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Research Article

Numerical Solution of Fredholm Integro-differential Equations By Using Hybrid Function Operational Matrix of Differentiation

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Abstract

In this paper, first, a numerical method is presented for solving a class of linear Fredholm integro-differential equation. The operational matrix of derivative is obtained by introducing hybrid third kind Chebyshev polynomials and Block-pulse functions. The application of the proposed operational matrix with tau method is then utilized to transform the integro-differential equations to the algebraic equations. Finally, show the efficiency of the proposed method is indicated by some numerical examples.

Keywords : Fredholm integro-differential equation; Hybrid function; Chebyshev polynomial; Block-pulse function; Operational matrix of derivative.

1 Introduction

In recent years, there has been a growing interest in the integro-differential equations, which provide an important tool for modeling numerous real world problem in engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatics. The kinds of equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. Therefore, many different methods are used to obtain the solution of the linear and nonlinear Integro-differential equations such as: the successive approximations, Adomian decomposition, Homotopy perturbation method, Chebyshev and taylor collocation, Hybrid function, Cas and Haar wavelet, Tau and Walsh series methods [1]-[8].

In this paper, a numerical method using hybrid of third kind Chebyshev polynomials and Block-pulse functions (HTKCPBPF) is presented for the following linear Fredholm integro-differential equations of the type:

$$\begin{cases} u'(x) = f(x) + u(x) + \lambda \int_0^1 K(x,t)u(t)dt, \\ u(0) = a, \end{cases} \tag{1.1}$$

where λ, a, are constants, $f(x))$ and $K(x,t)$ are Known and $u(t)$ is the unknown function to be determined. This method reduces the integral equation to a set of algebraic equations by expanding $u(x)$ as (HTKCPBPF) with unknown coefficients. The paper is organized as follows: In Section 2, we review briefly about Block-pulse functions and third kind Chebyshev polynomials and hybrid of them. Section 3 is devoted to function approximation. In Sections 4, we construct

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the operational matrices of derivative based on the (HTKCPBPF). Convergence analysis of the proposed method is done in Section 5. In Section 6 and 7, we show the validity and efficiency of the proposed method, we present some numerical examples. Finally, Section 8 concludes the paper.

2 Function and hybrid function

2.1 Block-pulse functions

A set of Block-pulse functions $b_i(x), i = 1, 2, ..., N$, on the interval $[0, 1)$ are defined as [9]:

$$b_i(x) = \begin{cases} 1, & \frac{i-1}{N} \leq x < \frac{i}{N} \\ 0, & otherwise, \end{cases} \tag{2.2}$$

These functions satisfy in the following properties:

i- disjointness

$$b_i(x)b_j(x) = \begin{cases} b_i(x), & fori = j \\ 0, & fori \neq j \end{cases}$$

ii- orthogonality

$$\int_0^1 b_i(x)b_j(x)dt = \frac{1}{N}\delta_{ij}$$

where $i, j = 1, 2, ..., N$, and δ_{ij} is the Kronecker delta,

iii- completeness

for every $f \in L^2[0, 1)$ when m approach to the infinity, parsevals identity hold:

$$\int_0^1 f^2(x)dx = \sum_0^\infty (f_i^2 \|b_i(x)\|^2),$$

where $f_i = N\int_0^1 f(x)b_i(x)dx$.

2.2 Third kind of Chebyshev polynomials

The third kind of Chebyshev polynomial $V_n(x)$ is a polynomial of degree n in x defined by [9] :

$$V_n(x) = \frac{\cos(n + \frac{1}{2})\theta}{\cos(\frac{1}{2})\theta}, \tag{2.3}$$

where $x = \cos\theta$. clearly from 2.2, fundamental recurrence relation as follows:

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \quad n = 2, 3, ...,$$

where

$$V_0(x) = 1, \quad V_1(x) = 2x - 1,$$

These polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $\omega(x) = \sqrt{\frac{1+x}{1-x}}$, that is

$$\int_{-1}^1 V_i(x)V_j(x)w(x)dx = \pi\delta_{ij}.$$

2.3 Hybrid functions

For $n = 1, ..., N$ and $m = 0, ..., M - 1$, the HTKCPBPF are defined as follows [13]:

$$\varphi_{nm}(x) =$$

$$\begin{cases} \sqrt{\frac{2}{N}}V_m(2Nx - 2n + 1), & \frac{n-1}{N} \leq x < \frac{n}{N} \\ 0, & otherwise, \end{cases} \tag{2.4}$$

with the following weight function

$$\omega_n(x) = \omega(2Nx - 2n + 1).$$

3 Function approximation

A function $f(x) \in L^2[0, 1)$ may be expanded as:

$$f(x) = \sum_{n=1}^\infty \sum_{m=0}^\infty c_{nm}\varphi_{nm}(x), \tag{3.5}$$

where

$$c_{nm} = \frac{\langle f(x), \varphi_{nm}(x)\rangle}{\langle \varphi_{nm}(x), \varphi_{nm}(x)\rangle} = \frac{N^2}{\pi}\int_0^1 \omega_n(x)\varphi_{nm}(x)f(t)dx. \tag{3.6}$$

