Science and Research Branch (IAU)

Application of the exact operational matrices for solving the Emden-Fowler equations, arising in Astrophysics

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Abstract

The objective of this paper is applying the well-known exact operational matrices (EOMs) idea for solving the Emden-Fowler equations, illustrating the superiority of EOMs over ordinary operational matrices (OOMs). Up to now, a few studies have been conducted on EOMs ; but the solved differential equations did not have high-degree nonlinearity and the reported results could not strongly show the excellence of this new method. So, we chose Emden-Fowler type differential equations and solved them utilizing this method. To confirm the accuracy of the new method and to show the preeminence of EOMs over OOMs, the norm 1 of the residual and error function for both methods are evaluated for multiple m values, where m is the degree of the Bernstein polynomials. We report the results by some plots to illustrate the error convergence of both methods to zero and also to show the primacy of the new method versus OOMs. The obtained results demonstrate the increased accuracy of the new method.

Keywords : Exact operational matrices; Bernstein polynomials; Emden-Fowler equation; Lane-Emden equation.

1 Introduction

In 1915, Galerkin [22] introduced a broad generalization of the Ritz [60] method to be used primarily for the approximate solution of variational and boundary value problems, including problems that cannot be reduced to variational problems [50]. His method was highly appreci-

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ated and thousands of problems have been solved, using the Galerkin method, since 1915.

For understanding the basic idea behind the Galerkin method, suppose that we have to find a solution (defined in domain U) for the following (nonlinear) differential equation which does not have any exact analytical solution

$$\mathcal{N}[u(x)] = 0, \quad x \in U, \quad (1.1)$$
$$B(u(s)) = 0, \quad s \in \partial U,$$

where the solution, at the boundary $B(u(s))$ of U, satisfies the homogeneous boundary conditions. Now, suppose $y_m(x)$ as an approximation for $u(x)$, which is made by the so-called trial functions $\beta_i(x)$

$$u(x) \approx y_m(x) = \sum_{i=0}^{\infty} c_i \beta_i(x), \quad (1.2)$$

$$B(y_m(s)) = 0, \quad s \in \partial U. \quad (1.3)$$

As we mentioned, $\mathcal{N}[u(x)]$ does not have any exact analytical solution; so, we are sure that $y(x)$ cannot satisfy the equation (1.1). Therefore, we attempt to minimize the residual function

$$R_m(x) = \mathcal{N}[y_m(x)] \neq 0. \qquad (1.4)$$

The main idea of the Galerkin method for minimizing the residual function is finding the coefficients c_i so that the following weighted means of the residual function becomes zero

$$(R_m, \beta_j(x))_\omega = \int_U R_m \beta_j(x) \omega(x) dx = 0, \ (1.5)$$
$$x \in U,$$

where $\omega(x)$ is a weight function.

However, an old and efficient technique for simplifying the computations in the residual function is using the operational matrices. In 1975, Chen and Hsiao [15] introduced an operational matrix to perform integration of Walsh functions. Chen [16] continued his work to introduce some operational matrices to do fractional calculus operations. In 1977, Sannuti et al. introduced operational matrix of integration for Block-Pulse basis functions. Based on Mouroutsos [62] statements, these studies continued at that time with the determination of integrating operational matrices for miscellaneous basis functions like the Laguerre, the Legendre, the Chebyshev and the Fourier trial functions. In 1988-1989, Razzaghi et al. [55, 57, 56] presented the integral and product operational matrices of Fourier series, time function Taylor series, and shifted-Jacobi series. In 1993, Bolek [9] presented a direct method for deriving an operational matrix of differentiation for Legendre polynomials. In 2000-2012, Yousefi et al. [58, 30, 34, 73, 72, 74], presented Legendre wavelets and Bernstein operational matrices for solving the variational problems and differential equations. In 2012, Kazem et al. [32], presented a general formulation for d-dimensional orthogonal functions, and their derivative and product matrices was presented. In 2013, the authors of [63] presented a modified form of the homotopy analysis method, based on Chebyshev operational matrices. In 2013, Kazem [31] derived the Jacobi integral operational matrix for the fractional calculus. In 2013-14, Toutounian and Tohidi et al. [66][67] introduced the derivative and integration matrices of Bernoulli basis function. In 2014, Saadatmandi [61] proposed Bernstein operational matrix for fractional derivative of order

in the Caputo sense. In 2015, Borhanifar [10] et al. proposed shifted Jacobi operational matrix of derivative, utilizing spectral tau, and collocation methods.

The Galerkin method can be implemented for solving differential equations by low-cost computations. However, we implement computations of the (1.4) residual functions by means of the corresponding operational matrices. But, the customary idea for derivation of the operational matrices does not guarantee the exactitude of the performed operations. For a more detailed description, suppose Θ_m to be the set of the base functions and the known functions $f(x)$ and $g(x)$ to be

$$\Theta_m = \{\beta_1(x), \beta_2(x), \cdots, \beta_m(x)\},$$
$$f(x) = \sum_{i=1}^{m} \kappa_i \beta_i(x) = k^T b_m(x),$$
$$g(x) = \sum_{i=1}^{m} \lambda_i \beta_i(x) = l^T b_m(x),$$

where

$$b_m(x) = \begin{bmatrix} \beta_1(x) & \beta_2(x) & \cdots & \beta_m(x) \end{bmatrix}^T,$$
$$k = \begin{bmatrix} \kappa_1(x) & \kappa_2(x) & \cdots & \kappa_m(x) \end{bmatrix}^T,$$
$$l = \begin{bmatrix} \lambda_1(x) & \lambda_2(x) & \cdots & \lambda_m(x) \end{bmatrix}^T.$$

Ordinary operational matrices (OOMs) employ the differentiation, integration and product of the basis vector(s) as

$$\int_0^x f(x)dx = k^T \int_0^x b_m(x)$$
$$\doteq k^T P_1 b_m(x),$$
$$\frac{d}{dx} f(x) = k^T \frac{d}{dx} b_m(x) \doteq k^T D_1 b_m(x),$$
$$f(x)g(x) = (k^T b_m(x))(b_m(x)^T l)$$
$$\doteq k^T \widehat{L_1} b_m(x),$$

where P_1, D_1 and $\widehat{L_1}$ are integration, differentiation and product matrices related to the basis Θ_m and the symbol ($\doteq$) stands for definition. Although both of the functions $f(x)$ and the $g(x)$ are located in the $Span(\Theta_m)$, it is quite probable that their integration or product do not remain in that space. In other words, operational matrices do not guarantee exactitude of the operations.

Recently, Parand et al. [43], have presented a solution for this problem. They introduced exact

operational matrices (EOMs) P_2, D_2 and $\widehat{L_2}$ so that

$$\int_0^x f(x)dx = k^T \int_0^x b_m(x)$$
$$= k^T P_2 b_{n_1}(x),$$
$$\frac{d}{dx}f(x) = k^T \frac{d}{dx}b_m(x) = k^T D_2 b_{n_2}(x),$$
$$f(x)g(x) = (k^T b_m(x))(b_m(x)^T l)$$
$$= k^T \widehat{L_2} b_{n_3}(x).$$

As it can be seen, all of the approximations have been removed and also the basis vector has been changed. $b_{n_i}(x)$ depends on the $b_m(x)$ and the respective operational matrix.

Parand et al. [43] implemented their idea to introduce exact operational matrices of the Bernstein polynomials and solved some simple differential equations by them. However, the solved problems did not show the real potential of the introduced method in improving the answer accuracy. The potency of the new method becomes clear, only if it solves nonlinear problems with high-degree nonlinearity. In this paper, we solve the Emden-Fowler type differential equations by the Bernstein polynomials to present a stronger proof for EOMs performance.

In section 2, we briefly introduce the Emden-Fowler equations. Section 3 presents a brief introduction to the EOMs introduced by [43]; EOMs like differentiation matrix D, integration matrix P, product matrix $\widehat{C}$, and the Galerkin matrix Q. The section, also, introduces the new "series operational matrices" for achieving the best approximation of $g(y(x))$ by the Bernstein polynomials, where $g(x)$ is a given function and $y(x)$ is the unknown function of the differential equation. At the end of the section, a summary of the solution error analysis proposed in [43] is presented, when the solutions are approximated by the Bernstein polynomials. In section 4, 7 Emden-Fowler type equations are solved by the EOM approach. Also, the results are compared with the results of OOM approach to prove the validity and applicability of EOMs and to show their superiority over OOMs. Finally, section 5, provides a conclusion, alongside some new suggestions for further studies.

2 Emden-Fowler equations

2.1 Introduction to the equations

Several problems in mathematical physics and astrophysics, occurring on semi-infinite interval are related to the diffusion of heat perpendicular to the parallel planes. They can be modeled by the heat equation [21, 14, 26]:

$$x^{-k}\frac{d}{dx}\left(x^k \frac{d}{dx}\right) + f(x)g(y(x)) = h(x) ,$$
$$x > 0, \ k > 0 ,$$

or equivalently

$$y''(x) + \frac{k}{x}y'(x) + f(x)g(y(x)) = h(x) ,$$
$$x > 0, \ k > 0 ,$$

where $y(x)$ represents the temperature. For the steady-state case and when $k = 2$ and $h(x) = 0$, this equation is called generalized Emden-Fowler equation [21, 14, 26, 13, 18, 59]

$$y''(x) + \frac{2}{x}y'(x) + f(x)g(y(x)) = 0 ,$$
$$x > 0 , \tag{2.6}$$

subject to the conditions

$$y(0) = a , \qquad y'(0) = b ,$$

where $f(x)$ and $g(y(x))$ are two given functions.

By setting $f(x) = 1$, Eq. (2.6) will be reduced to the Lane-Emden equation by which several phenomena in mathematical physics and astrophysics are modeled (for different $g(y(x))$ values). It is used in theory of stellar structure, theory of thermionic currents, modeling the thermal behavior of a spherical cloud of gas, modeling isothermal gas sphere, and so on.

$$y''(x) + \frac{2}{x}y'(x) + g(y(x)) = 0 ,$$
$$x > 0 , \tag{2.7}$$
$$y(0) = a, \ y'(0) = b .$$

For $g(y(x)) = y^p(x)$, $a = 1$ and $b = 0$, the equation (2.7) yields the standard Lane-Emden equation which is used to model thermal behavior of a spherical cloud of gas, acting under the mutual attraction of its molecules, and subject to the classical laws of thermodynamics [13, 2, 1].

$$y''(x) + \frac{2}{x}y'(x) + y^p(x) = 0 ,$$
$$x > 0 , \tag{2.8}$$
$$y(0) = 1 , \ y'(0) = 0 ,$$

where $p > 0$ is a constant. By substituting the p value by $0, 1$, and 5 in Eq. (2.8), the following equations will be, respectively, the exact solutions of $y(x)$

$$y(x) = 1 - \frac{1}{3!}x^2 , \qquad y(x) = \frac{\sin{(x)}}{x} ,$$

$$y(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}} . \qquad (2.9)$$

For other p values, there is no analytical exact solution for the standard Lane-Emden equation.

