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Research Article

Mean value theorem for integrals and its application on numerically solving of fredholm integral equation of second kind with Toeplitz plus Hankel Kernel

N. Mikaeilvand *†, S. Noeiaghdam ‡

Abstract

The subject of this paper is the solution of the Fredholm integral equation with Toeplitz, Hankel and the Toeplitz plus Hankel kernel. The mean value theorem for integrals is applied and then extended for solving high dimensional problems and finally, some example and graph of error function are presented to show the ability and simplicity of the method.

Keywords : Fredholm Integral Equations; Toeplitz plus Hankel Kernel; Mean Value Theorem for Integrals.

1 Introduction

The integral equations with a Toeplitz, Hankel or Toeplitz plus Hankel kernel attracts attention of many authors as they have practical applications in such diverse fields as scattering theory, fluid dynamics, linear filtering theory, and inverse scattering problems in quantum-mechanics, problems in radiative wave transmission, and further applications in Medicine and Biology [1, 2, 6, 7, 8, 11, 15, 18, 19, 20, 21].

Many different powerful methods have been proposed to obtain exact and approximate solution of integral equation with a Toeplitz plus Hankel kernel. Solvability of the integral equations with TPH kernel considered in ([12]). Fredholm integral equations with TPH kernel have been solved numerically in ([13]). Also, Fredholm integral equations solved by variational iteration method ([17]), homotopy perturbation method (HPM) ([5, 14]), Adomian decomposition method (ADM) ([4, 9]).

In [3], Avazzadeh et al. introduced a new method for solving Fredholm integral equation by using integral mean value method (IMVM)and M. Heydari et al. [10] extended their method for high dimensional Fredholm integral equations. Based on their works, in this paper, (IMVM) is used for solving integral equations with Toeplitz, Hankel and Toeplitz plus Hankel kernel.

The paper organized as follow : Section 2 introduces the main idea of method for solving Fredholm integral equation with TPH kernel and also, some exampels, graph of error function and comparison between exact and approximate solution for one dimensional Fredholm integral equations with TPH kernel. Our idea is devoted to generalized the method for solving two and also, high dimensional integral equations in Sections 3 and 4. Section 4, includes the extented formulae to clarify the generalization process and some numerical

*Corresponding author. Mikaeilvand@iauardabil.ac.ir
†Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran.
‡Young Researchers and Elite Club, Tabriz Branch, Islamic Azad University, Tabriz, Iran.

example, graph of error functions and comparison between exact and approximate solution to describe the method, and Section 5 is discussion and conclusion.

2 Fredholm integral equation with TPH kernel

Consider the following nonlinear Fredholm integral equation of second kind:

$$u(x) = f(x)+\lambda \int_a^b [P(x-t)+Q(x+t)]F(u(t))dt, \tag{2.1}$$

whereλ is a real number and also F, f, P and Q are given continuous functions, and u is unknown function to be determined.

For solving TPH kernel with IMVM, we use mean value theorem for integrals as follows:

Theorem 2.1 *(mean value theorem for integrals [22]). If $w(x)$ is continuous in $[a, b]$, then there is a point $c \in [a, b]$, such that*

$$\int_a^b w(x)dx = (b-a)w(c), \tag{2.2}$$

Now, we illustrate the main idea of our method. By applying the above theorem to Eq. (2.1) we have

$$\begin{aligned} u(x) &= f(x) + \lambda(b-a) \\ &\quad [p(x-c) + q(x+c)]F(u(c)), \\ c &\in [a, b] \end{aligned} \tag{2.3}$$

now we must just find c and $u(c)$ as unknowns. Substitution of c into Eq. (2.3)

$$u(c) = f(c)+\lambda(b-a)[p(0)+q(2c)]F(u(c)) \tag{2.4}$$

For constructing another equation concerning c and $u(c)$ we substitute Eq. (2.3) into Eq. (2.1)

$$\begin{aligned} u(x) &= f(x) + \lambda \int_a^b [p(x-t) + q(x+t)] \\ &\quad F(f(t) + \lambda(b-a)[p(t-c) \\ &\quad +q(t+c)]F(u(c)))dt \end{aligned} \tag{2.5}$$

substituting $x = c$ in to (2.5):

$$\begin{aligned} u(c) &= f(c) + \lambda \int_a^b [p(c-t) + q(c+t)] \\ &\quad F(f(t) + \lambda(b-a)[p(t-c) \\ &\quad +q(t+c)]F(u(c)))dt \end{aligned} \tag{2.6}$$

Now, we solve Eqs. (2.4) and (2.6) simultaneously. For solving the above nonlinear system, various methods can be used.

Some different examples of Fredholm integral equation with Toeplitz, Hankel and Toeplitz plus Hankel kernel are chosen to illustrate the presented method. The results show the ability and simplicity of the method.