In 3.6, $\langle .,. \rangle_{L^2_\omega[0,1)}$ denotes the inner product in $L^2_\omega[0,1)$, with weight function $w_n(x)$. If the infinite series in 3.5 is truncated, then equation 3.5 can be written as:

$$f(x) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm}\varphi_{nm}(x) = C^T\varphi(x), \quad (3.7)$$

where C and $\varphi(x)$ are $NM \times 1$ matrices given by:

$$C = [c_{10}, c_{11}, c_{12}, ..., c_{1,M-1}, c_{20}, ..., c_{N,M-1}]^T, \quad (3.8)$$

$$\varphi(x) = [\varphi_{10}(x), \varphi_{11}(x), ..., \varphi_{1,M-1}(x), \varphi_{20}(x) \quad (3.9)$$
$$..., \varphi_{N,M-1}(x)]^T.$$

The differentiation of vector $\varphi(x)$ can be obtained by:

$$\frac{d\varphi(x)}{dx} = D\varphi(x). \quad (3.10)$$

We derive the matrix D in the following section for some particular values of N and M.

4 Operational matrix of derivative

In this section, we figure out the precise derivative of the HTKCPBPF with $N = 2$ and $M = 3$. In this case, the six basis functions are given by :

$$\begin{aligned}\varphi_1 &= \varphi_{10}(x) = 1,\\ \varphi_2 &= \varphi_{11}(x) = 8x - 3,\\ \varphi_3 &= \varphi_{12}(x) = 64x^2 - 40x + 5,\end{aligned} \quad (4.11)$$

for $t \in [0, \frac{1}{2})$, and

$$\begin{aligned}\varphi_4 &= \varphi_{20}(x) = 1,\\ \varphi_5 &= \varphi_{21}(x) = 8x - 7,\\ \varphi_6 &= \varphi_{22}(x) = 64x^2 - 104x + 41,\end{aligned} \quad (4.12)$$

for $t \in [\frac{1}{2}, 1)$. Let $\varphi_6(t) = (\varphi_{10}(t) \ \ \varphi_{11}(t) \ \ \varphi_{12}(t) \ \ \varphi_{20}(t) \ \ \varphi_{21}(t) \ \ \varphi_{22}(t))$. By differentiation 4.12, 4.13 from 0 to t, and representing them in the matrix form, we obtain

$$\begin{aligned}\frac{d\varphi_1}{dx} &= 0,\\ \frac{d\varphi_2}{dx} &= 8 = 8\varphi_{10},\\ \frac{d\varphi_3}{dx} &= 128x - 40 = 16\varphi_{11} + 8\varphi_{10},\\ \frac{d\varphi_4}{dx} &= 0,\\ \frac{d\varphi_5}{dx} &= 8 = 8\varphi_{20},\\ \frac{d\varphi_6}{dx} &= 128x - 104 = 16\varphi_{21} + 8\varphi_{20}.\end{aligned}$$

Thus, we have

$$\frac{d\varphi(x)}{dx} = D_{6\times6}\varphi(x). \quad (4.13)$$

Where

$$D_{6\times6} = 2\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 4 & 0 & 0 & 0 & 0 & 0\\ 4 & 8 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 4 & 0 & 0\\ 0 & 0 & 0 & 4 & 8 & 0\end{bmatrix}$$

The matrix $D_{6\times6}$ can be written as

$$D_{6\times6} = 2\begin{bmatrix}F_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & F_{3\times3}\end{bmatrix}$$

where

$$F_{3\times3} = \begin{bmatrix}0 & 0 & 0\\ 4 & 0 & 0\\ 4 & 8 & 0\end{bmatrix}$$

In general, for $M \geq 4$, we have

$$\frac{d\varphi(x)}{dx} = D\varphi(x), \quad (4.14)$$

where $\varphi(x)$ is given in 3.10 and D is a $NM \times NM$ matrix given by

$$D = N\begin{bmatrix}F & 0 & 0 & \cdots & 0\\ 0 & F & 0 & \cdots & 0\\ 0 & 0 & F & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & F\end{bmatrix},$$

where $F = a_{(ij)}$ is $M \times M$ matrices, whose the elements are given explicitly by:

$$a_{ij} = \begin{cases} 2(i+j-1), & i > j, (i+j) odd,\\ 2(i-j), & i > j, (i+j) even,\\ 0, & otherwise.\end{cases} \quad (4.15)$$

