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Research Article

Computational technique of linear partial differential equations by reduced differential transform method

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Abstract

This paper presents a class of theoretical and iterative method for linear partial differential equations. An algorithm and analytical solution with a initial condition is obtained using the reduced differential transform method. In this technique, the solution is calculated in the form of a series with easily computable components. There test modeling problems from mathematical mechanic, physic, electronic and so on, and are discussed to illustrate the effectiveness and the performance of the our method.

Keywords : Reduced differential transform method; Taylor series; Parabolic equations; Hyperbolic equations.

1 Introduction

Linear partial differential equations (LPDEs) arise in the formulation of fundamental laws of nature and in the mathematical analysis of a wide variety of problems in applied mathematics, mathematical physic, mechanic and engineering science. For instance,

1- The heat or diffusion equation: this equation describes the diffusion of thermal energy in a medium. It can be used to model the flow of a quantity, such as heat, or a concentration of particles. It is also used as a model equation for growth and diffusion, in general, and growth of a solid tumor, in particular.

2- The wave equation: this equation describes the propagation of a wave (or disturbance), and it arises in a wide variety of physical problems. Some of these problems include a vibrating string, longitudinal vibrations of an elastic rod or beam, transmission of electric signals along a cable.

3- The telegraph equation: this equation arises in the study of propagation of electrical signals in a cable of a transmission line.

4- And so on [15].

Many problems of physical are described by LPDEs with appropriate initial and/or boundary conditions. In this paper, the reduced differential transformation method (RDTM) is presented for LPDEs with initial conditions in a general form. Also, a recursive formula of the RDTM is introduced that is applied for a wide variety of LPDEs specially of the types parabolic and hyperbolic equations.

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2 Differential transform method (DTM)

The DTM was first proposed by Zhou [16], who solved linear and nonlinear initial value problems in electric circuit analysis, and was used heavily in the literature successfully applied to eigenvalue problems [7], one-dimensional planar Bratu problem [1], higher-order initial value problems [2, 8], systems of ordinary and partial differential equations [3, 5], high index differential-algebraic equations [14], integro-differential equations [4].

2.1 Two dimensional the differential transform method

The basic definitions and fundamental operations of the two-dimensional differential transform are introduced in [9] as the following

$$W(k,h)=\frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial t^h}w(x,t)\right]_{(0,0)}, \qquad (2.1)$$

where $w(x,t)$ is the original function and $W(k,h)$ is the transformed function. The differential inverse transform of $W(k,h)$ is

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^kt^h, \qquad (2.2)$$

and from Eqs. (2.1) and (2.2) can be concluded

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial t^h}w(x,t)\right]_{(0,0)}x^kt^h. \qquad (2.3)$$

In Table 1 has listed the fundamental mathematical operations of two-dimensional differential transform. The proofs of Table 1 are available in [6].

2.2 The reduced differential transform method

The basic definitions and operations of the RDTM [11, 12, 13] are defined as follows:

Definition 2.1 *If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let*

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \qquad (2.4)$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.

Definition 2.2 *The reduced differential transform of a sequence $\{U_k(x)\}_{k=0}^{\infty}$ is introduced as follows:*

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k. \qquad (2.5)$$

To combining equation (2.4) and (2.5), we have

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}t^k. \qquad (2.6)$$

Some basic properties of the reduced differential transformation obtained from definitions (2.4) and (2.6) are summarized in Table 2. The proofs of Table 2 and the basic definitions of the RDTM are available in [10].

3 The RDTM for LPDE

Our main result is an application of the RDTM for LPDE. Consider the following LPDE

$$\sum_{n=0}^{N-1}a_n(x,t)G_nu(x,t)+G_Nu(x,t)=\sum_{m=1}^{M}b_m(x,t)H_mu(x,t)+f(x,t), \qquad (3.7)$$

with the initial conditions

$$G_nu(x,0)=g_n(x), \qquad 0\le n\le N-1, \qquad (3.8)$$

where

$$G_n=\frac{\partial^n}{\partial t^n}, \quad 0\le n\le N,$$
$$H_m=\frac{\partial^m}{\partial x^m}, \quad 1\le m\le M.$$