Example 2.1 *(Fredholm integral equation with Toeplitz kernel): Consider the integral equations*

$$u(x) = \frac{1}{3}(-1+e)e^x + e^{2x} - \frac{1}{3}\int_0^1 e^{x-t}u(t)dt$$

for which the exact solution is $u(x) = e^{2x}$. Using the presented method leads to the following system of equations:

$$\begin{cases} u(c) = \frac{1}{3}(-1+e)e^c + e^{2c} - \frac{1}{3}u \\ u(c) = (\frac{-1}{3} + \frac{e}{3})e^c + e^{2c} + \frac{1}{3}((\frac{4}{3} + \frac{4e}{3})e^c + \frac{1}{3}u) \end{cases}$$

By mentioned method, values of c and $u(c)$ are found as follows:

$c = 0.5413248546, \qquad u(c) = 2.9524924420$

Hence, we have $u(x) = e^{2x} - 1.11022 \times 10^{-16}$. The error function is demonstrated in Figure 1.

Note that the absolute error is -1.11022×10^{-16} with considering 16 digits and it is equivalent to the exact solution.The comparison between exact solution and approximate solution showned in Figure 2.

Example 2.2 *(Fredholm integral equation with Hankel kernel):*

$$u(x) = -(-2+e)e^x + x^2 + \int_0^1 e^{x+t}u(t)dt$$

for which the exact solution is $u(x) = x^2$.Using the presented method leads to the following system of equations:

$$\begin{cases} u(c) = c^2 + (2-e)e^c + e^{2c}u \\ u(c) = c^2 + (2+e)e^c + \frac{1}{2}e^c \\ (-6 - (-3+e)e(1+e) + e^c(-1+e^2)u) \end{cases}$$

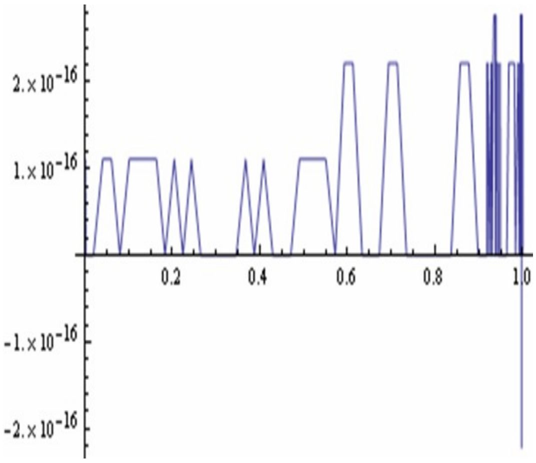

Figure 1: The error function of Example 2.1.

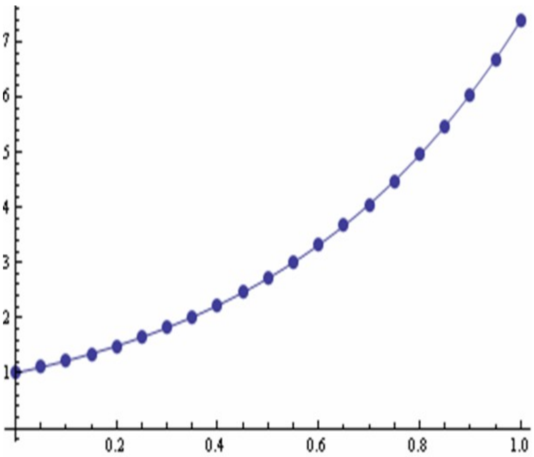

Figure 2: The comparison between exact solution and approximate solution of Example 2.1.

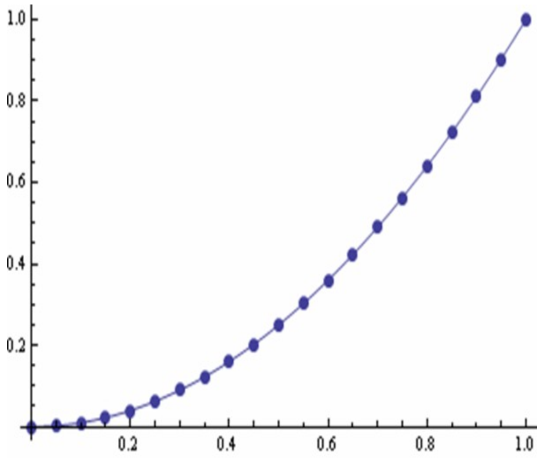

Figure 3: The error function of Example 2.2.

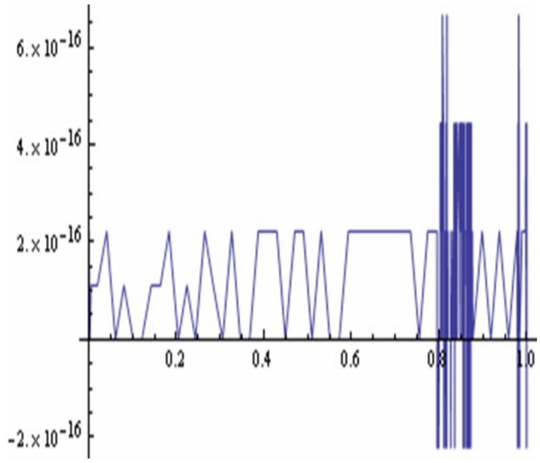

Figure 4: The comparison between exact solution and approximate solution of Example 2.2.