For example if $M = 7$ as follows:

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 & 0 \\ 8 & 4 & 12 & 0 & 0 & 0 & 0 \\ 8 & 12 & 4 & 16 & 0 & 0 & 0 \\ 12 & 8 & 16 & 4 & 20 & 0 & 0 \\ 12 & 16 & 8 & 20 & 4 & 24 & 0 \end{bmatrix}_{7\times 7},$$

Using the above procedure, the operational matrix of nth derivative can be derived as:

$$\frac{d^n\varphi(x)}{dx^n} = D^n\varphi(x), \tag{4.16}$$

The integration of two HTKCPBPF vectors is obtained as

$$E = \int_0^1 \varphi(t)\varphi(t)^T dt, \tag{4.17}$$

where E is a $NM \times NM$ symmetric matrix. For example if $N = 1$ and $M = 6$ as follows:

$$E = \frac{1}{N^2}.$$

$$\begin{bmatrix} 2 & -2 & \frac{2}{3} & -\frac{2}{3} & \frac{2}{5} & -\frac{2}{5} \\ -2 & \frac{14}{3} & -\frac{10}{3} & \frac{26}{15} & -\frac{22}{15} & \frac{38}{35} \\ \frac{2}{3} & -\frac{10}{3} & \frac{86}{15} & -\frac{62}{15} & \frac{254}{105} & -\frac{214}{105} \\ -\frac{2}{3} & \frac{26}{15} & -\frac{62}{15} & \frac{674}{105} & -\frac{494}{105} & \frac{922}{315} \\ \frac{2}{5} & -\frac{22}{15} & \frac{254}{105} & -\frac{494}{105} & \frac{2182}{315} & -\frac{1622}{315} \\ -\frac{2}{5} & \frac{38}{35} & -\frac{214}{105} & \frac{922}{315} & -\frac{1622}{315} & \frac{25402}{3465} \\ \frac{2}{7} & -\frac{34}{35} & \frac{502}{315} & -\frac{782}{315} & \frac{11542}{3465} & -\frac{19102}{3465} \end{bmatrix}$$

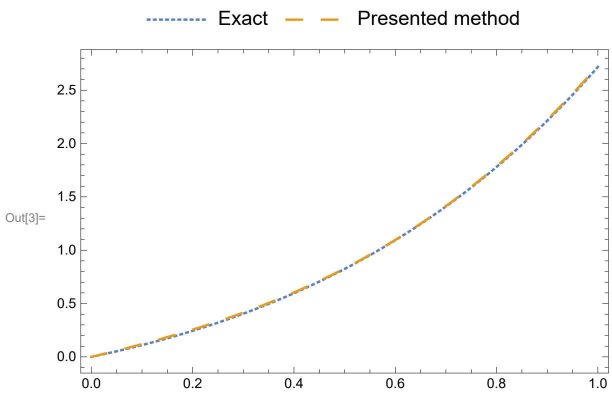

Figure 1: The exact and Presented method solution of Example 7.2

5 Convergence analysis

The following theorem gives the convergence and accuracy estimation of HTKCPBPF.