The approximate solution using the t partial solution is given by

$$u(x,t)=\psi+G_N^{-1}\left(f(x,t)\right)+G_N^{-1}\left(\sum_{m=1}^{M}b_m(x,t)H_mu(x,t)\right)-G_N^{-1}\left(\sum_{n=0}^{N-1}a_n(x,t)G_nu(x,t)\right), \qquad (3.9)$$

where

$$\psi=u(x,0)+tu_t(x,0)+\cdots+t^{N-1}\frac{\partial^{N-1}u(x,0)}{\partial t^{N-1}}=g_0(x)+tg_1(x)+\cdots+\frac{1}{(N-1)!}t^{N-1}g_{N-1}(x)=\sum_{l=0}^{N-1}\frac{1}{l!}t^lg_l(x),$$

Table 1: Two dimensional differential transformation

Orig. Fun.	Transformed Fun.
$u(x,t) \pm v(x,t)$	$U(k,h) \pm V(k,h)$
$cu(x,t)$	$cU(k,h)$
$\frac{\partial u(x,t)}{\partial x}$	$(k+1)U(k+1,h)$
$\frac{\partial u(x,t)}{\partial t}$	$(h+1)U(k,h+1)$
$\frac{\partial^{r+s} u(x,t)}{\partial x^r \partial t^s}$	$\frac{(k+r)!}{k!}\frac{(h+s)!}{h!}$ $U(k+r,h+s)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k}\sum_{s=0}^{h}$ $U(r,h-s)V(k-r,s)$

Table 2: Basic operations of RDTM

Orig. Fun.	Transformed Fun.
$u(x,t)$	$U_k(x)$
$u(x,t) \pm v(x,t)$	$U_k(x) \pm V_k(x)$
$cu(x,t)$	$cU_k(x)$ c is a cons.
$x^m t^n$	$x^m \delta(k-n)$
$x^m t^n u(x,t)$	$x^m U_{k-n}(x)$
$\frac{\partial}{\partial x} u(x,t)$	$\frac{\partial}{\partial x} U_k(x)$
$\frac{\partial^r}{\partial t^r} u(x,t)$	$\frac{(k+r)!}{k!} U_{k+r}(x)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k} U_r(x) V_{k-r}(x)$

and

$$G_N^{-1} = \int_0^t \int_0^t \cdots \int_0^t (.) \underbrace{dtdt\cdots dt}_{N times}. \quad (3.10)$$

According to the RDTM, we consider the transformations of the functions $u(x,t)$, $f(x,t)$, $a_n(x,t)$ and $b_m(x,t)$ as the following

$$\begin{aligned} u(x,t) &= \textstyle\sum_{k=0}^{\infty} U_k(x) t^k, \\ f(x,t) &= \textstyle\sum_{k=0}^{\infty} F_k(x) t^k, \\ a_n(x,t) &= \textstyle\sum_{k=0}^{\infty} A_{n,k}(x) t^k, \\ b_m(x,t) &= \textstyle\sum_{k=0}^{\infty} B_{m,k}(x) t^k, \end{aligned}$$

where

$$\begin{aligned} U_k(x) &= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}, \\ F_k(x) &= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} f(x,t)\right]_{t=0}, \\ A_{n,k}(x) &= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} a_n(x,t)\right]_{t=0}, \\ B_{m,k}(x) &= \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} b_m(x,t)\right]_{t=0}, \end{aligned} \quad (3.11)$$

and $0 \le n \le N-1$, $1 \le m \le M$. To substitute the relations (3.11) into Eq. (3.9), we have

$$\begin{aligned} \textstyle\sum_{k=0}^{\infty} U_k(x) t^k = \sum_{l=0}^{N-1} \frac{1}{l!} t^l g_l(x) + \\ G_N^{-1}\left(\textstyle\sum_{k=0}^{\infty} F_k(x) t^k\right) \\ +G_N^{-1}\left(\textstyle\sum_{m=1}^{M}\sum_{k=0}^{\infty} \Omega_{m,k}(x) t^k\right) \\ -G_N^{-1}\left(\textstyle\sum_{n=0}^{N-1}\sum_{k=0}^{\infty} \Phi_{n,k}(x) t^k\right), \end{aligned} \quad (3.12)$$

where by Table 2, $\Omega_{m,k}(x)$ and $\Phi_{n,k}(x)$ are as follows

$$\begin{aligned} \Omega_{m,k}(x) &= \textstyle\sum_{r=0}^{k} B_{m,r}(x) H_m U_{k-r}(x), \\ \Phi_{n,k}(x) &= \textstyle\sum_{r=0}^{k} \frac{(k-r+n)!}{(k-r)!} A_{n,r}(x) U_{k-r+n}(x). \end{aligned}$$