In this paper, we are going to solve the following Emden-Fowler equations whose $f(x)$ and $g(y(x))$ functions are given as

$$f(x) = 1 , \qquad g(y(x)) = y^p(x) , \qquad p \in \mathbb{N} ,$$
$$[a, \, b] = [1, \, 0] , \qquad (2.10)$$

$$f(x) = 1 , \qquad g(y(x)) = y^p(x) , \qquad p \notin \mathbb{N} ,$$
$$[a, \, b] = [1, \, 0] , \qquad (2.11)$$

$$f(x) = 1 , \qquad g(y(x)) = e^{y(x)} ,$$
$$[a, \, b] = [0, \, 0] , \qquad (2.12)$$

$$f(x) = 1 , \qquad g(y(x)) = \sinh{(y(x))} ,$$
$$[a, \, b] = [1, \, 0] , \qquad (2.13)$$

$$f(x) = 1 , \qquad g(y(x)) = \sin{(y(x))} ,$$
$$[a, \, b] = [1, \, 0] , \qquad (2.14)$$

$$f(x) = 1 , \qquad g(y(x)) = 4\left(2e^{y(x)} + e^{\frac{y(x)}{2}}\right) ,$$
$$[a, \, b] = [0, \, 0] , \qquad (2.15)$$

$$f(x) = -2\left(2x^2 + 3\right) , \qquad g(y(x)) = y(x) ,$$
$$[a, \, b] = [1, \, 0] . \qquad (2.16)$$

It is notable that except the 3 above-mentioned equations and the equations (2.15) and (2.16) for which the exact analytical solutions are respectively

$$y(x) = -2\ln{(1 + x^2)}, \qquad (2.17)$$

$$y(x) = e^{x^2}, \qquad (2.18)$$

none of the above-listed equations has analytical exact solution.

Several researchers have studied these equations. In 1989, Bender [4] presented a perturbative method to solve some nonlinear differential equations like the Lane Emden equation. In 1993, Shawagfeh[64] used an approximate analytical approach for solving this equation, by employing the Adomian Decomposition Method (ADM). In 2001, Mandelzweig et al [38] used the quasi-linearization approach for the solution of this equation. To conquer the difficulty of the singular point, Wazwaz [68] applied ADM with an alternative framework and solved the above-mentioned equations. After that, he applied the [69] modified decomposition method for analytic

treatment of such equations to speed up the quick convergence of the series and reduce the work size and present the solution after few iterations, without requiring any Adomian polynomial. In 2003, Liao [35] presented an analytic algorithm for the above equations which logically contained ADM. In that research, unlike its previous analytical techniques, the algorithm itself presented a convenient way to adjust convergence regions for the researcher, even without Padé technique. Using the method of semi-inverse, J. He [27] reached a variational principle for the above equations. it gave much convenience in numerical computations by applying finite element or the Ritz methods. In 2004, Razzaghi and Parand [47] provided a numerical technique on the basis of a rational Legendre Tau method to solve higher ODEs such as the above equations. In that study, the derivative and product operational matrices of the Legendre functions together with the Tau method were applied for reducing the solution of some physical problems to the solution of an algebraic equations system. Ramos, in his studies [52, 54, 53], solved the equations applying different methods. He presented linearization methods for singular IVPs, in second order ODEs such as the above equations. Those methods provided linear constant-coefficient ODEs which were analytically integrable; thus, the methods resulted in piecewise analytical solutions and globally smooth solutions[52]. Then, in [53], the writer provided a series solutions of the above equations on the basis of either a formulation of Volterra integral equation or dependent variable expansion in the main ODE. Moreover, he developed piecewise-adaptive decomposition methods to solve nonlinear ODEs [54]. Those methods presented series solutions in intervals that were subject to continuity conditions at each end point. In 2006, Yousefi [71] converted the above equations to integral equations using integral operator. Then he applied Legendre wavelet approximations. In that study, the Legendre wavelet alongside the Gaussian integration method were applied for reducing the integral equations to some algebraic equations. In 2008, Aslanov [2] presented a recurrence relation for the approximate solution components. Then, he sought for convergence conditions for the Emden-Fowler type equations and improved the final results on the series solution convergence radius. Dehghan

and Shakeri [19] studied the above equations, applying the variational iteration method, and illustrated the effectiveness and the applicability of their procedure in solution of this equation. Their technique did not need discretization, linearization or small perturbations; so, it reduced the computations volume. In 2009, Chowdhury et al. [17] achieved the analytical solutions of the generalized Emden-Fowler in the second order ODEs, by using homotopy-perturbation method (HPM). It was a mixture of the perturbation and the homotopy method. The HPM primary property is deforming difficult problems into a set of easier solvable problems. In 2009, Bataineh et al. [3] gained the analytical solutions of singular IVPs of the above equations, using homotopy analysis method (HAM). Their solutions included an auxiliary parameter providing a simple way to control the series solutions convergence region. They showed that the obtained solutions by ADM and HPM, are only special cases of the solutions of HAM. Parand et al. [49] presented a pseudospectral technique for solving the above equations on a semi-infinite domain. It was based on the rational Legendre functions and Gauss-Radau integration. That method reduced the nonlinear ODE solution to the solution of a system of nonlinear algebraic equations.

Moreover, recently, a great couple of studies have been performed on the above equations; the interested reader is referred to the following papers including the authors' recent studies [45, 29, 23, 42, 41, 48, 46, 33, 24, 25, 40, 11, 39, 28].

3 Bernstein polynomials (B-polynomials)

3.1 Overview of B-polynomials

The Bernstein polynomials (B-polynomials) [74], are widely used polynomials defined on $[0, 1]$. Bernstein polynomials of degree m form a basis for the power polynomials of the same degree [70]. They are continuous over the domain. They satisfy the symmetry

$$B_{i,m}(x) = B_{m-i,m}(1-x),$$

positivity

$$\forall x \in [0, 1], \quad B_{i,m}(x) \geq 0,$$

and normalization or unity of partition [70]

$$\sum_{i=0}^{m} B_{i,m}(x) = 1.$$

Also, $B_{i,m}(x)$ in which $i \notin \{0, m\}$ has a single unique local maximum of

$$i^i m^{-m} (m-i)^{m-i} \binom{m}{i}$$

occurring at $t = \frac{i}{m}$. All of the B-polynomial bases take 0 value at $x = 0$ and $x = 1$, except the first polynomial at $x = 0$ and the last one at $x = 1$, which are equal to 1. It can provide the flexibility, applicable to impose boundary conditions at the end points of the interval.

We are going to present the solutions of the discussed equations, by a linear combination of these polynomials $P(x) = \sum_{i=0}^{m} c_i B_{i,m}(x)$ in which the coefficients c_i are determined using the Galerkin method. In recent years, the B-polynomials have attracted the attention of many researchers. These polynomials have been utilized for solving different equations by taking advantage of various approximate methods. B-polynomials have been used for solving the Fredholm integral equations [12, 37], Volterra integral equations [7], Volterra-Fredholm-Hammerstein integral equations [36], differential equations [74, 65, 5, 8], integro-differential equations [6], parabolic equation subject to specification of mass [73] and so on. Singh et al. [65] and Yousefi et al. [74] have proposed operational matrices in different ways for solving differential equations. In [65], the B-polynomials have been firstly orthonormalized using Gram-Schmidt orthonormalization process and then the operational matrix of integration has been obtained. By the expansion of B-polynomials in terms of Taylor basis, Yousefi and Behroozifar [74] have found the operational matrices of differentiation, integration and product for B-polynomials.

3.2 EOM-related matrices

3.2.1 B-polynomials

As we mentioned, m-degree B-polynomials[8] are a set of polynomials defined on $[0, 1]$:

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \qquad 0 \leq i \leq m,$$

where $\binom{m}{i}$ means

$$\frac{m!}{i!\,(m-i)!}\; .$$

In this paper, we use the $\psi_m(x)$ notation to show

$$\psi_m(x) = \begin{bmatrix} B_{0,m}(x) & B_{1,m}(x) & \cdots & B_{m,m}(x) \end{bmatrix}^T \; .$$

We should remind that [74]:

$$\psi_m(x) = A_m \times T_m(x), \qquad (3.19)$$

where

$$T_m(x) = \begin{bmatrix} x^0 & x^1 & \cdots & x^m \end{bmatrix}^T , \qquad (3.20)$$

and the $(i+1)^{\text{th}}$ row of matrix A is

$$A_{i+1} = \begin{bmatrix} \overbrace{0 \;\; \cdots \;\; 0}^{i \text{ times}} & (-1)^0 \binom{m}{i} & \cdots & (-1)^{m-i}\binom{m}{i}\binom{m-i}{m-i} \end{bmatrix} . \tag{3.21}$$

Matrix A is an upper triangular matrix and $\det(A) = \prod_{i=0}^{m} \binom{m}{i}$. So, A is an invertible matrix. The writers of [43] have presented the following relation for A^{-1}.

$$\{\mathbf{A}^{-1}\}_{i,j=0}^{m} = \begin{cases} \dfrac{\binom{m-i}{j-i}}{\binom{m}{j}} , & j \geq i \; , \\[2mm] 0 , & j < i \; . \end{cases} \qquad (3.22)$$

Here, we propose a short introduction to exact operational matrices [43], which is the backbone of this paper.

3.2.2   A general formula for x^i

The term x^i is a very common term in differential equations. So, [43] proposed a general formula for x^i to be written as linear combination of the Bernstein polynomials

$$x^i = d_{i,m}^T \psi_m(x) , \qquad m \geq i , \qquad (3.23)$$

$$d_{i,m} = \left(\begin{bmatrix} \overbrace{0 \;\; 0 \;\; \cdots \;\; 0}^{i} & 1 & \overbrace{0 \;\; 0 \;\; \cdots \;\; 0}^{m-i} \end{bmatrix} A_m^{-1} \right)^T .$$

3.2.3   K matrices

For the future applications, we consider the two following simple matrices (called K-matrices), proposed in [43]:

$$K_{m,i} = \begin{bmatrix} I_m & 0_{m \times i} \end{bmatrix}_{m \times m+i} , \qquad (3.24)$$

$$K'_{m,i} = \begin{bmatrix} 0_{m \times i} & I_m \end{bmatrix}_{m \times m+i} . \qquad (3.25)$$

3.2.4   The increaser matrix

Suppose that we are going to solve the differential equation (1.1), using the Galerkin method. To implement (1.4) by EOMs, we apply EOMs on the equation and sum all of the terms to reach the residual function. For becoming able to factor out a base vector from all of the different $b_n(x)$-sized terms in the residual function, [43] introduced the so-called increaser matrix $E_{i,j}$, by which:

$$b_i(x) = E_{i,j}.b_j(x).$$

Using this matrix, we can convert the basis vector (existing in each term) to the largest $b_{maxNum}(x)$. By factoring out the $b_{maxNum}(x)$, we can write the residual function as $R_{maxNum}b_{maxNum}(x)$. So, solving the problem, will be reduced to solving the following equation

$$R_{maxNum}b_{maxNum}(x) = 0, \quad maxNum \geq m. \tag{3.26}$$

Also, [43] has proposed the increaser matrix of the Bernstein basis vector

$$\psi_m(x) = E_{m,i}\psi_{m+i}(x) , \qquad (3.27)$$
$$E_{m,i} = A_m K_{m+1,i} A_{m+i}^{-1} .$$

The $E_{m,i}$ matrix size is $(m+1) \times (m+1+i)$.