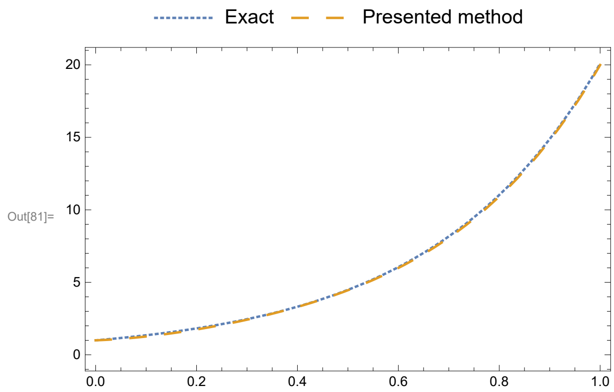

Figure 2: The exact and Presented method solution of Example 7.3

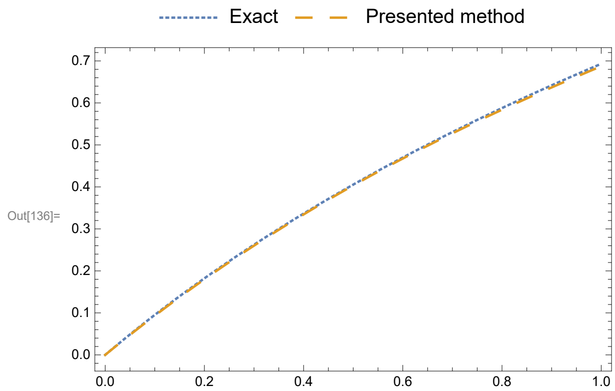

Figure 3: The exact and Presented method solution of Example 7.4

Theorem 5.1 *Let $f(x)$ be a second-order derivative square-integrable function defined on $[0, 1)$ with bounded second-order derivative, say $|f''(x)| \leq A$ for some constant A, then*

(i) $f(x)$ can be expanded as an infinite sum of the HTKCPBPF and the series converges to $f(x)$ uniformly, that is

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\varphi_{nm}(t),$$

where $c_{nm} = \langle f(x), \varphi_{nm}(x)\rangle_{L^2_\omega[0,1)}$.
(ii)

$$\beta_{f,n,M} \leq \frac{\pi A^2}{8}\sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(m-1)^4},$$

where

$$\beta_{f,n,M} =$$

$$[\int_0^1 |f(x) - \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm}\varphi_{nm}(x)|^2\omega_n(x)dx]^{\frac{1}{2}}.$$

Table 1: shows some valuse of the solutions and absolute errors

x	Hybrid function	Exact solution	Absolute error
0.0	1.11022×10^{-16}	0.0000000000	1.11022×10^{-16}
0.1	0.1195898162	0.1105170918	0.00907270
0.2	0.2556578961	0.2442805516	0.01137730
0.3	0.4147980500	0.4049576422	0.00984041
0.4	0.6036040883	0.5967298790	0.00687421
0.5	0.8286698212	0.8243606353	0.00430919
0.6	1.0965890592	1.0932712802	0.00331778
0.7	1.4139556126	1.4096268952	0.00432872
0.8	1.7873632916	1.7804327427	0.00693055
0.9	2.2234059067	2.2136428000	0.00976311
1	2.7286772681	2.7182818284	0.01039540

Table 2: shows some value of the solutions and absolute errors

x	Hybrid function	Exact solution	Absolute error
0.0	1	1	0
0.2	1.7471755121	1.8221188003	0.074943679
0.4	3.3022086903	3.3201169222	0.017908232
0.6	5.9881829326	6.0496474644	0.061464531
0.8	10.917173846	11.023176380	0.106002533
1	19.990251205	20.085536923	0.095285717

Table 3: shows some value of the solutions and absolute errors

x	Hybrid function	Exact solution	Absolute error
0.0	1.14492×10^{-16}	0.0000000000	1.4492×10^{-16}
0.2	0.1820778778	0.1823215567	0.000243679
0.4	0.3363045784	0.3364722366	0.000167658
0.6	0.4696357572	0.4700036292	0.000367872
0.8	0.5872265339	0.5877866642	0.000560131
1	0.6924314924	0.6931471805	0.000715688

Proof. To prove (i), we have:

$$(i) c_{nm} = \langle f(x), \varphi_{nm}(x) \rangle_{L^2_\omega[0,1)}$$

$$= \frac{N^2}{\pi} \int_0^1 \omega_n(x) \varphi_{nm}(x) f(x) dx$$

$$= \frac{N^2}{\pi} \int_{\frac{n-1}{N}}^{\frac{n}{N}} f(x) \sqrt{\frac{2}{N}} V_m(2Nx - 2n + 1)$$

$$\omega(2Nx - 2n + 1) dx.$$

Let $t = (2Nx - 2n + 1)$ then $dt = 2Ndx$. Clearly, we have

$$c_{nm} = \frac{N}{2\pi} \sqrt{\frac{2}{N}} \int_{-1}^{1} f(\frac{t + 2n - 1}{2N}) V_m(t) \sqrt{\frac{1+t}{1-t}} dt.$$