We now perform the integrations (3.10) on the Eq. (3.12) to write

$$\begin{aligned} &\sum_{k=0}^{\infty} U_k(x)t^k = \\ &\sum_{l=0}^{N-1} \frac{t^l}{l!} g_l(x) + \sum_{k=0}^{\infty} \frac{k!\ t^{k+N}}{(k+N)!} F_k(x) \\ &+ \sum_{k=0}^{\infty} \sum_{r=0}^{k} \sum_{m=1}^{M} \frac{k!\ t^{k+N}}{(k+N)!} \\ &\qquad B_{m,r}(x) H_m U_{k-r}(x) \\ &- \sum_{k=0}^{\infty} \sum_{r=0}^{k} \sum_{n=0}^{N-1} \frac{(k-r+n)!\, k!\, t^{k+N}}{(k-r)!(k+N)!} \\ &\qquad A_{n,r}(x) U_{k-r+n}(x). \end{aligned} \tag{3.13}$$

Let $k \to k - N$ on the right side, then

$$\begin{aligned} &\sum_{k=0}^{\infty} U_k(x)t^k = \\ &\sum_{l=0}^{N-1} \frac{t^l}{l!} g_l(x) + \sum_{k=N}^{\infty} \frac{(k-N)!\ t^k}{k!} F_{k-N}(x) \\ &+ \sum_{k=N}^{\infty} \sum_{r=0}^{k-N} \sum_{m=1}^{M} \frac{(k-N)!\ t^k}{k!} \\ &\qquad B_{m,r}(x) H_m U_{k-N-r}(x) \\ &- \sum_{k=N}^{\infty} \sum_{r=0}^{k-N} \sum_{n=0}^{N-1} \frac{(k-N-r+n)!\ (k-N)!\ t^k}{(k-r)!k!} \\ &\qquad A_{n,r}(x) U_{k-N-r+n}(x). \end{aligned} \tag{3.14}$$

At last, equation coefficients of the same powers of t, we obtain the recursive formula for coefficients as the following

$$\begin{aligned} &U_0(x) = g_0(x),\ \ U_1(x) = g_1(x),\ \ldots, \\ &U_{N-1}(x) = \tfrac{1}{(N-1)!} g_{N-1}(x), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &U_{k+N}(x) = \tfrac{k!}{(k+N)!} F_k(x) + \\ &\sum_{r=0}^{k} \sum_{m=1}^{M} \tfrac{k!}{(k+N)!} B_{m,r}(x) H_m U_{k-r}(x) \\ &- \sum_{r=0}^{k} \sum_{n=0}^{N-1} \tfrac{(k-r+n)!\ k!}{(k-r)!(k+N)!} \\ &\qquad A_{n,r}(x) U_{k-r+n}(x), \\ &k = 0, 1, 2, \ldots. \end{aligned} \tag{3.16}$$

Substituting (3.15) into (3.16) and by a straight forward iterative calculations, we obtain the following $U_k(x)$ values. So, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{p}$ give approximate solution as

$$u_p(x,t) \approx \sum_{k=0}^{p} U_k(x) t^k,$$

where p is order of approximation solution. In result,the exact solution of problem is given by

$$u(x,t) = \lim_{p\to\infty} u_p(x,t).$$

Let us consider the error functional for p-order approximate solution as the following

$$\begin{aligned} &Error(x,t) = \\ &|\sum_{n=0}^{N-1} a_n(x,t) G_n u_p(x,t) + G_N u_p(x,t) \\ &- \sum_{m=1}^{M} b_m(x,t) H_m u_p(x,t) - f(x,t)|. \end{aligned} \tag{3.17}$$

4 Applications

The recursive formula (3.16) with (3.15) applies for a rather wide class of the LPDEs to the initial conditions. As application, we consider the examples of parabolic and hyperbolic equations.