3.2.5   The differentiation matrix

D_m is the operational matrix of differentiation for the Bernstein basis vector, introduced in [43]

$$\frac{d}{dx}\psi_m(x) = D_m \psi_{m-1}(x) , \qquad (3.28)$$

$$D_m = m \left(K'^{T}_{m,1} - K^{T}_{m,1} \right) .$$

3.2.6   The integration Matrix

[43] has introduced the Bernstein polynomials integration operational matrix P_m as

$$\int_0^x \psi_m(x)dx = P_m \psi_{m+1}(x) , \qquad (3.29)$$

where

$$P_m = \begin{bmatrix} p_{0,m} & \cdots & p_{m,m} \end{bmatrix}^T_{(m+1)\times(m+2)} ,$$

$$p_{i,m} = \frac{1}{m+1} \begin{bmatrix} \overbrace{0 \;\; \cdots \;\; 0}^{i+1} & \overbrace{1 \;\; \cdots \;\; 1}^{m+1-i} \end{bmatrix}^T .$$

3.2.7 The product matrix

For an arbitrary vector c, we can write:

$$c^T \psi_m(x)\psi_n^T(x) = \psi_{m+n}^T(x) \times \tilde{C}_{m,n} , \quad (3.30)$$

where $\tilde{C}_{m,n}$, introduced in [43], is an $(m + n + 1) \times (n + 1)$ product operational matrix, for the vector c

$$\left[\tilde{C}_{m,n}\right]_{i,j} = \begin{cases} 0 , & i \notin [j, j + m] , \\ c_{i-j}a_{(i-j),(j-1),m,n} , & o.w. \end{cases}$$

Moreover, by transposing (3.30), we have

$$\psi_n(x)\psi_m^T(x)c = \widehat{C_{n,m}}\psi_{m+n}(x) , \quad (3.31)$$

$$\widehat{C_{n,m}} = \tilde{C}_{m,n}^T ,$$

$\widehat{C_{n,m}}$ is, also, called the product operational matrix for the vector c [43].

3.2.8 The power matrix

Suppose that $y(x) = c^T\psi_m(x)$; [43] has introduced $\overline{C_{m,p}}$ operational matrix by which $y^p(x) = \overline{C_{m,p}}\phi_{p \cdot m}(x)$ and named it the power operational matrix for the Bernstein polynomials

$$\overline{C_{m,p}} = c^T \prod_{i=1}^{p-1} \widehat{C_{i \cdot p,m}} , \qquad p \geq 2 . \quad (3.32)$$

3.2.9 The Series matrix

Suppose that $y(x) = c^T\psi_m(x)$; here, we propose a matrix to approximate $f(y(x))$ function, where power series of $f(x)$ can be written as

$$f(x) = \sum_{i=0}^{\infty} e_i x^i,$$

in which $|x| < R$ and R is the convergence radius of the power series. So, we can write

$$f(x) \simeq \sum_{i=0}^{N} e_i x^i.$$

Substituting $y(x)$ by x, we have

$$f(y(x)) \simeq \sum_{i=0}^{N} e_i \left(c^T\psi_m(x)\right)^i,$$

where $|y(x)| < R$. Using the equation (3.32), we can write

$$f(y(x)) \simeq e_0 \cdot d_{0,m \cdot N}^T \psi_{m \cdot N} + e_1 \cdot c^T \psi_m(x)$$
$$+ \sum_{i=2}^{N} e_i \overline{C_{m,i}}\psi_{m \cdot i}(x),$$

where $N \geq 2$. Also, applying the equation (3.27), we have

$$f(y(x)) \simeq \left(e_0 \cdot d_{0,m \cdot N}^T + e_1 \cdot c^T E_{m,m(N-1)} \right.$$
$$+ \left. \sum_{i=2}^{N} e_i \overline{C_{m,i}} E_{m \cdot i, m(N-i)} \right)\psi_{m \cdot N}(x),$$

and finally

$$f(y(x)) \simeq \widetilde{C_{e_i,m,N}}\psi_{m \cdot N}(x), \quad (3.33)$$

where

$$\widetilde{C_{e_i,m,N}} = e_0 \cdot d_{0,m \cdot N}^T + e_1 \cdot c^T E_{m,m(N-1)}$$
$$+ \sum_{i=2}^{N} e_i \overline{C_{m,i}} E_{m \cdot i, m(N-i)},$$

and $|y(x)| < R$ and $N \geq 2$.

Up to now, we have obtained a relation for approximating the function $f(y(x))$. In the next part, we seek for the optimal e_i coefficients.

3.2.10 Truncated Taylor series

In the previous part, we based the series matrix on the power series. However, although we know that $f(x) = \sum_{i=0}^{\infty} c_i x^i$, $c_i = \frac{f^{(i)}}{i!}$, after truncating the series to N terms, not only the truncated series will not be exactly equal to $f(x)$, but also the c_is would not be, necessarily, the best coefficients for approximating $f(x)$. To reach the best coefficients, the following theorem is required.

Theorem 3.1 *Consider the Hilbert space $H = L^2[a, b]$ and one of its finite-dimensional subspaces:*

$$F = Span\{f_0(x), f_1(x), \cdots, f_N(x)\},$$

with inner product defined by

$$\langle f(t), g(t) \rangle = \int_a^b f(t)g(t)dt.$$

1. *For any arbitrary function $q(x) \in H$, there exists a unique best approximation $p(x)$ (in respect to the defined inner product) for the $q(x)$ function*

2.

$$\langle (q(x) - p(x)), f \rangle = 0,$$
$$f = \begin{bmatrix} f_0(x) & f_1(x) & \cdots & f_N(x) \end{bmatrix}^T,$$

where

$$\langle f(x), f \rangle = \begin{bmatrix} \langle f(x), f_0(x) \rangle & \cdots & \langle f(x), f_N(x) \rangle \end{bmatrix}.$$

Refer to [74] Consider the Hilbert space $H = L^2[a, b]$. Approximating the function $f(x) \in H$ by a truncated series in the interval $\begin{bmatrix} a & b \end{bmatrix}$ is equal to approximating it by a function $g(x)$ in the T_m subspace of H:

$$T_m = Span\{1, x, \cdots, x^N\},$$

where

$$g(x) = e^T t_N,$$
$$e = \begin{bmatrix} e_0 & e_1 & \cdots & e_N \end{bmatrix}^T,$$
$$t_N = \begin{bmatrix} 1 & x & x^2 & \cdots & x^N \end{bmatrix}^T.$$

Using the theorem, we have

$$\langle (f(x) - e^T t_N), t_N \rangle = 0,$$
$$e^T \langle t_N, t_N \rangle = \langle f(x), t_N \rangle,$$

where $\langle t_N, t_N \rangle$ is an $(N+1) \times (N+1)$ matrix and is said dual matrix of t_N. Let

$$U_{t_N} = \langle t_N, t_N \rangle = \int_a^b t_N t_N^T dx,$$

then,

$$e^T = \left(\int_a^b f(x) t_N^T(x) dx \right) U_{t_N}^{-1}. \qquad (3.34)$$

So, the best approximation for $f(x)$ (in T_m) will be

$$f(x) = \sum_{i=0}^{N} e_i x^i,$$

where e_is are the elements of the vector e, obtained above.

3.2.11   The Q Matrix

In the subsection 3.2.4, we converted the solution of the differential equation (1.1) to the solution of the algebraic equations system (3.26). In (3.26), $b_{maxNum}(x)$ is a basis vector and its elements (functions) are linearly-independent. So, we can solve the following equation, instead

$$R_{maxNum} = 0, \qquad maxNum \geq m.$$

To overcome the problems of solving a system with $maxNum$-equations and m unknown variables, [43] has introduced the Galerkin matrix $Q_{maxNum,m}$, which reduces the number of equations to m, based on the Galerkin method:

$$R_{1 \times (maxNum+1)} \times Q(maxNum, m) = 0, \qquad (3.35)$$

where $Q(maxNum, m)$ is an $(maxNum + 1) \times (m + 1)$ matrix

$$Q(maxNum, m) = [q_{ij}]_{maxNum+1 \times m+1},$$
$$i \leq maxNum, j \leq m,$$

$$q_{ij} =$$
$$\frac{\binom{maxNum}{i-1}\binom{m}{j-1}(maxNum + m + 2 - (i+j))! + (i+j-2)!}{(maxNum + m + 1)!}.$$

Then, using Eq. (3.35), we solve

$$R_m^* = 0, \qquad (3.36)$$

where

$$R_m^* = R_{1 \times (maxNum+1)} \times Q(maxNum, m).$$

By solving the resulting algebraic system, we will find the $m + 1$ unknown coefficients c_i, in (1.2), and, finally, find the $y_m(x)$.

3.3   The solution error bound

For analyzing the solution error bound of their method; For this purpose, [43] has discussed about the solution analysis of the Bernstein polynomials EOMs in detail. Here, we present the gist of that discussion and refer the interested reader to that paper.

Definition 3.1 *Let $f(x) \in C[0, 1]$ and $\phi : [0, 1] \to \mathbb{R}$ be an admissible step-weight function (for details see [20]), the Ditzian-Totik modulus of second order smoothness of $f(x)$ on $[0, 1]$, $w_\phi^2(\delta)$, is defined by*

$$w_\phi^2(\delta) = \sup_{|h| \leq \delta} \sup_{x \pm h\phi(x) \in [0,1]}$$
$$|f(x - \phi(x)h) - 2f(x) + f(x + \phi(x)h)|, \qquad \delta > 0.$$

Theorem 3.2 *Let* $\varphi(x) = \sqrt{x(1-x)}$ *and* $\phi :$ $[0,1] \to \mathbb{R}$ *($\phi \neq 0$) be an admissible step-weight function of the Ditzian-Totik modulus of smoothness such that* ϕ^2 *and* φ^2/ϕ^2 *are concave. Suppose the two functions* $f : [a,b] \to \mathbb{R}$ *and* $Y =$ $\text{Span}\{B_{0,m}(x), B_{1,m}(x), ..., B_{m,m}(x)\}$, *if* $c^T\psi_m(x)$ *is the best approximation to* $f(x)$ *in* Y, *then the mean error bound will be*

$$||f(x) - c^T\psi_m(x)||_2 \leq Cw_\phi^2(\frac{1}{\sqrt{m}}\frac{\varphi(x)}{\phi(x)}) ,$$
$$x \in [0,1] .$$

4   Applications

To show the efficiency and accuracy of the present method on ordinary differential equations, seven examples are presented. The exact solution is not available for most of them; therefore, we solved them using a seventh-eighth order continuous Runge-Kutta method as an almost exact solution, using the Maple$^{\copyright}$ dverk78 function, for checking the accuracy of the EOM method results. For achieving the approximate solutions, we applied the present method for different m valued $y_m(x)$s in (1.2). Then, we compared them with the almost exact solutions and computed the norm1 for the residual and the error function of each one.

The numerical implementation and all of the executions are performable by maplesoft$^{\copyright}$.maple.16.x64, 64-bit Microsoft$^{\copyright}$ Windows7 Ultimate Operating System, alongside hardware configuration: Laptop 64-bit Core i3 M380 CPU, 8 GBs of RAM.