Solving the obtained system leads to
$c = 0.5413248546, \quad u(c) = 2.9524924420$
with considering 10-digit accuracy.Hence, approximate solution is $u(x) = x^2 - 4.12812 \times 10^{-16}$ *and absolute error is* -4.12812×10^{-16}. *In Figure 3. show the comparison between exact solution and approximate solution and The error function is demonstrated in Figure 4.*

Example 2.3 *(Fredholm integral equation with TPH kernel):*

$$\begin{aligned} u(x) &= x^3 - 2(3 + \cos(1) - 4\sin(1)) \\ &\times (\cos(x) + \sin(x)) \\ &+ \int_0^1 [\sin(x+t) + \cos(x-t)]u(t)dt \end{aligned}$$

for which the exact solution is $u(x) = x^3$.*and system of equations is:*

$$\begin{cases} u(c) = c^3 - 2(3 + \cos(1) - 4\sin(1)) \\ (\cos(c) + \sin(c)) \\ -1/2(\cos(c) + \sin(c)) \\ (4(3 + \cos(1) - 4\sin(1))\sin(1)^2 \\ +u(-3 + \cos(2))(\cos(c) + \sin(c))) \\ \\ u(c) = c^3 - 2(3 + \cos(1) - 4\sin(1)) \\ (\cos(c) + \sin(c)) \\ +(\sin(c+c) + \cos(c-c))u(c) \end{cases}$$

By solving of the related system, we have

$$c = 0.6296960723676109$$

and

$$u(c) = 0.6296960723676109.$$

Hence, approximate solution is $u(x) = x^3 + 2.22045 \times 10^{-16}$ *and absolute error is* $2.22045 \times$

10^{-16}. *Figure5 is comparison between exact and approximate solution and The error function is demonstrated in Figure 6.*

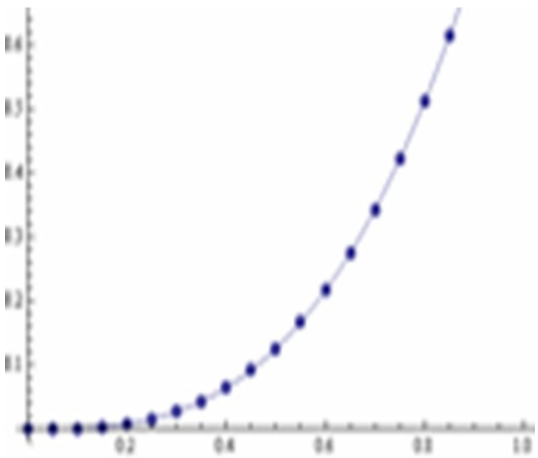

Figure 5: The error function of Example 2.3.

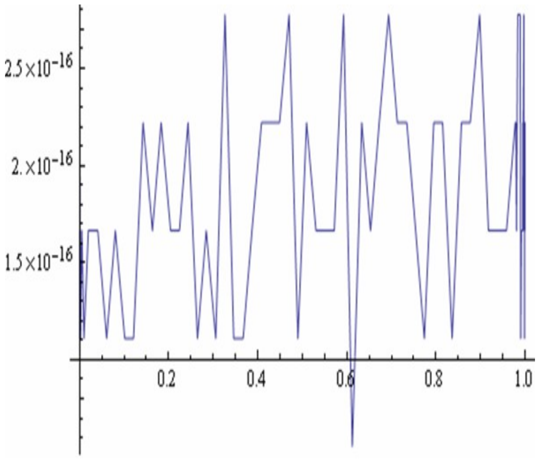

Figure 6: The comparison between exact solution and approximate solution of Example 2.3.

3 Solving of two dimensional Fredholm integral equation with TPH kernel via IMVM

Consider the following two dimensional Fredholm integral equation of the second kind:

$$u(x,y) = f(x,y) + \lambda \int_a^b \int_c^d (P[(x,y)-(s,t)] + Q[(x,y)+(s,t)]F(u(s,t))dsdt \tag{3.7}$$

For solving the above equation, we apply the integral mean value theorem. However, the mean value theorem is valid for double integrals, we apply one dimensional integral mean value theorem directly to fulfill required linearly independent equations.

