, D. Feng, L. Qi, *A new similarity measure of generalized fuzzy numbers and its application to pattern recognition*, Pattern Recognition Letters 25 (2004) 875-883.