4.1   Solution of the Emden Fowler equations

In the subsection 2.1 we got familiar with the Emden-Fowler equations. Multiplying the Emden-Fowler equation (2.6) in x, we would have

$$N[y(x)] = xy''(x) + 2y'(x)+$$
$$xf(x)g(y(x)) = 0,$$
$$\text{(4.37)}$$
$$0 \leq x \leq \lim_{M\to\infty} M,$$

$$y(0) = a, \quad y'(0) = b.$$

Neglecting the large values of M, we can solve the equation for $0 \leq x \leq M$. So, we solve the problem in the domain $[0, M]$. But the domain of Bernstein polynomials is [0,1] and are not applicable directly for solving this problem. One approach for overcoming this difficulty is changing variables. We use the following mapping relations

$$s = \frac{x}{M} , \qquad v(s) = \frac{y(x)}{M} .$$

Applying the above-mentioned mapping, we have

$$y(x) = Mv(s), \qquad \text{(4.38)}$$
$$\frac{d}{dx}y(x) = \frac{d}{ds}v(s),$$
$$\frac{d^2}{dx^2}y(x) = \frac{1}{M}\frac{d^2}{ds^2}v(s),$$
$$v(0) = \frac{a}{M},$$
$$v'(0) = b.$$

Now, substituting the mapped function $v(s)$ by the function $y(x)$ in the equation (4.37), we have

$$N[y(x)] = sv''(s) + 2v'(s)$$
$$+ sMf(sM)g(Mv(s)) = 0,$$
$$\text{(4.39)}$$
$$0 \leq s \leq 1, s = \frac{x}{M},$$

$$v(0) = \frac{a}{M}, \quad v'(0) = b.$$

So, we can start solving the above equation, using the Bernstein functions. To estimate the first term of the equation (4.39) suppose that

$$z''(s) \simeq v''(s), \qquad \text{(4.40)}$$

where

$$z''(s) = c^T\psi_m(s). \qquad \text{(4.41)}$$

Then, using equations (3.23) and (3.31) we can write

$$\begin{cases} sz''(s) = c^T\psi_m(s)\psi_1^T(s)d_{1,1} \\ \qquad = c^T(\widehat{D_{1,1}})_{m,1}\psi_{m+1}(s), \\ z'(s) - z'(0) = c^TP_m\psi_{m+1}(s). \end{cases}$$

Also, for estimating the second term of the equation (4.39), by integrating the equation (4.41) and applying (3.23), we will have

$$z'(s) = g^T\psi_{m+1}(s), \qquad \text{(4.42)}$$
$$g = P_m^Tc + b \cdot d_{0,m+1}. \qquad \text{(4.43)}$$

Finally, to estimate the last equation, we require estimation of the functions $z(s)$, $f(sM)$ and $g(Mv(s))$. For approximating $z(s)$, we integrate the equation (4.42)

$$z(s) - z(0) = g^T P_{m+1} \psi_{m+2}(s).$$

Then, using the equation (3.23) we can write

$$z(s) = h^T \psi_{m+2}(s), \qquad (4.44)$$
$$h = P_{m+1}^T g + \frac{a}{M} \cdot d_{0,m+2}.$$

Now, it only remains estimating the functions $f(sM)$ and $g(Mv(s))$. Depending on the type of the Emden-Fowler problem ((2.10) to (2.16)), there exist two vectors k and $s(c)$, (elements of the vector $s(c)$ depend on the elements of the vector c) and integers i_0 and j_0 by which we can express $sMf(sM) \simeq k^T \psi_{i_0}(s)$ and $g(Mz(s)) \simeq s^T(c)\psi_{j_0}(s)$. So, utilizing the equation (3.31), we can write

$$\begin{aligned} g(Mz(s)) \cdot sMf(sM) \\ = s^T(c)\psi_{j_0}(s)\psi_{i_0}^T(s)k \\ = s^T(c)\widehat{K_{j_0,i_0}}\psi_{i_0+j_0}(s). \end{aligned}$$
$$(4.45)$$

Using (4.37), we can write the Residual(s) as

$$\begin{aligned} \text{Residual}(s) = N[z(s)] = sz''(s) + 2z'(s) \\ + sMf(sM)g(Mz(s)). \end{aligned}$$

Then, applying the equation (4.45), we can write

$$\begin{aligned} \text{Residual}(s) = c^T(\widehat{D_{1,1}})_{m,1}\psi_{m+1}(s) \\ + 2g^T\psi_{m+1}(s) + s^T(c)\widehat{K_{j_0,i_0}}\psi_{i_0+j_0}(s). \end{aligned}$$

By factoring out the entire $\psi(s)$ functions, we will have

$$\text{Residual}(s) = R_{1\times maxNum}\psi_{maxNum}(s),$$

where $maxNum = \max(m+1, i_0+j_0)$. The value of the vector R, above, depends on the values of i_0 and j_0. Applying the equation (3.27), we will have

$$R = \begin{cases} c^T(\widehat{D_{1,1}})_{m,1}E_{m+1,i_0+j_0-(m+1)} \\ \quad + 2g^T E_{m+1,i_0+j_0-(m+1)} + s^T(c)\widehat{K_{j_0,i_0}}, & i_0+j_0 > m+1, \\ c^T(\widehat{D_{1,1}})_{m,1} + 2g^T \\ \quad + s^T(c)\widehat{K_{j_0,i_0}}E_{i_0+j_0,m+1-(i_0+j_0)}, & i_0+j_0 < m+1, \\ c^T(\widehat{D_{1,1}})_{m,1} + 2g^T + s^T(c)\widehat{K_{j_0,i_0}}, & i_0+j_0 = m+1. \end{cases}$$
$$(4.46)$$

Using (3.36), we solve the following system to find the unknown c_is (elements of the vector c)

$$R_m^* = R \times Q(maxNum, m) = 0. \qquad (4.47)$$

Still, we are not able to solve the R_m^*. Because (as it is obvious from (4.46)) the vectors $s(c)$ and k are unknown. As it was declared, the value of these two vectors depend on the type of the Emden-Fowler equation and the related $f(x)$ and $g(y(x))$ functions, all of which are listed in (2.10)-(2.16). The $f(x)$ function, in the equations (2.10)-(2.15), is the constant function 1, whereas, in (2.16), $f(x)$ is equal to $-2(2x^2+3)$. So, using the equation (3.23), we can write

$$sMf(sM) = \begin{cases} M \cdot d_{1,1}^T\psi_1(s), & (2.10)-(2.15), \\ -2M(2M^2d_{3,3}^T + 3d_{1,3}^T)\psi_3(x), & \text{for Eq. } (2.16). \end{cases}$$

By means of the above equations, all of the k vectors will be found. Now, we have to find the $s(c)$ vectors. For the equations (2.10) and (2.16), $g(Mz(s))$ can be found by using (4.44) and (3.32)

$$g(Mz(s)) = \begin{cases} M^p z^p(s) = M^p\overline{H_{m+2,p}}\psi_{(m+2)p}(s), \\ \quad p \in \mathbb{N}, \text{ for Eq. } (2.10), \\ Mz(s) = Mh^T\psi_{m+2}(s), \\ \quad \text{for Eq. } (2.16), \end{cases}$$

and for the other $g(Mz(s))$ functions ((2.11) − (2.15)), using (4.44) besides (3.33), we can write

$$g(y(x)) \simeq \widetilde{H_{e_i,m,N}}\psi_{(m+2)N}(x).$$

Regarding the equations (3.33) and (3.34), we know that for estimating the function $g(y(x))$, the best coefficients e_is are the elements of the following vector

$$e_{(N+1)\times 1} = \left(U_{\psi_N(x)}^{-1}\right)^T \int_a^b g(x)\psi_N(x)dx.$$

Now, we are able to solve the system (4.47) and find the elements of the vector c. Finding the vector c, results in finding the vector h in (4.44) and the vector h, alongside the equations (4.38) (4.40) and (4.44), outputs the function $y(x)$

$$\begin{aligned} y(x) = Mv(s) \\ \simeq Mz(s) \\ = Mh^T\psi_{m+2}(s). \end{aligned}$$

It is, also, worth mentioning that solving the (4.47) system of nonlinear equations has several difficulties (even by using the Newton's method) when the number of algebraic equations grows up. The main difficulty with such systems is choosing

the initial guess to manage the Newton method for sufficing to low-order computations. So, we applied a technique, introduced in [43], to choose an appropriate initial guess.

However, we solved the entire (2.10)-(2.16) Emden-Fowler equations and the results are depicted in the figures (1)-(17). All of the problems, except the (2.12) are solved for varying m values

$$m \in \{2, 3, \cdots, E\},$$

where, E is the largest m for which the results are reported. To evaluate the solution accuracy for each m, it is not needed to check them one by one; reporting the norm of the residual and the error functions is enough for evaluation. So, we present two categories of figures. The first category contains the figures which report the norm of the residual or the error function for the entire m values, and the second category includes the plots of the error or the residual function for the largest m value (E).

Looking at the first category plots, we find out that both of the residual and the error function values decreases by increasing the m value and it confirms the convergence of both methods. Also, these plots demonstrate the absolute priority of EOMs. The second category of plots shows that, almost for all x values, both of the residual and the error functions, related to the EOM method, takes less values than those related to OOM. Meanwhile, the second category plots are semi-logarithmic because, according to the high superiority of EOMs over OOMs, the error of the EOM method results can not, obviously, be seen in non-logarithmic plots. It is also worth to mention that, for having appropriate approximations of $g(y(x))$, we have chosen some large N values (e.g. 8, 10, 12, and 15) and the acceptable accuracy of the result reveals the efficient selection of N.

However, before explaining the figures, we should remind that the problem is solved in the interval $x \in [0, M]$ and we have not selected the value of M, yet. However, in the conducted studies about these problems (e.g. [42]), only the positive $y(x)$ values were the matter of interest. Therefore, since all of our problems have positive $y(0)$ initial values, M should be chosen as $y(M) = 0$. Remember that we have access to this value because we have the (almost) exact solutions of our entire problems. Nevertheless, when $M > 5$, we set it to be 5.

Now, it is the turn of explaining the results, one by one. The analytical solution of the equation (2.10), for $p = 0$, is so that we can exactly write the solution as a linear combination of the Bernstein polynomials; so, we do not report the related figures. However, for $p = 1$ and $p = 5$ the results are depicted in the figures (1)-(4). Looking at the figure (5), we see less accuracy, in comparison with other problems. It shows that the related N value is not sufficiently large to approximate $g(y(x))$ well. However, it can be considered as a good example for indicating the performance of the truncated series, in function approximation. In the figure (8) (related to (2.16)), the analytical solution is compared with both EOM and OOM solutions and the results show the preeminence of the former over the latter. About the problem (2.11), it has been solved twice, once for $g(y(x)) = y^{\frac{3}{2}}(x)$ and once for $g(y(x)) = y^{\frac{5}{2}}(x)$. The figures (10) and (12) show the ineligible results for the residual function of the OOM method. As an example, in $x = 0$, the error is even more than 1, whereas even in those points, the EOM method presents some acceptable results. The interesting point in the solution of the problem (2.12), for $m = 2$ (figure (13)), is the capability of the method to reach such high accuracy approximations, even for this small m value, when the N value is chosen large-enough (in this case, we set it to 15). The figures (15) and (17) (problems (2.13) and (2.14)), again, show some almost ineligible results for OOM method, whereas EOM has some acceptable results.