Corollary 3.1 *(mean value theorem for integrals). If $w(x,y)$ is continuous in $[a,b]\times[c,d]$, then there are points $c_1 \in [a,b]$ and $c_2 \in [c,d]$, such that*

$$\int_a^b w(s,t)ds = (b-a)w(c_1,t) \tag{3.8}$$

and

$$\int_c^d w(s,t)dt = (d-c)w(s,c_2) \tag{3.9}$$

Proof:*It is clear by using(2.2)*

Theorem 3.1 *(mean value theorem for double integrals). If $w(x,y)$ is continuous in $[a,b]\times[c,d]$, then there are points $c_1 \in [a,b]$ and $c_2 \in [c,d]$, such that*

$$\int_a^b \int_c^d w(s,t)dsdt = (b-a)(d-c)w(c_1,c_2). \tag{3.10}$$

Proof: *It is clear using Theorem (2.1).*

By applying (3.8) and (3.9) for the right hand of (3.7), since the integral equation (3.7) depends on x and y, c_1 and c_2 will be functions with respect to x and y and here we write them as $c_1(x;y)) \in [a,b]$ and $c_2(x;y)) \in [c,d]$. To be able to implement our algorithm, we take $c_1(x;y))$ and $c_2(x;y))$ as constants. Now to find the solution of integral equation we describe the following algorithm:

Algorithm

1. Apply (3.9) in (3.7) to get

$$u(x,y) = f(x,y) + \lambda(d-c)\int_a^b (P[(x,y) - (s,c_2)] + Q[(x,y)+(s,c_2)])F(u(s,c_2)))ds \tag{3.11}$$

2. Apply (3.8)in (3.11)and Replace the obtained equation in 3.11 as

$$u(x,y) = f(x,y) + \lambda(b-a)(d-c) (P[(x,y)-(c_1,c_2)] + Q[(x,y)+(c_1,c_2)]) F(u(c_1,c_2)) \tag{3.12}$$

3. Let $(x, y) = (c_1, c_2)$in the (3.12). It is obtained as

$$u(c_1, c_2) = f(c_1, c_2) + \lambda(b-a)(d-c) \\ (P[(c_1, c_2) - (c_1, c_2)] + Q[(c_1, c_2) \\ +(c_1, c_2)])F(u(c_1, c_2)) \tag{3.13}$$

4. Replace (3.12) into (3.7) and then let$(x, y) = (c_1, c_2)$ in the obtained formula.

$$u(c_1, c_2) = f(c_1, c_2) + \lambda \int_a^b \int_c^d (P[(c_1, c_2) \\ -(s,t)] + Q[(c_1, c_2) + (s,t)])F(f(s,t) \\ +\lambda(b-a)(d-c)(P[(s,t) - (c_1, c_2)] \\ +Q[(s,t) + (c_1, c_2)])F(u(c_1, c_2)))dsdt \tag{3.14}$$

5. Substitute (3.12) into (3.11). We have

$$u(x, y) = f(x, y) + \lambda(d-c) \int_a^b (P[(x, y) \\ -(s, c_2)] + Q[(x, y) + (s, c_2)]F(f(s, c_2) \\ +\lambda(b-a)(d-c)(P[(s, c_2) - (c_1, c_2)] \\ +Q[(s, c_2) + (c_1, c_2)])F(u(c_1, c_2)))) \; ds \tag{3.15}$$

6. Let$(x, y) = (c_1, c_2)$ to be in the above equation

$$u(c_1, c_2) = f(c_1, c_2) + \lambda(d-c) \int_a^b (P[(c_1, c_2) \\ -(s, c_2)] + Q[(c_1, c_2) + (s, c_2)])F(f(s, c_2) \\ +\lambda(b-a)(d-c)(P[(s, c_2) - (c_1, c_2)] \\ +Q[(s, c_2) + (c_1, c_2)])F(u(c_1, c_2))) \; ds \tag{3.16}$$

7. Solve the Eqs. (3.13), (3.14) and (3.16) simultaneously as the system including 3 equations and 3 unknowns $c_1; c_2$ and $u(c_1, c_2)$.

4 Solving of high dimensional integral equations via IMVM

Consider the second kind high dimensional integral equation

$$u(\mathbf{x}) = f(\mathbf{x}) + \lambda \int_{a_1}^{b_1} \int_{a_2}^{b_2} ... \int_{a_n}^{b_n} [P(\mathbf{x} - \mathbf{t}) \\ +Q(\mathbf{x} + \mathbf{t})]F(u(\mathbf{t})) \; d\mathbf{t} \tag{4.17}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{t} = (t_1, t_2, ..., t_n)$. Similar to the previous section, instead of using mean value theorem for multiple integral, we apply one dimensional integral mean value theorem directly to provide needful linearly independent equations.

Theorem 4.1 *(mean value theorem for multiple integrals): If $s(\mathbf{x})$ is continuous in $[a_i, b_i]^n, i = 1, 2, ..., n$, then there are points $c_i \in [a_i, b_i], i = 1, 2, ..., n$ such that*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} ... \int_{a_n}^{b_n} (s(\mathbf{x})) \; dx_n \; dx_{n-1} \; ... \; dx_1 \\ = \prod_{j=0}^{n-1} (b_{n-j} - a_{n-j}) s(c_1, c_2, ..., c_n) \tag{4.18}$$

In the similar way, to find $u(x_1, x_2, ..., x_n)$, we have to obtain $c_1, c_2, ..., c_n$ and $u(c_1, c_2, ..., c_n)$. Follow the consecutive substituting in the following algorithm which lead to the system including $(n+1)$ unknowns and $(n+1)$ linearly independent equations.