By letting $t = cos\theta$ and the definition of the HTKCPBPF, it follows that

$$c_{nm} = \frac{N}{2\pi} \sqrt{\frac{2}{N}} \int_0^\pi f(\frac{cos\theta + 2n - 1}{2N})$$

$$(cos\ m\theta + cos(m+1)\theta) d\theta$$

$$= \frac{N}{2\pi} \sqrt{\frac{2}{N}} [\int_0^\pi f(\frac{cos\theta + 2n - 1}{2N}) cos\ m\theta$$

$$+ \int_0^\pi f(\frac{cos\theta + 2n - 1}{2N}) cos(m+1)\theta d\theta].$$

Using the integration by parts, we have

$$c_{nm} = \sqrt{\frac{2}{N}} \frac{1}{4\pi} [\frac{1}{m} \int_0^\pi f'(\frac{cos\theta + 2n - 1}{2N})$$

$$(sin\ m\theta sin\theta) d\theta +$$

$$\frac{1}{m+1}\int_0^{\pi} f'(\frac{cos\theta+2n-1}{2N})$$
$$(sin(m+1)\theta sin\theta)d\theta] = \sqrt{\frac{2}{N}}\frac{1}{4\pi}[I_1+I_2], \quad (5.18)$$

where

$$I_1 = \frac{1}{m}\int_0^{\pi} f'(\frac{cos\theta+2n-1}{2N})(sin\ m\theta sin\theta)d\theta,$$

and

$$I_2 = \frac{1}{m+1}\int_0^{\pi} f'(\frac{cos\theta+2n-1}{2N})(sin(m+1)\theta sin\theta)d\theta.$$

Now, we estimate I_1 and I_2, respectively. A simple computation shows that

$$I_1 = \frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta+2n-1}{2N})[cos(m-1)\theta - cos(m+1)\theta]d\theta$$
$$= \frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta+2n-1}{2N})[cos(m-1)\theta d\theta - \frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta+2n-1}{2N})cos(m+1)\theta]d\theta = I_{11} - I_{12},$$

where

$$I_{11} = [\frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta+2n-1}{2N})cos(m-1)\theta d\theta,$$

and

$$I_{12} = \frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta+2n-1}{2N})cos(m+1)\theta d\theta.$$

By using the integration by parts, and for $m > 1$, we get

$$I_{11} = \frac{1}{4mN(m-1)}\int_0^{\pi} f''(\frac{cos\theta+2n-1}{2N})[sin(m-1)\theta sin\theta]d\theta$$
$$= \frac{1}{8mN(m-1)}\int_0^{\pi} f''(\frac{cos\theta+2n-1}{2N})[cos(m-2)\theta d\theta - cos\ m\theta]d\theta,$$

$$I_{12} = \frac{1}{4mN(m+1)}\int_0^{\pi} f''(\frac{cos\theta+2n-1}{2N})[sin(m+1)\theta sin\theta]d\theta$$
$$= \frac{1}{8mN(m+1)}\int_0^{\pi} f''(\frac{cos\theta+2n-1}{2N})[cos\ m\theta - cos(m+2)\theta]d\theta.$$

Thus, for $m > 1$,we conclude that

$$I_1 = \frac{1}{8mN}\int_0^{\pi} f''(\frac{cos\theta+2n-1}{2N})[\frac{cos(m-2)\theta - cos\ m\theta}{(m-1)} - \frac{cos\ m\theta - cos(m+2)\theta}{(m+1)}]d\theta,$$

and hence

$$|I_1|^2 = |\frac{1}{8mN}\int_0^{\pi} f''(\frac{cos\theta+2n-1}{2N})[\frac{cos(m-2)\theta - cos\ m\theta}{(m-1)} - \frac{cos\ m\theta - cos(m+2)\theta}{(m+1)}]d\theta|^2$$
$$= \frac{1}{64m^2N^2}|\int_0^{\pi} f''(\frac{cos\theta+2n-1}{2N})[\frac{cos(m-2)\theta - cos\ m\theta}{(m-1)} - \frac{cos\ m\theta - cos(m+2)\theta}{(m+1)}]d\theta|^2.$$