# 5 Concluding Remarks and suggestions for further study

In this paper, we implemented the recently introduced exact operational matrices (EOMs) [43] of the Bernstein polynomials to solve the well-known Emden-Fowler equations, using the Galerkin method. For the sake of reaching more accurate results, we proposed a relation for finding the best coefficients e_i for approximating a function $g(x)$ in the form of $\sum_{i=1}^{N} e_i x^i$. Then, we introduced the new so-called "series operational matrix" for finding the coefficients s_i to write the function $g(y(x))$ in the form of $\sum_{i=0}^{m \cdot N} s_i B_{i,m \cdot N}(x)$, where $B_{i,m \cdot N}(x)$s are the Bernstein polynomials of degree $m \cdot N$.
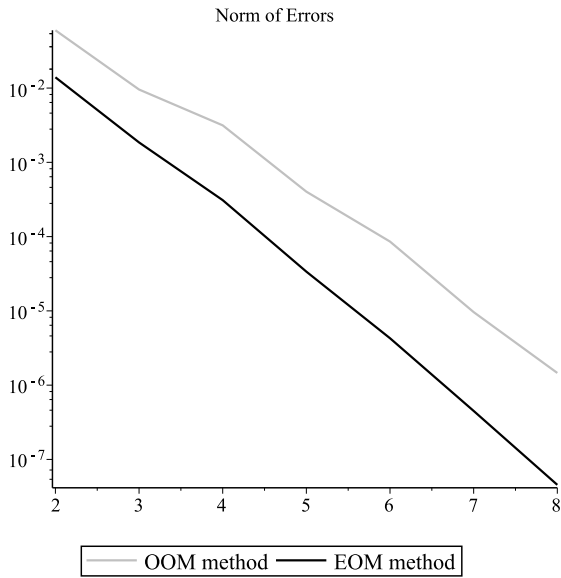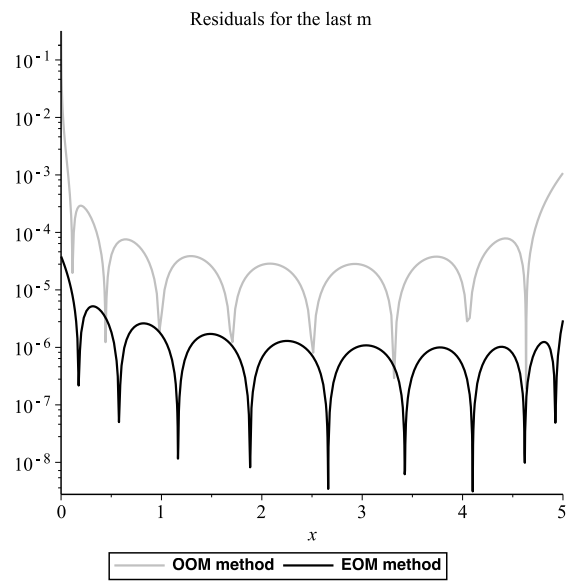
(a) $\|Error\,(y_m(x))\|_1$
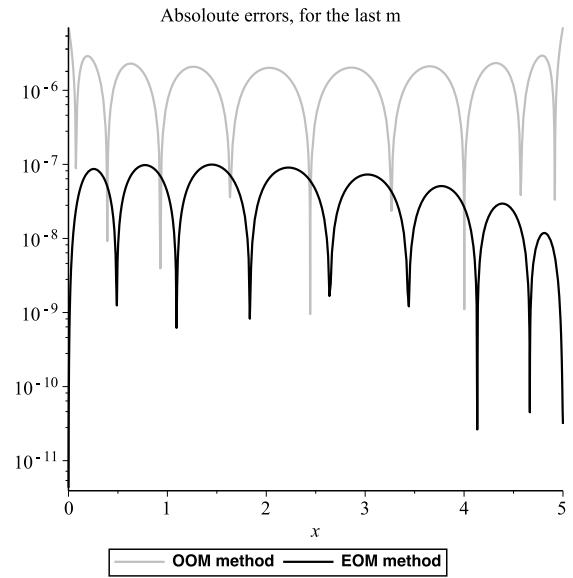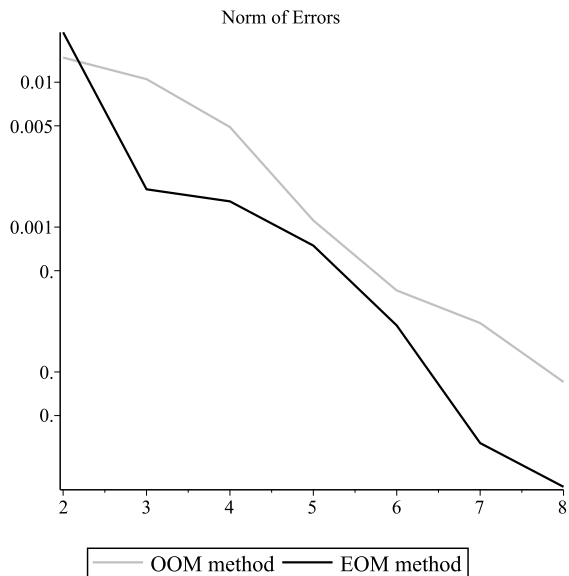

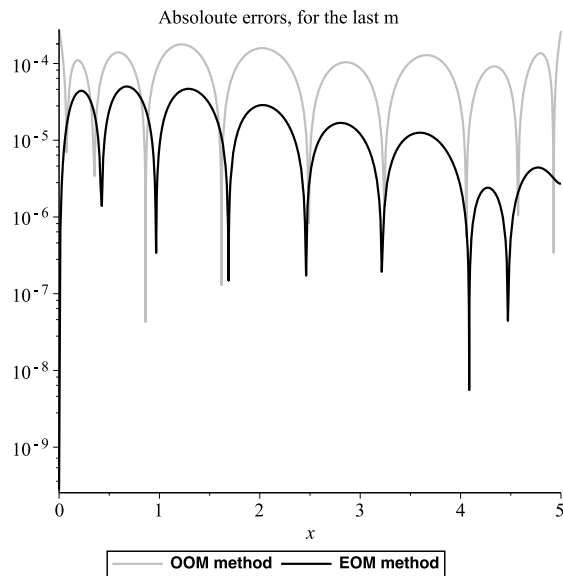(a) $\|Error\,(y_M(x))\|_1$
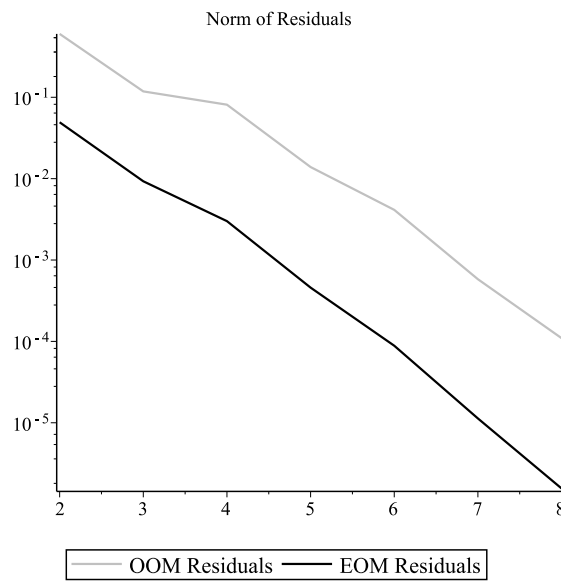
(b) $\|Residual\,(y_m(x))\|_1$