Algorithm

1. Apply the integral mean value theorem for interval$[a_n, b_n]$:

$$\int_{a_n}^{b_n} [p(\mathbf{x} - \mathbf{t}) + q(\mathbf{x} + \mathbf{t})]F(u(\mathbf{t}))dt_n \\ = (b_n - a_n)[p(\mathbf{x} - \xi_n) + q(\mathbf{x} + \xi_n)]F(u(\xi_n)) \tag{4.19}$$

   where $\xi_n = (t_1, t_2, ..., t_{n-1}, c_n)$.

2. Substitute (4.19) into (4.17) to obtain :

$$u(\mathbf{x}) = \\ f(\mathbf{x}) + \lambda(b_n - a_n) \int_{a_1}^{b_1} ... \int_{a_{n-1}}^{b_{n-1}} [p(\mathbf{x} - \xi_n) \\ +q(\mathbf{x} + \xi_n)]F(u(\xi_n))dt_{n-1} \cdots dt_1 \tag{4.20}$$

3. Again, we use the integral mean value theorem with $i = n - 1$,as follows

$$\int_{a_{n-1}}^{b_{n-1}} [p(\mathbf{x} - \xi_n) + q(\mathbf{x} + \xi_n)]F(u(\xi_n))dt_{n-1} \\ = (b_{n-1} - a_{n-1})[p(\mathbf{x} - \xi_{n-1}) + q(\mathbf{x} \\ +\xi_{n-1})]F(u(\xi_{n-1})) \tag{4.21}$$

   where $\xi_{n-1} = (t_1, t_2, ..., c_{n-1}, c_n)$.

4. Replace the above equation into (4.20). So, we have

$$u(\mathbf{x}) = f(\mathbf{x}) + \lambda(b_n - a_n)(b_{n-1} - a_{n-1}) \\ \int_{a_1}^{b_1} \cdots \int_{a_{n-2}}^{b_{n-2}} [p(\mathbf{x} - \xi_{n-1}) \\ +q(\mathbf{x} + \xi_{n-1})]F(u(\xi_{n-1}))dt_{n-2} \cdots dt_1 \tag{4.22}$$

5. Repeat the process for $i = 3, ..., n-1$, as

$$u(\mathbf{x}) = f(\mathbf{x}) + \lambda(b_n - a_n) \cdots (b_{n-i+1} \\ -a_{n-i+1}) \int_{a_1}^{b_1} \cdots \int_{a_{n-i}}^{b_{n-i}} [p(\mathbf{x} - \xi_{n-i+1}) \\ +q(\mathbf{x} + \xi_{n-i+1})]F(u(\xi_{n-i+1}))dt_{n-i} \cdots dt_1 \tag{4.23}$$

where $\xi_{n-i+1} = (t_1, t_2, ..., t_{n-i}, c_{n-i+1}, c_{n-i+2}, ..., c_n)$. Also, the nth step of the above process leads to

$$\begin{aligned} u(\mathbf{x}) &= f(\mathbf{x}) + \lambda(b_n - a_n)\cdots(b_1 - a_1) \\ &\quad [p(\mathbf{x} - \xi_1) + q(\mathbf{x} + \xi_1)]F(u(\xi_1)) \end{aligned} \tag{4.24}$$

where $\xi_1 = (c_1, c_2, ..., c_n)$.

6. Let $\mathbf{x} = \xi_1$ into 4.24 to get

$$\begin{aligned} u(\xi_1) &= f(\xi_1) + \lambda \prod_{j=1}^{n}(b_j - a_j)[p(0) \\ &\quad + q(2\xi_1)]F(u\xi_1)) \end{aligned} \tag{4.25}$$

Now the first equation of the proposed system including $(n+1)$unknowns and $(n+1)$ equations is constructed.

7. To make the other equations, the demonstrated process must be repeated such that the ith achieved equation is as follows (use (4.23) and (4.24))

$$u(\mathbf{x}) =$$

$$\begin{aligned} & f(\mathbf{x}) + \lambda \textstyle\prod_{j=0}^{i-1}(b_{n-j} - a_{n-j}) \int_{a_1}^{b_1} \cdots \int_{a_{n-i}}^{b_{n-i}} \\ & [p(\mathbf{x} - \xi_{n-i+1}) + q(\mathbf{x} + \xi_{n-i+1})] \\ & F([f(\xi_{n-i+1}) + \lambda \textstyle\prod_{j=0}^{i-1}(b_{n-j} - a_{n-j}) \\ & [p(0) + q(2\xi_{n-i+1})]F(u(\xi_1))])dt_{n-i}\cdots dt_1 \end{aligned} \tag{4.26}$$