4.1 The parabolic equations

Example 4.1 *A parabolic equation that describes heat transfer in a quiescent medium (solid body) in the case where thermal diffusivity is an exponential function of the coordinate as the following*

$$\frac{\partial u}{\partial t} = a(e^{\beta x} \frac{\partial^2 u}{\partial x^2} + \beta e^{\beta x} \frac{\partial u}{\partial x}), \tag{4.18}$$

where a and β are constant. By attention to (3.7), we have

$$\begin{aligned} &N = 1, M = 2,\ a_0(x,t) = 0,\ f(x,t) = 0, \\ &b_1(x,t) = a\beta e^{\beta x},\ \ b_2(x,t) = a e^{\beta x}, \end{aligned}$$

and assume that the initial condition is $u(x,0) = e^{-x}$. Therefore, from (3.16) we obtain

$$\begin{aligned} &U_0(x) = e^{-x}, \\ &U_{k+1}(x) = \sum_{r=0}^{k} \tfrac{1}{k+1} (B_{1,r}(x) \tfrac{\partial}{\partial x} U_{k-r}(x) \\ &\qquad + B_{2,r}(x) \tfrac{\partial^2}{\partial x^2} U_{k-r}(x)) \\ &\qquad - \sum_{r=0}^{k} \tfrac{1}{k+1} A_{0,r}(x) U_{k-r}(x), \\ &k = 0, 1, 2, \ldots, \end{aligned} \tag{4.19}$$

where $B_{1,r}(x)$, $B_{2,r}(x)$ and $A_{0,r}(x)$ are determined by (3.11). We consider

$$a = 0.05, \qquad \beta = -1, \qquad p = 15,$$

and obtain the approximate solution by (4.19). The approximate solution and error functional have been shown in figures 1 and 2, respectively.

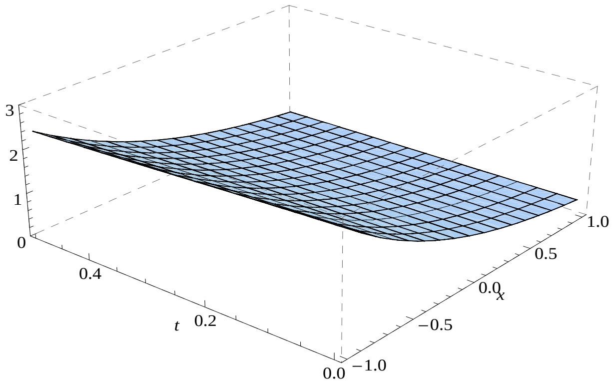

Figure 1: Plot of $u(x,t)$ in Example 4.1.

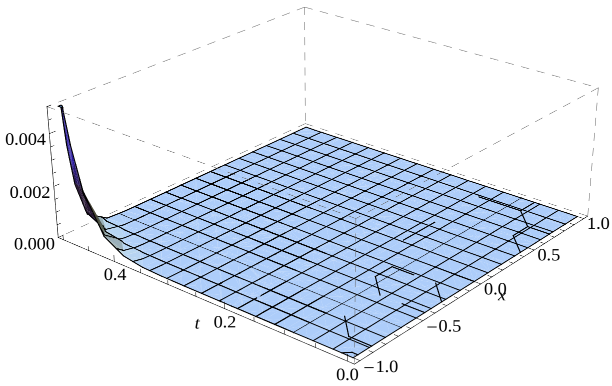

Figure 2: Plot of $Error(x,t)$ in Example 4.1.

Example 4.2 *Another parabolic equation is containing trigonometric function and arbitrary parameters as the following*

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + (c\cos^h \omega t + s)u, \quad (4.20)$$

where a, b, c, h, s and ω are constant. From (3.7), we have

$$N = 1,\ M = 2,\ a_0(x,t) = c\cos^h \omega t + s,$$
$$b_1(x,t) = b,\ b_2(x,t) = a,\ f(x,t) = 0,$$

and suppose that the initial condition is $u(x,0) = \sin t$. Hence, by (3.16) we obtain

$$\begin{aligned} &U_0(x) = \sin t,\\ &U_{k+1}(x) = \textstyle\sum_{r=0}^{k} \frac{1}{k+1}(B_{1,r}(x)\frac{\partial}{\partial x}U_{k-r}(x)\\ &\qquad + B_{2,r}(x)\frac{\partial^2}{\partial x^2}U_{k-r}(x))\\ &\qquad - \textstyle\sum_{r=0}^{k} \frac{1}{k+1}A_{0,r}(x)U_{k-r}(x),\\ &k = 0, 1, 2, \dots, \end{aligned} \quad (4.21)$$

where $B_{1,r}(x)$, $B_{2,r}(x)$ and $A_{0,r}(x)$ are determined by (3.11). Let us consider

$$a = 0.5,\ \ b = 0.5,\ \ c = 0.1,\ \ s = -0.5,$$
$$h = 2,\ \ \omega = \pi,\ \ p = 10.$$

By attention to (4.21), the approximate solution and error functional have been shown in figures (3) and (4), respectively.