(b) $\|Residual\,(y_M(x))\|_1$

Figure 1: The norm1 of the residual and error function plots for several m values and $f(x) = 1$, $g(x) = y(x)$, $N = 8$
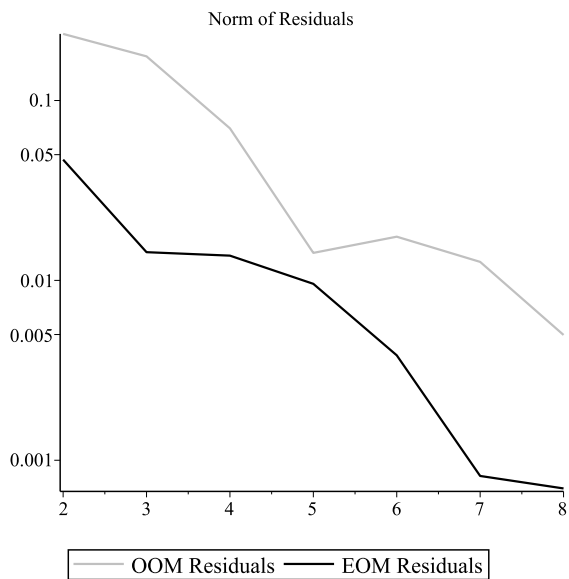
Figure 2: The norm1 of the residual and error function plots for the largest m value (E) of the figure (1) for $f(x) = 1$, $g(x) = y(x)$, $N = 8$
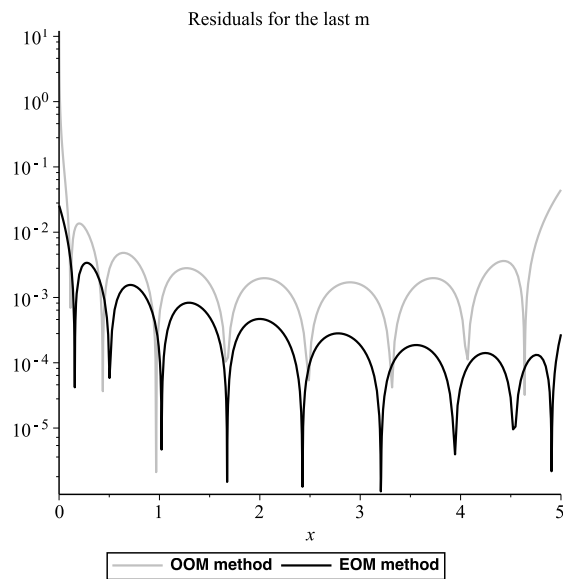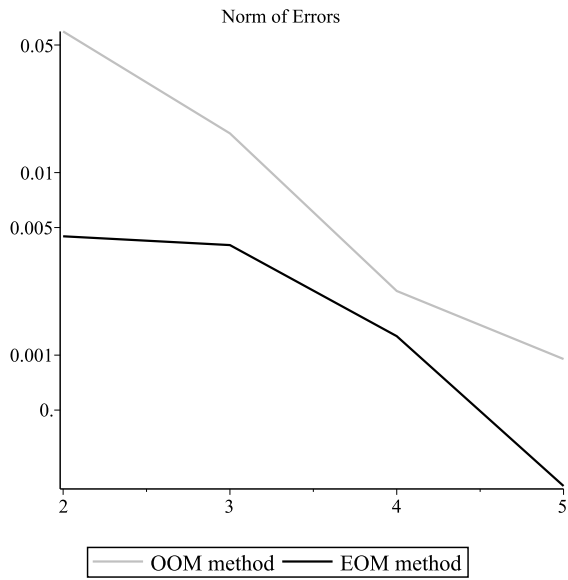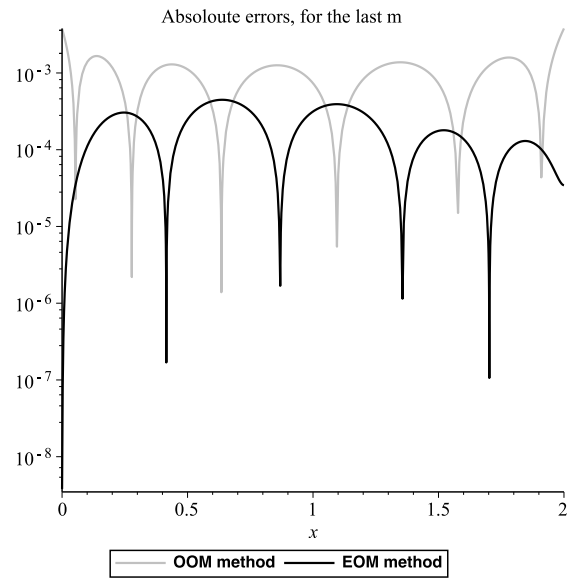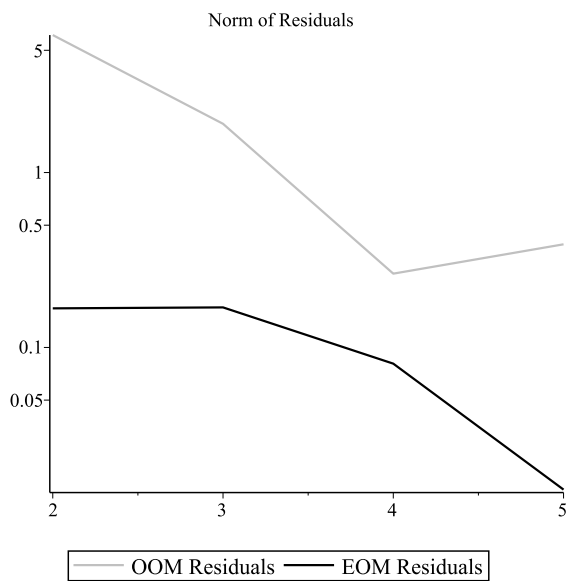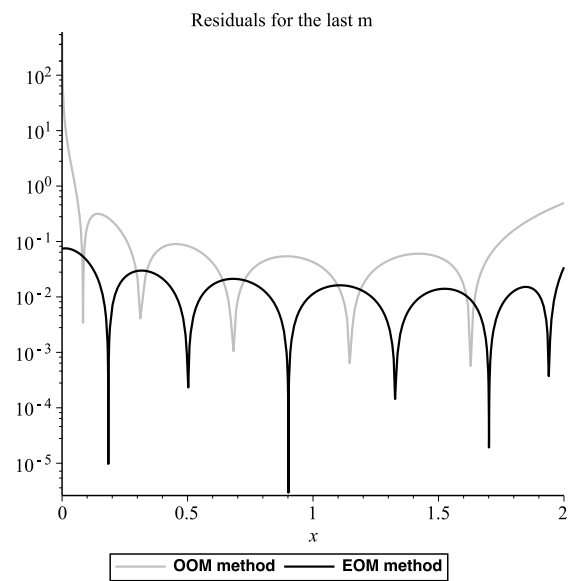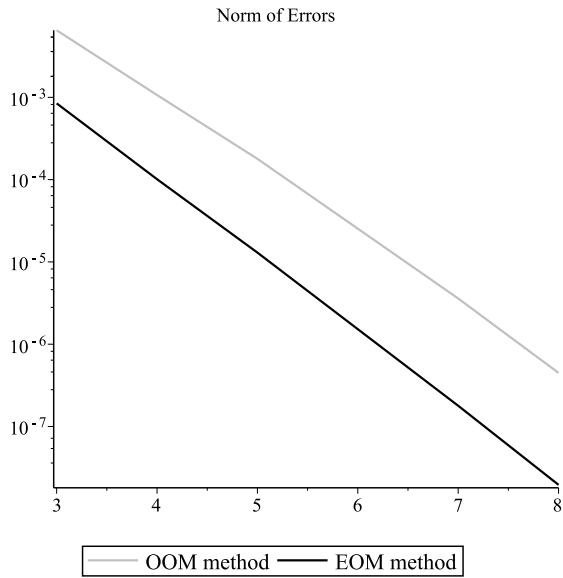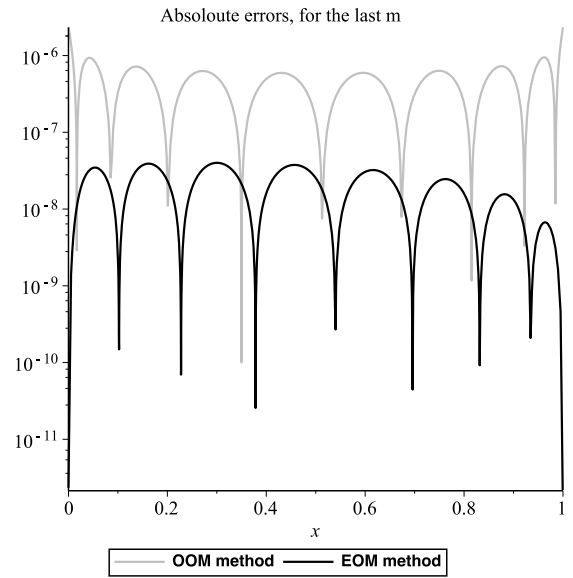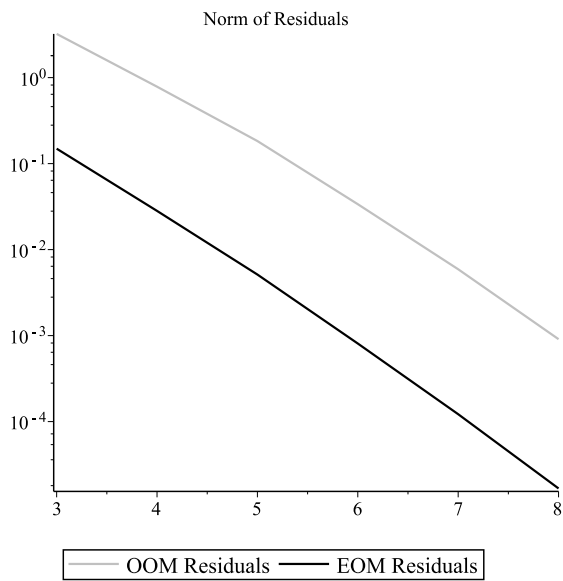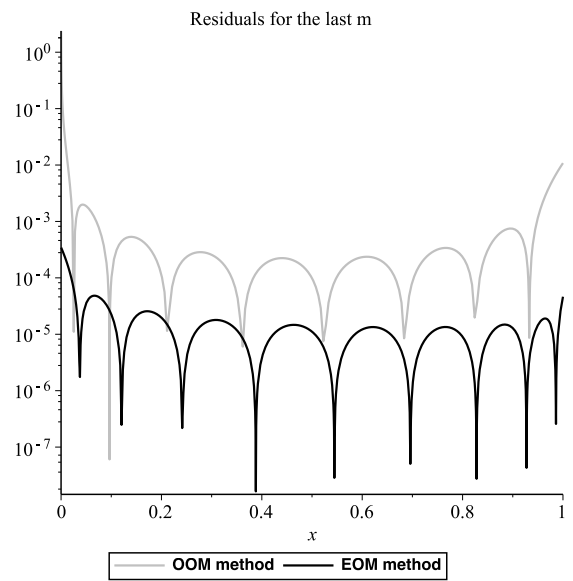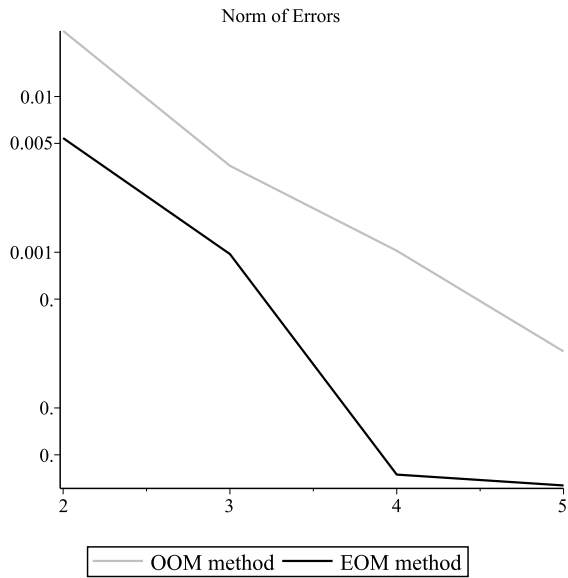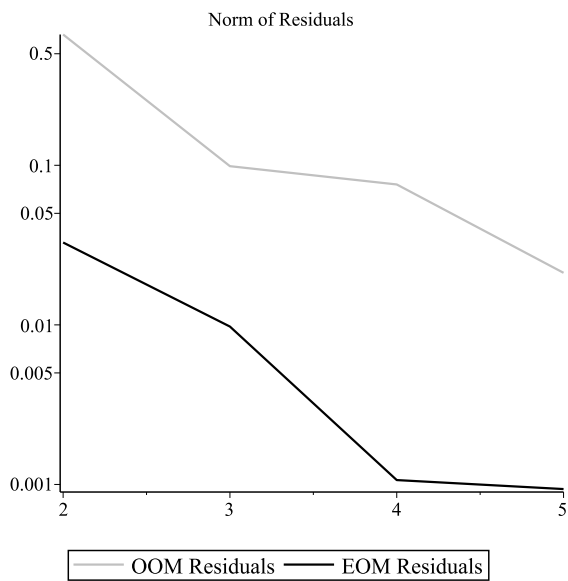
(a) $\|Error\,(y_m(x))\|_1$