where $i = 1, ..., n-1$, and gives

$$u(\xi_1) =$$

$$\begin{aligned} & f(\xi_1) + \lambda \textstyle\prod_{j=0}^{i-1}(b_{n-j} - a_{n-j}) \int_{a_1}^{b_1} ... \int_{a_{n-i}}^{b_{n-i}} \\ & [p(\xi_1 - \xi_{n-i+1}) + q(\xi_1 + \xi_{n-i+1})] \\ & F([f(\xi_{n-i+1}) + \lambda \textstyle\prod_{j=0}^{i-1}(b_{n-j} - a_{n-j}) \\ & [p(\xi_{n-i+1} - \xi_1) + q(\xi_{n-i+1} + \xi_1)] \\ & F(u(\xi_1))])dt_{n-i}\cdots dt_1 \end{aligned} \tag{4.27}$$

as the ith equation. Therefore, we obtain the $(n-1)$, new equations when $i = 1, ..., n-1$, for final mentioned system including $(n+1)$unknowns and $(n+1)$ equations. Also, 4.25 can be provided 4.27 with $i = n$.

8. Implement the previous step to obtain the nth equation as

$$\begin{aligned} u(\mathbf{x}) &= f(\mathbf{x}) + \lambda \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \\ &\quad [P(\mathbf{x} - \mathbf{t}) + q(\mathbf{x} + \mathbf{t})] \\ &\quad F(f(\mathbf{t}) + \lambda \prod_{j=0}^{n-1}(b_{n-j} - a_{n-j}) \\ &\quad [p(\mathbf{t} - \xi_1) + q(\mathbf{t} + \xi_1)] \\ &\quad F(u(\xi_1)))dt_n\cdots dt_1 \end{aligned} \tag{4.28}$$

which lead to

$$\begin{aligned} u(\xi_1) &= f(\xi_1) + \lambda \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \\ &\quad [P(\xi_1 - \mathbf{t}) + q(\xi_1 + \mathbf{t})] \\ &\quad F(f(\mathbf{t}) + \lambda \prod_{j=0}^{n-1}(b_{n-j} - a_{n-j}) \\ &\quad [p(\mathbf{t} - \xi_1) + q(\mathbf{t} + \xi_1)] \\ &\quad F(u(\xi_1)))dt_n\cdots\ dt_1 \end{aligned} \tag{4.29}$$

Finally, the last equation of the proposed system including $(n+1)$ unknowns and $(n+1)$ equations is constructed.

9. Solve the nonlinear system including (4.25), (4.27) and (4.29) with the Newtons method or other efficient methods.

Remark 4.1 The equations of the final system include the numerous integrals containing long terms. It is recommended to use the numerical integration rules such as Gauss quadrature rule or trapezoidal integration method.

Some different examples of two dimensional Fredholm integral equation with Toeplitz ,Hankel and Toeplitz plus Hankel kernel are chosen to illustrate the presented method.

Example 4.1 *(Two dimensional Fredholm integral equation with Hankel kernel) Consider the integral equations:*

$$u(x, y) = f(x, y) + \int_0^1 \int_0^1 2^{x+y+s+t}u(s, t)\ ds\ dt$$

where

$$f(x, y) = x + y - \frac{2^{x+y}(-2 + \log(16))}{\log(2)^3}$$

for which the exact solution is $u(x,y) = x + y$. Using the presented method leads to the following system of equations:

$$u(c_1,c_2) =$$

$$c_1 + c_2 + \frac{1}{\log(2)^4}\, 2^{-1+c_1+c_2}$$
$$(2c_2\log(2)^3 + 3\times 2^{c_1+2c_2}u\log(2)^3$$
$$+2(-3\times 2^{c_2} + \log(2)^2)(-1+\log(4)))$$
$$-\frac{2^{c_1+c_2}(-2+\log(16))}{\log(2)^3}$$

$$u(c_1,c_2) =$$

$$c_1 + c_2 - \frac{2^{c_1+c_2}(-2+\log(16))}{\log(2)^3}$$
$$+\frac{1}{\log(2)^5}2^{-2+c_1+c_2}(9\times 2^{c_1+c_2}u(c_1,c_2)\log(2)^3$$
$$+(-2+\log(16))(-9+\log(2)\log(16)))$$

$$u(c_1,c_2) = \quad c_1 + c_2 - \frac{2^{c_1+c_2}(-2+\log(16))}{\log(2)^3} + 2^{c_1+c_2+c_1+c_2}u(c_1,c_2)$$

By mentioned method, values of c_1, c_2 and $u(c_1,c_2)$ are found as

$$c_1 = 0.5449840956,\ c_2 = 0.5449055202$$

and

$$u(c_1,c_2) = 2.9524924420$$

. Hence, $u(x,y) = x + y + 3.55271\times 10^{-15}$. The error function is demonstrated in Figure 7. Note that the absolute error is 3.55271×10^{-15} with considering 15 digits and it is equivalent to the exact solution.The comparison between exact solution and approximate solution showned in Figure 8.