By the fact that $|f''(x)|\leq A$ and Schwartz inequality, it follows that

$$|I_1|^2 \leq \frac{\pi A^2}{64m^2N^2(m-1)^2(m+1)^2}$$

$$\int_0^\pi |(m+1)cos(m-2)\theta + 2mcos\ m\theta +$$

$$(m-1)cos(m+2)\theta|^2 d\theta$$

$$= \frac{\pi A^2}{64m^2N^2(m-1)^2(m+1)^2}\times$$

$$[\int_0^\pi (m+1)^2 cos^2(m-2)\theta d\theta +$$

$$\int_0^\pi 4m^2 cos^2 m\theta d\theta +$$

$$\int_0^\pi (m-1)^2 cos^2(m+2)\theta d\theta]$$

$$= \frac{\pi A^2}{64m^2N^2(m-1)^2(m+1)^2}$$

$$[\frac{\pi}{2}(m+1)^2 + \frac{\pi}{2}4m^2 + \frac{\pi}{2}(m-1)^2]$$

$$= \frac{\pi^2 A^2(3m^2+1)}{64m^2N^2(m-1)^2(m+1)^2}$$

$$\leq \frac{\pi^2 A^2}{4N^2(m-1)^4}.$$

For $m > 2$, we obtain

$$|I_1| \leq \frac{\pi A}{2N(m-1)^2} \tag{5.19}$$

In a similar way, we will have

$$|I_2| \leq \frac{\pi A}{2N(m-1)^2} \tag{5.20}$$

Therefore, for $m > 2$, we conclude that

$$\begin{aligned} |c_{nm}| &= |\frac{1}{4\pi}\sqrt{\frac{2}{N}}[I_1 + I_2]| \\ &\leq \frac{1}{4\pi}\sqrt{\frac{2}{N}}\frac{\pi A}{N(m-1)^2} \leq \frac{A}{2\sqrt{2}}\frac{1}{n^{\frac{3}{2}}(m-1)^2} \end{aligned} \tag{5.21}$$

Note that $f'(x)$ is bounded on $[0,1)$ due to the fact that $|f''(x)|\leq A$, indeed, by the Differential Mean Value Theorem and for any $t \in (0,1)$, there exists some $\gamma_x \in (0,x)$ such that

$$f'(x) - f'(0) = f''(\gamma_x)x,$$

So

$$|f'(x)| \leq |f'(0)| + A,$$

for $x \in (0,1)$. Thus $f'(x)$ is bounded on $[0,1)$,say $|f'(x)|\leq \tilde{A}$ for some constant $\tilde{A}$. Hence, by 5.18, we have

$$\begin{aligned} |c_{n,1}| &\leq \sqrt{\frac{2}{N}}\frac{1}{4\pi}[\int_0^\pi |f'(\frac{cos\theta + 2n - 1}{2N})|d\theta \\ &+ \frac{1}{2}\int_0^\pi |f'(\frac{cos\theta + 2n - 1}{2N})|d\theta] \\ &= \sqrt{\frac{2}{N}}\frac{1}{4\pi}\frac{3}{2}\int_0^\pi |f'(\frac{cos\theta + 2n - 1}{2N})|d\theta \\ &\leq \sqrt{\frac{2}{N}}\frac{1}{4\pi}\frac{3\pi\tilde{A}}{2} = \frac{3\tilde{A}}{4\sqrt{2}n^{\frac{1}{2}}} \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} |c_{n,2}| &\leq \sqrt{\frac{2}{N}}\frac{1}{4\pi}[\frac{1}{2}\int_0^\pi |f'(\frac{cos\theta + 2n - 1}{2N})|d\theta \\ &+ \frac{1}{3}\int_0^\pi |f'(\frac{cos\theta + 2n - 1}{2N})|d\theta] \\ &= \sqrt{\frac{2}{N}}\frac{1}{4\pi}\frac{5}{6}\int_0^\pi |f'(\frac{cos\theta + 2n - 1}{2N})|d\theta \\ &\leq \sqrt{\frac{2}{N}}\frac{1}{4\pi}\frac{5\pi\tilde{A}}{6} = \frac{5\tilde{A}}{12\sqrt{2}n^{\frac{1}{2}}} \end{aligned} \tag{5.23}$$

Relations 5.21-5.23 show that the series$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}$ is absolutely convergent. For $m = 0$ and according to the definition of $\varphi_{n,0}(x)$, the series $\sum_{n=1}^{\infty} c_{n,0}\varphi_{n,0}(x)$ is convergent. Therefore, the series$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\varphi_{nm}(x)$ converges to $f(x)$ uniformly.

(ii)

$$\beta^2_{f,n,M} = \int_0^1 |f(x) - \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm}\varphi_{nm}(x)|^2\omega_n(x)dx$$
$$= \int_0^1 |\sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty} y_{nm}\varphi_{nm}(x)|^2\omega_n(x)dx$$
$$= \sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty} |y_{nm}|^2.$$
$$(\sqrt{\frac{2}{N}})^2 \int_{\frac{n-1}{N}}^{\frac{n}{N}} V_m(2Nx-2n+1)^2.$$
$$\sqrt{\frac{1+(2Nx-2n+1)}{1-(2Nx-2n+1)}}dx.$$

Let $t = 2Nx - 2n + 1$ then $dt = 2ndx$.