(a) $\|Error\,(y_M(x))\|_1$

(b) $\|Residual\,(y_m(x))\|_1$

(b) $\|Residual\,(y_M(x))\|_1$

Figure 3: The norm1 of the residual and error function plots for several m values and $f(x) = 1$, $g(x) = y^5(x)$, $N = 8$
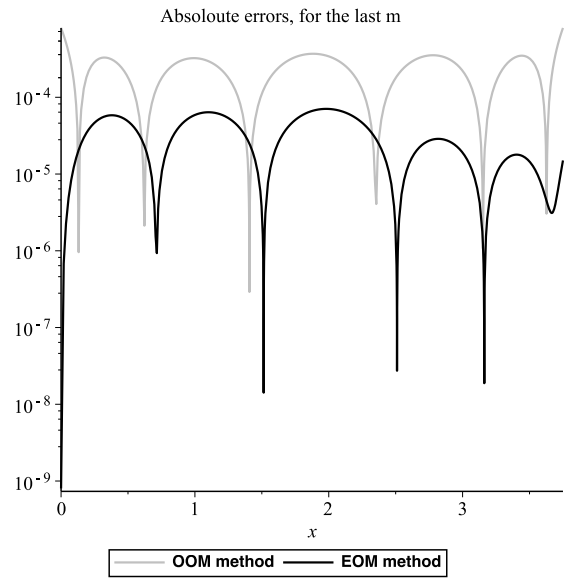
Figure 4: The norm1 of the residual and error function plots for the largest m value (E) of the figure (3) for $f(x) = 1$, $g(x) = y^5(x)$, $N = 8$
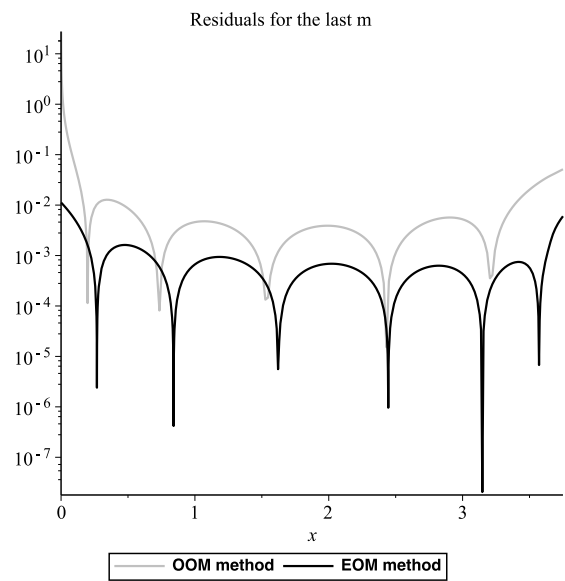
(a) $\|Error\left(y_m(x)\right)\|_1$

(b) $\|Residual\left(y_m(x)\right)\|_1$

Figure 5: The norm1 of the residual and error function plots for several m values a $f(x) = 1$, $g(x) = 4\left(2e^{y(x)} + e^{\frac{y(x)}{2}}\right)$, $N = 10$

(a) $\|Error\left(y_M(x)\right)\|_1$

(b) $\|Residual\left(y_M(x)\right)\|_1$

Figure 6: The norm1 of the residual and error function plots for the largest m value (E) of the figure (5) for $f(x) = 1$, $g(x) = 4\left(2e^{y(x)} + e^{\frac{y(x)}{2}}, \ N = 10\right)$

(a) $\|Error\,(y_m(x))\|_1$

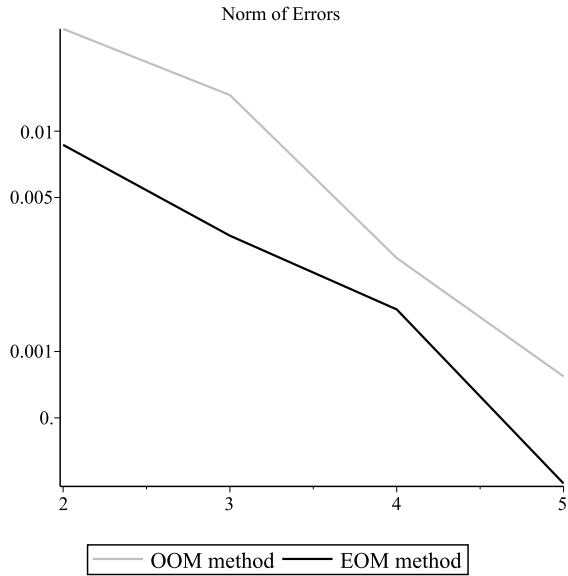(a) $\|Error\,(y_M(x))\|_1$

(b) $\|Residual\,(y_m(x))\|_1$

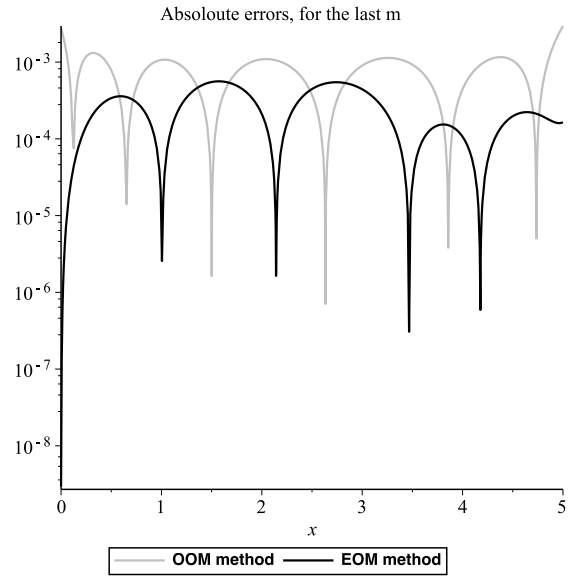

(b) $\|Residual\,(y_M(x))\|_1$

Figure 7: The norm1 of the residual and error function plots for several m values and $f(x) = -2\left(2x^2 + 3\right)$, $g(x) = y(x)$, $N = 8$

Figure 8: The norm1 of the residual and error function plots for the largest m value (E) of the figure (7) for $f(x) = -2\left(2x^2 + 3\right)$, $g(x) = y(x)$, $N = 8$

(a) $\|Error(y_m(x))\|_1$

(a) $\|Error(y_M(x))\|_1$

(b) $\|Residual(y_m(x))\|_1$

(b) $\|Residual(y_M(x))\|_1$

Figure 9: The norm1 of the residual and error function plots for several m values and $f(x) = 1$, $g(x) = y^{\frac{3}{2}}(x)$, $N = 12$
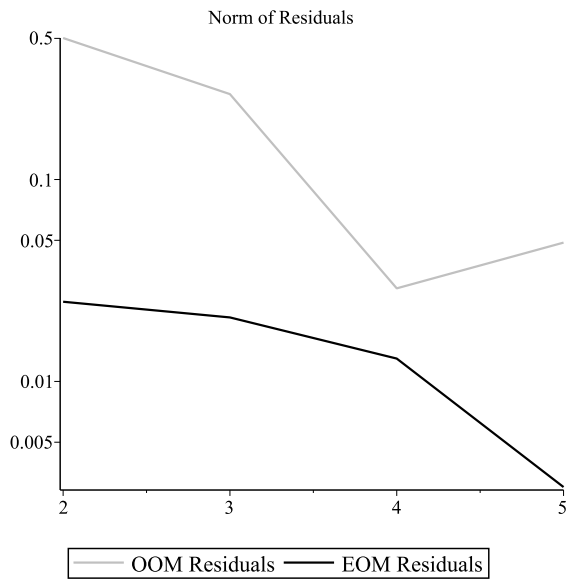
Figure 10: The norm1 of the residual and error function plots for the largest m value (E) of the figure (9) for $f(x) = 1$, $g(x) = y^{\frac{3}{2}}(x)$, $N = 12$
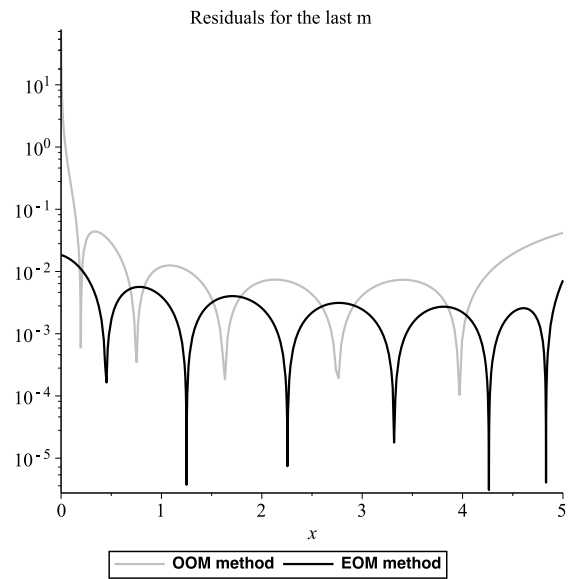
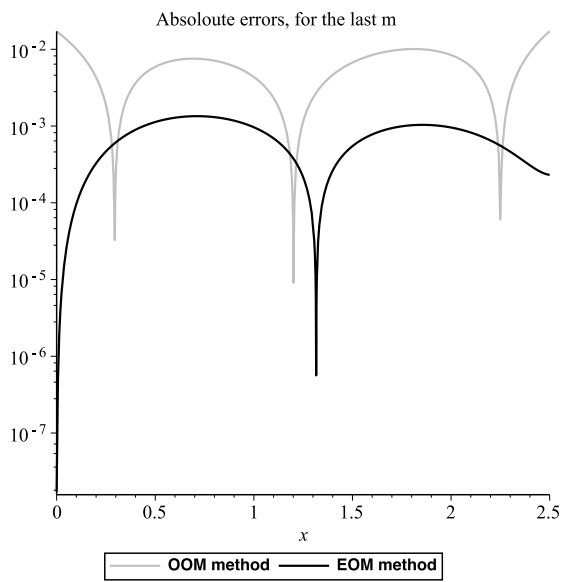(a) $\|Error\,(y_m(x))\|_1$

(a) $\|Error\,(y_M(x))\|_1$
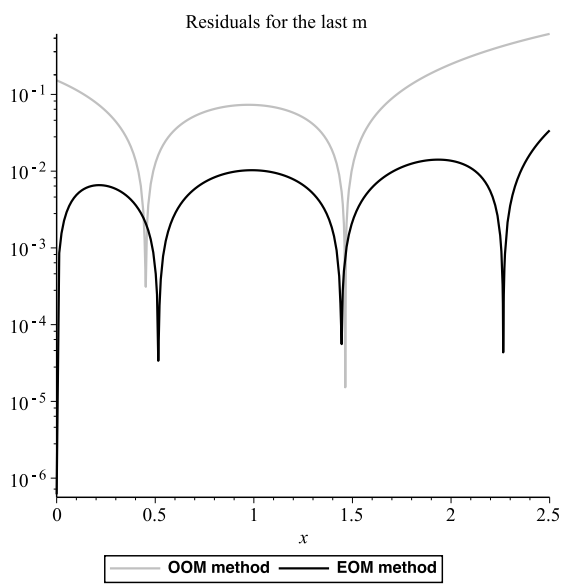

(b) $\|Residual\,(y_m(x))\|_1$

(b) $\|Residual\,(y_M(x))\|_1$

Figure 11: The norm1 of the residual and error function plots for several m values and $f(x) = 1$, $g(x) = y^{\frac{5}{2}}(x)$, $N = 12$