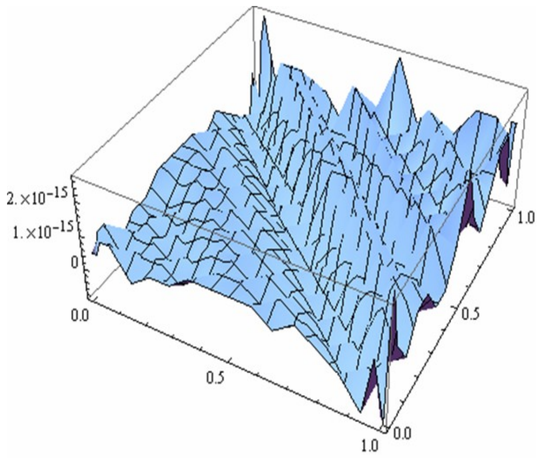

Figure 7: The error function of Example .

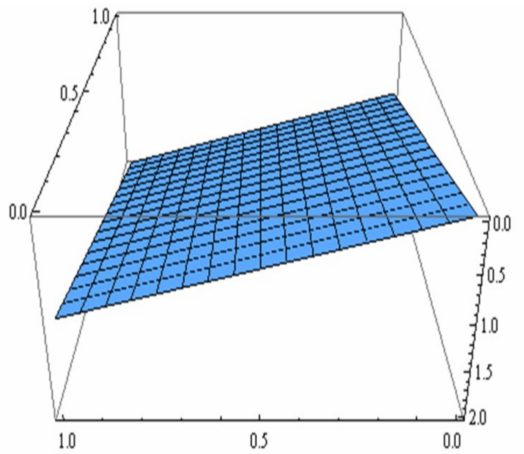

Figure 8: The comparison between exact solution and approximate solution of Example .

Example 4.2 *(Two dimensional Fredholm integral equation with Toeplitz kernel)*

$$u(x,y) = f(x,y) + \int_0^1\int_0^1 e^{(2x+2y)-(s+t)}u(s,t)\,ds\,dt$$

where

$$f(x,y) = x - y$$

for which the exact solution is $u(x,y) = x - y$. Using the presented method leads to the following system of equations:

$$u(c_1,c_2) = c_1 - c_2 + e^{-1+c_1+c_2}(e^{c_1}(-2+c_2+e-c_2e) + (-1+e)e^{1+c_2}u(c_1,c_2)),$$

$$u(c_1,c_2) = c_1 - c_2 + (-1+e)^2e^{c_1+c_2}u(c_1,c_2),$$

$$u(c_1,c_2) = c_1 - c_2 + e^{(2c_1+2c_2)-(c_1+c_2)}u(c_1,c_2).$$

Solving the obtained system leads to

$$c_1 = 0.4180232931, \qquad c_2 = 0.4180232931$$

and

$$u(c_1, c_2) = -2.9338723141 \times 10^{-27}$$

with considering 10-digit accuracy.Hence, approximate solution is $u(x, y) = x - y + 6.94271 \times 10^{-26}$ *and absolute error is* 6.94271×10^{-26}.*In Figure 9. show he comparison between exact solution and approximate solution and The error function is demonstrated in Figure 10.*

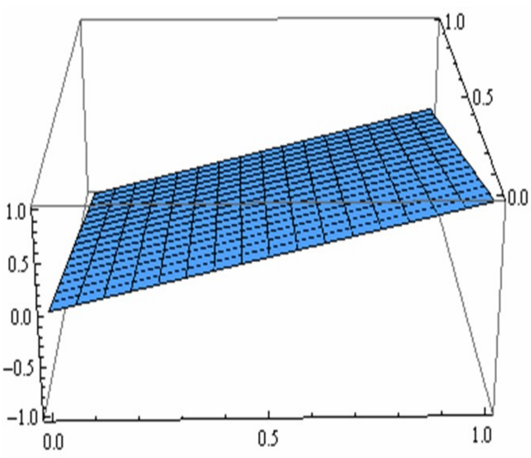

Figure 9: The error function of Example 4.2.

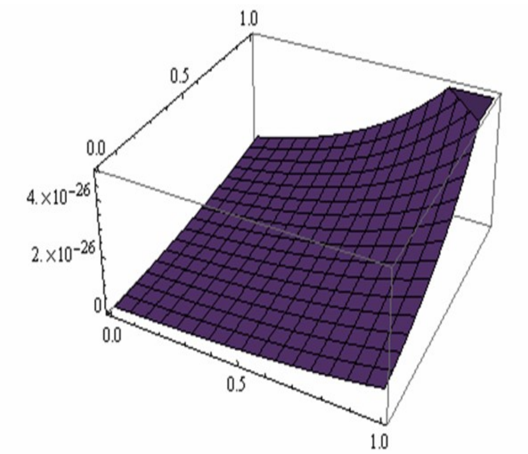

Figure 10: The comparison between exact solution and approximate solution of Example 4.2.