Therefore

$$\beta^2_{f,n,M} = \sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty} |c_{nm}|^2 \frac{1}{N^2}\int_{-1}^{1} V_m^2(t)\sqrt{\frac{1+t}{1-t}}dt,$$

we have

$$\int_{-1}^{1} V_m^2(t)\sqrt{\frac{1+t}{1-t}}dt = \pi,$$

where the last equality follows due to the orthogonality of $\varphi_{nm}(x)$. Together with 5.21 we get

$$\beta^2_{f,n,M} \le \frac{\pi A^2}{8}\sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(m-1)^4}.$$

6 Solution of Fredholm integro-differential equation

Consider the linear Fredholm integro-differential equation given by 1.1. We approximate $u(x)$,$K(x,t)$,$f(x)$ by the way mentioned in section 2 as:

$$u(x) = C^T\varphi(x),\ K(x,t) = \varphi^T(x)K\varphi(t),$$
$$f(x) = F^T\varphi(x), \qquad (6.24)$$

by using 4.14 we have

$$u'(x) = C^T D\varphi(x). \qquad (6.25)$$

With substituting in 1.1 we have

$$C^T D\varphi(x) = F^T\varphi(x) + C^T\varphi(x) + \int_0^1 \varphi^T(x)K\varphi(t)\varphi^T(t)Cdt. \qquad (6.26)$$

Applying 4.16, the residual $R(x)$ for 1.1 can be written as:

$$R(x) = C^T D\varphi(x) - F^T\varphi(x) - C^T\varphi(x) - \varphi^T(x)KEC. \qquad (6.27)$$

As in a typical Tau method, we generate $NM-1$ linear equations by applying

$$\int_0^1 \omega(x)R(x)\varphi_i(x)dx = 0,\ \imath = 0,1,...,NM-1. \qquad (6.28)$$

Also, by substituting initial conditions 1.1 we have

$$u(0) = C^T\varphi(0) = a, \qquad (6.29)$$

Eqs. 6.28-6.29 generate NM set of linear equations. These linear equations can be solved for unknown coefficients of the vector C.

7 Numerical Examples

In this Section, linear Fredholm integro-differential equation have been solved using the proposed method.

Example 7.1 *Consider the integro-differential equation*

$$\begin{cases} u'(x) = 1 - \frac{1}{3}x + \int_0^1 xtu(t)dt, \\ u(0) = 0. \end{cases} \qquad (7.30)$$

with the exact solution $u(x) = x$. We apply the method that was explained in Section 6 for $N = 1, M = 3$. After performing some manipulations, the components of the vector C are given by

$$u(x) = C^T\varphi(x).$$
$$c_0 = \frac{3}{4\sqrt{2}}, \quad c_1 = \frac{1}{4\sqrt{2}}, \quad c_2 = 0.$$

Thus

$$u(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + c_2\varphi_2(x) \qquad (7.31)$$
$$= \begin{pmatrix} \frac{3}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} & 0 \end{pmatrix}.$$
$$\begin{pmatrix} \sqrt{2} \\ \sqrt{2}(4x-3) \\ \sqrt{2}(16x^2-20x+5) \end{pmatrix} = x, \qquad (7.32)$$

which is the exact solution.

Example 7.2 *Consider the integro-differential equation*

$$\begin{cases} u'(x) = xe^x + e^x - x + \int_0^1 xu(t)dt, \\ u(0) = 0. \end{cases} \quad (7.33)$$

with the exact solution $u(x) = xe^x$. We apply the method that was explained in Section 6 for $N = 1, M = 4$. After performing some manipulations, the components of the vector C are given by

$$c_0 = 1.24557, \quad c_1 = 0.564918,$$
$$c_2 = 0.106836, \quad c_3 = 0.012142,$$

Thus

$$u(x) = 1.11022 \times 10^{-16} + 1.13549x + 0.494223x^2 + 1.09897x^3. \quad (7.34)$$

Table 1 shows some values of the solutions and absolute errors at some x, and plot of the exact and approximate solutions are shown in Figure 1.

Example 7.3 *Consider the integro-differential equation*

$$\begin{cases} u'(x) = 3e^{3x} - \frac{1}{3}(2e^3 + 1)x + \int_0^1 3xtu(t)dt, \\ u(0) = 1. \end{cases} \quad (7.35)$$

with the exact solution $u(x) = e^{3x}$. We apply the method that was explained in Section 6 for $N = 1, M = 5$. After performing some manipulations, the components of the vector C are given by

$$c_0 = 8.27245, \quad c_1 = 4.1692,$$
$$c_2 = 1.32412, \quad c_3 = 0.312728,$$
$$c_4 = 0.0567528.$$

Thus

$$u(x) = 1 + 1.26846x + 15.1003x^2 - 17.9252x^3 + 20.5467x^4. \quad (7.36)$$

Table 2 shows some values of the solutions and absolute errors at some x, and plot of the exact and approximate solutions are shown in Figure 2.