Figure 12: The norm1 of the residual and error function plots for the largest m value (E) of the figure (11) for $f(x) = 1$, $g(x) = y^{\frac{5}{2}}(x)$, $N = 12$
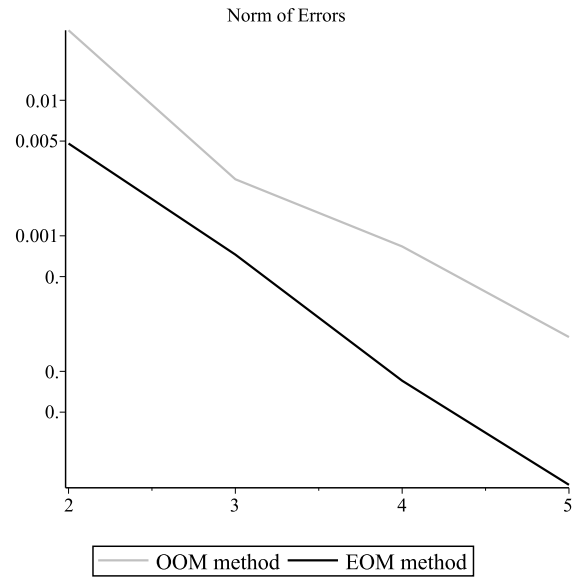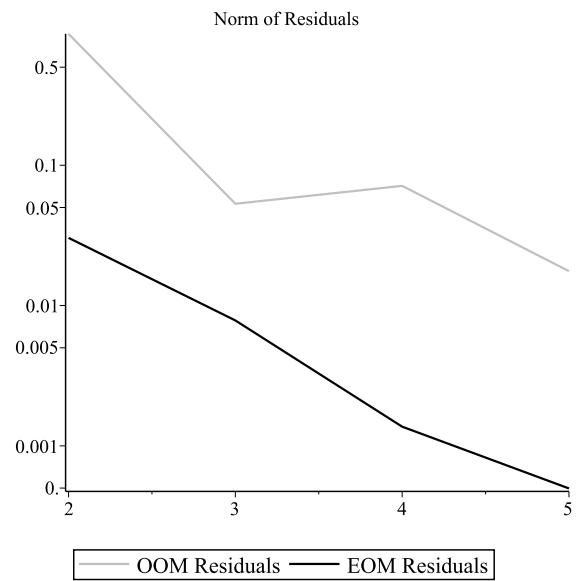
(a) $\|Error\,(y_M(x))\|_1$

(b) $\|Residual\,(y_M(x))\|_1$

Figure 13: The norm1 of the residual and error function for $M = 2$ and $f(x) = 1$, $g(x) = e^{y(x)}$, $N = 15$
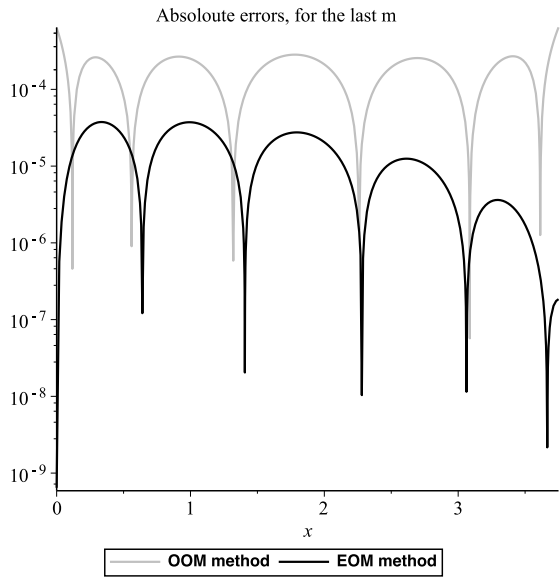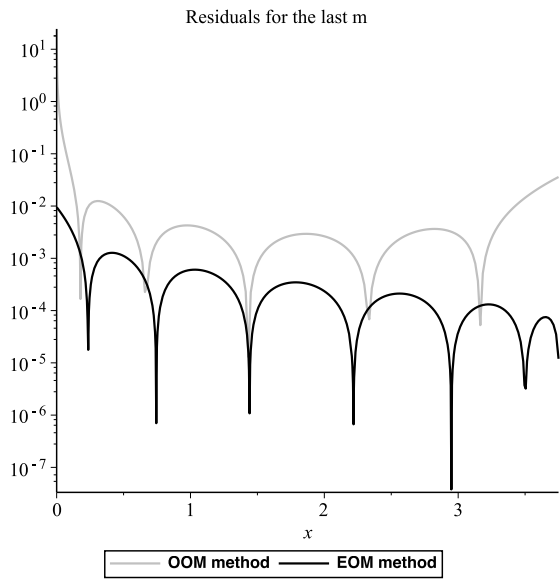

(a) $\|Error\,(y_m(x))\|_1$

(b) $\|Residual\,(y_m(x))\|_1$

Figure 14: The norm1 of the residual and error function plots for several m values and $f(x) = 1$, $g(x) = \sinh(y(x))$, $N = 10$

(a) $\|Error\,(y_M(x))\|_1$

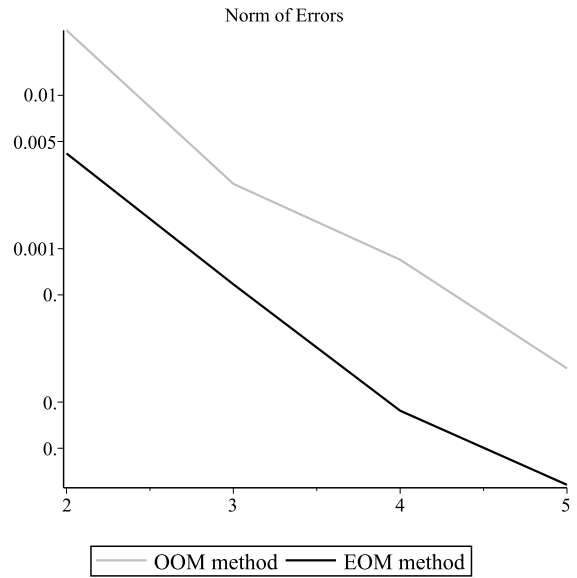
(a) $\|Error\,(y_m(x))\|_1$

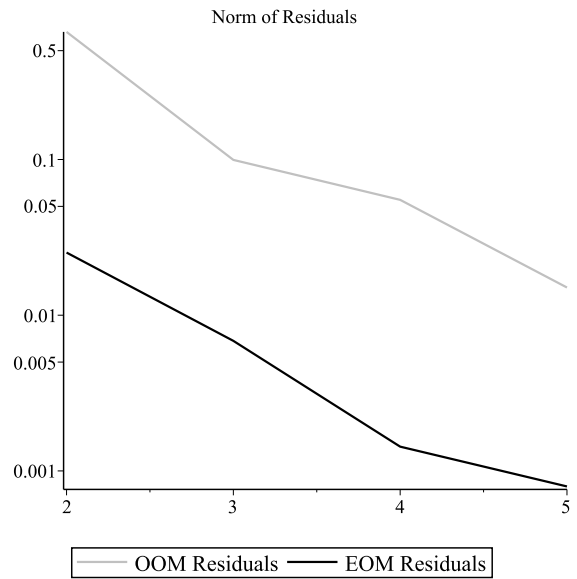

(b) $\|Residual\,(y_M(x))\|_1$

(b) $\|Residual\,(y_m(x))\|_1$

Figure 15: The norm1 of the residual and error function plots for the largest m value (E) of the figure (14) for $f(x) = 1$, $g(x) = \sinh(y(x))$, $N = 10$
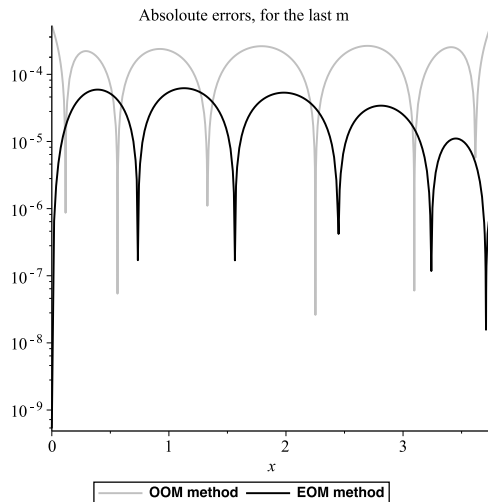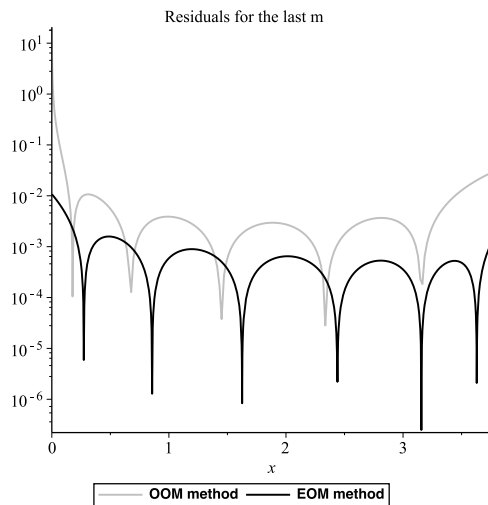
Figure 16: The norm1 of the residual and error function plots for several m values and $f(x) = 1$, $g(x) = \sin(y(x))$, $N = 12$

The differential equations on which the EOM idea were applied to solve, hitherto, were either linear or low-order nonlinear problems and the reported results in [43] did not indicate, completely, the potential superiority of the new method. Therefore, we chose Emden-Fowler type equations, which have a high-order nonlinearity (for presence of the $g(y(x)) \simeq \sum_{i=0}^{m \cdot N} e_i B_{i,m \cdot N}(x)$), to be solved by EOMs.

For solving the differential equations by the Galerkin method, using the EOMs, we reached an almost exact residual function ($Residual(x)$) which was never obtainable by the old "ordinary operational matrices" (OOMs) in most of the problems. We converted the residual function to a system of algebraic equations, using the Galerkin operational matrix [43] and solved them to find the unknown function $y(x)$. To have an appropriate criterion for the results accuracy of the problems without exact solution, we solved them, taking advantage of a seventh-eighth order continuous Runge-Kutta method (as an almost exact solution), using the Maple$^{©}$ dverk78 function.

To see the convergence of the resulting errors to zero, we applied the Galerkin method repeatedly, each new step with a larger $\psi_m(x)$. Then, using the (almost) exact solution, we reported the norm1 of the errors of both methods for each m value and showed the descending/converging behavior of error norm plot. However, the convergence speed was faster, in EOMs. We did, also, the same for residual functions. Moreover, we reported the norm and the error of the results for the largest m value in which the superiority of EOMs over OOMs was evident.

However, the Emden-Fowler problem domain is $[0, \infty)$. Therefore, for solving it by the Bernstein polynomials which are defined on the domain $[0, 1]$, we firstly truncated the problem domain to $[0, M]$ and then used a mapping technique to change its variable and solved it. However, the limited domain of the Bernstein EOMs can be considered as a disadvantage, in applying them for solving problems with semi-infinite/infinite domains. To overcome this limitation, we suggest extracting EOMs of other basis functions, with semi-infinite or infinite domains, for future works. Moreover, solving differential equations which have closer solutions to the vector space (made by the Bernstein polynomials and a spe-

Absolute errors, for the last m

(a) $\|Error(y_M(x))\|_1$

Residuals for the last m

(b) $\|Residual(y_M(x))\|_1$

Figure 17: The norm1 of the residual and error function plots for the largest m value (E) of the figure (16) for $f(x) = 1$, $g(x) = \sin(y(x))$, $N = 12$

cific norm) can illustrate the EOM efficiency, more clearly.

Acknowledgements

The respected anonymous referees have carefully reviewed this paper. As a result of their careful analysis, our paper has been improved. The authors would like to express their thankfulness to them for their helpful constructive comments.

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