Example 4.3 *(Two dimensional Fredholm integral equation with TPH kernel)*

$$u(x, y) = f(x, y) + \int_0^1 \int_0^1 (e^{(2x+2y)-(s+t)} + e^{(x+y)+(s+t)}\ u(s, t)\ ds\ dt$$

where

$$f(x, y) = x^2 - y^2$$

for which the exact solution is $u(x, y) = x^2 - y^2$.*and system of equations is:*

$$\begin{aligned} u(c_1, c_2) &= c_1^2 - c_2^2 + \frac{1}{6}e^{-1+c_2} \\ &(-6(5 + c_2^2(-1 + e) - 2e)e^{2c_1} \\ &+3e^{1+2c_1+3c_2}(-1 + e^2)u(c_1, c_2) + 2e^{1+2c_2} \\ &(-1 + e^3 + 3e3c_1)u(c_1, c_2) \\ &-6e^{1+c_1+c_2}(2 + c_2^2(-1 + e) \\ &+u(c_1, c_2) - e(1 + u(c_1, c_2)))), \end{aligned}$$

$$\begin{aligned} u(c_1, c_2) &= c_1^2 - c_2^2 + \frac{1}{36}(36(-1 + e)^2 \\ &e^{c_1+c_2} + 36e^{3(c_1+c_2)} + 9e^2 \\ &(c_1 + c_2)(-1 + e^2)^2 + 4 \\ &(-1 + e^3)^2)u(c_1, c_2), \end{aligned}$$

$$\begin{aligned} u(c_1, c_2) &= c_1^2 - c_2^2 + (e^{(2c_1+2c_2)-(c_1+c_2)} \\ &+e^{c_1+c_2+c_1+c_2})u(c_1, c_2). \end{aligned}$$

By solving of the related system, we have

$$c_1 = 0.6114976933,\ \ c_2 = 0.6114976933$$

and

$$u(c_1, c_2) = 5.6991080150 \times 10^{-21}$$

Hence, approximate solution is $u(x, y) = (x^2 - y^2) - 2.34655 \times 10^{-19}$ *and absolute error is* -2.34655×10^{-19}. *Figure 11. is comparison between exact and approximate solutions and The error function is demonstrated in Figure 12.*

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5 Conclusions

In this paper, we used the new efficient method for solving linear and nonlinear Fredholm integral equation of the second kind (different dimensional problems)with Toeplitz , Hankel and

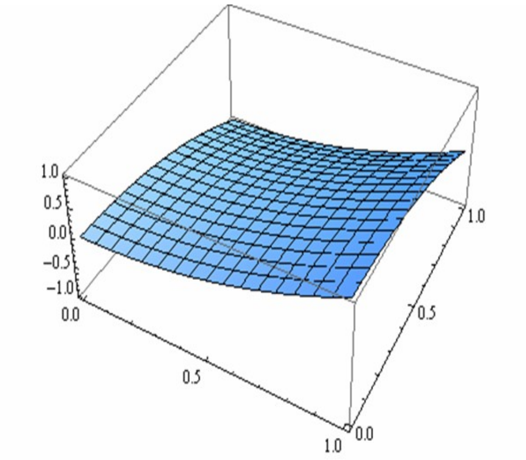

Figure 11: The error function of Example 4.3.

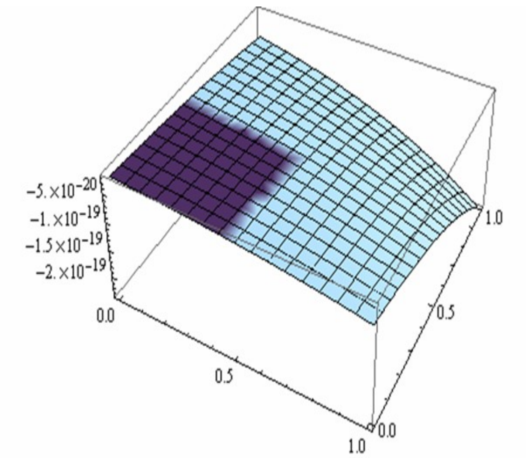

Figure 12: The comparison between exact solution and approximate solution of Example 4.3.

Toeplitz plus Hankel kernel.We exhibit the applied main idea that it just is the famous integral mean value theorem. Exampels that were presented show the ability of the model. The results confirm that the method is very effective and simple.

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Nasser Mikaeilvand is Assistance Professor of applied mathematics at Ardabil branch, Islamic Azad University, Ardabil, Iran. Interested in numer- ical solutions of Differential and Integral Equations, numerical linear algebra and Fuzzy mathematics.

Samad Noeiaghdam is currently PH.D. student in Applied Mathematics at Central Tehran branch, Islamic Azad University, Tehran, Iran. His research interests are in the fields of Integral Equations, Stochastic Arithmetic and solving ill-posed problems.