Example 7.4 *Consider the integro-differential equation*

$$\begin{cases} u'(x) = u(x) - \frac{1}{2}x + \frac{1}{1+x} - ln(1+x) + \frac{1}{(ln2)^2} \int_0^1 \frac{x}{1+t} y(t)dt, \\ u(0) = 0, \end{cases} \quad (7.37)$$

with the exact solution $u(x) = ln(1 + x)$. We apply the method that was explained in Section 6 for $N = 1, M = 5$. After performing some manipulations, the components of the vector C are given by

$$c_0 = 0.38715, \quad c_1 = 0.110789,$$
$$c_2 = -0.00924358, \quad c_3 = 0.00105708,$$
$$c_4 = -0.000129514.$$

Thus

$$u(x) = 1.14492 \times 10^{-16} + 0.993861x - 0.455717x^2 + 0.201176x^3 - 0.046889x^4. \quad (7.38)$$

Table 3 shows some values of the solutions and absolute errors at some x, and plot of the exact and approximate solutions are shown in Figure 3.

8 Conclusion

In this paper, we constructed operational matrix of derivative of hybrid the third kind Chebyshev polynomials and Block-pulse functions. Also, we applied these matrices to convert Fredholm integro-differential equations to system of linear algebraic equations. As to validity and efficiency of the proposed method, we presented some numerical examples.

References

[1] S. H. Behiry, Solution of nonlinear Fredholm integro-differential equations using a hybrid of Block pulse functins and normalized Bernstein polynomials, *Comput. Appl. Math.* 260 (2014)258-265

[2] H. Danfu, Sh. Xufeng, Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration, *Appl. Math. Comput.* 194 (2007) 460-466

[3] M. Dehghan, F. Shakeri, Approimate soluation of differential equation arising in astrophysics using the variation method, *New Astronomy* 13 (2008) 53-59

[4] P. Darania, A. Ebadian, A method for the numerical solution of the integro-differential equations, *Appl. Math. Comput.* (2006), http://doi:10.1016/j.amc.2006.10.046/.

[5] R. Ezzati, S. Najafalizadeh, Numerical solution of nonlinear Volterra-Fredholm integral equation by using Chebyshev polynomials, *Appl. Math. Comput.* (2003) 1530-1539.

[6] A. Golbabai, M. Javidi, Application of He's homotopy perturbation method for nth-order integro-differential equations, *Appl. Math. Comput.* 190 (2007) 1409-1416.

[7] J. Hou, Ch. Yang, Numerical method in solving Fredholm integro-differential equations by using hybrid functin operational matrix of derivative, *Journal of Information and Computational Science* (2013) 2757-2764

[8] H. Jaradat, O. Alsayyed, S. Al-Shara, Numerical solution of linear integro-differential equations, *Journal of Mathematics and Statistics* 11 (2008) 250-254.

[9] Z. H. Jiang, W. Schaufelberger, 1991, Block-pulse Functions and their Applications in Contorl Systems, *Springer-Verlag Berlin Heidelberg New York* 37.

[10] A. Karamete, M. Sezer, A Taylor collocation method for the solution of linear integro-differential equations, *Int. J. Comput. Math.* 79 (2002) 987-1000

[11] S. J. Liao, An explicit analytic solution to the Thomas-Fermi equation, *Appl. Math. Comput.* 144 (2003) 495-506

[12] K. Maleknejad, F. Mirzaee, Numerical solution of integro-differential equations by using rationalized Haar functions method, Kybernetes, *Int. J. Syst. Math.* 35 (2006) 1735-1744

[13] K. Maleknejad, M. Tavassoli Kajani, A hybrid collocation metod based on combining the third kind Chebyshev polynomials and Block-pulse functions for solving higer-order initial value problems, *Kuwait journal of Science* (2016).

[14] J. C. Mason, D. C. Handscomb, *Chebyshev polynomials* (2002) ISBN:0-8493-0355-9.

[15] S. Yeganeh, Y. Ordokhani, A. Saadatmandi, A Sinc-collocation method for second-order boundary value problems of nonlinear integro-differential equation, *J. Inf. Comput. Sci.* 7 (2012) 151-160.

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