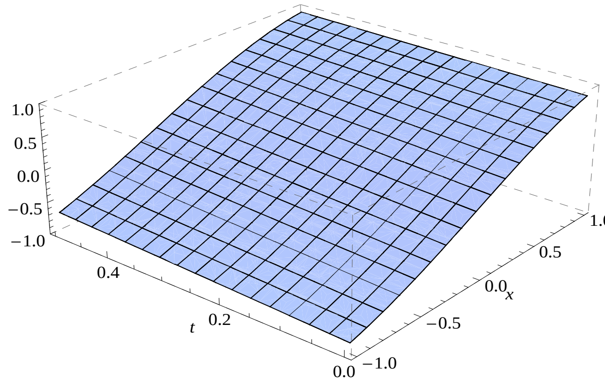

Figure 3: Plot of $u(x,t)$ in Example 4.2.

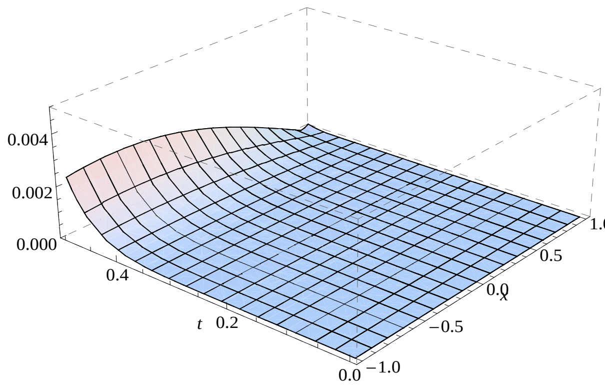

Figure 4: Plot of $Error(x,t)$ in Example 4.2.

4.2 The hyperbolic equations

Example 4.3 *Let consider the hyperbolic equation*

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial t}, \quad (4.22)$$

where α and β are constant. This equation governs free transverse vibration of a string, and also longitudinal vibration of a rod in a resisting medium with a velocity-proportional resistance coefficient. By attention to (3.7), we have

$$N = 2, M = 2,\ a_0(x,t) = 0,\ a_1(x,t) = \beta,$$
$$b_1(x,t) = 0,\ b_2(x,t) = \alpha^2,\ f(x,t) = 0,$$

and also assume that the initial conditions are

$$u(x,0) = \sin x^2,$$
$$u_t(x,0) = e^{x^2}.$$

So, from (3.16) we obtain

$$\begin{aligned} &U_0(x) = \sin x^2, \\ &U_1(x) = e^{x^2}, \\ &U_{k+2}(x) = \\ &\quad \sum_{r=0}^{k} \frac{1}{(k+1)(k+2)} B_{2,r}(x) \frac{\partial^2}{\partial x^2} U_{k-r}(x) \\ &\quad - \sum_{r=0}^{k} \frac{k-r+1}{(k+1)(k+2)} A_{1,r}(x) \frac{\partial}{\partial t} U_{k-r+1}(x), \\ &k = 0, 1, 2, \ldots, \end{aligned} \tag{4.23}$$

where $B_{2,r}(x)$ and $A_{1,r}(x)$ are determined by (3.11). We consider

$$\alpha = 1, \qquad \beta = 1, \qquad p = 10,$$

and obtain the approximate solution by (4.23). The approximate solution and error functional have been shown in figures (5) and (6), respectively.

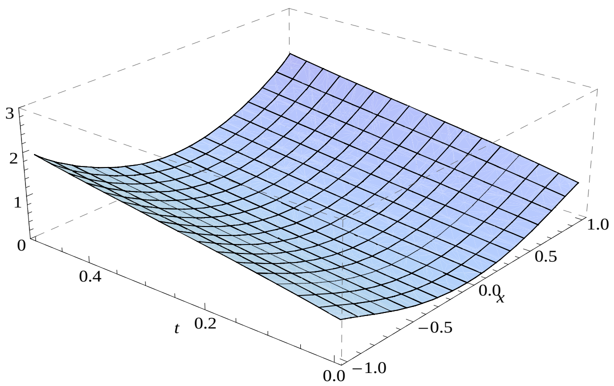

Figure 5: Plot of $u(x,t)$ in Example 4.3.

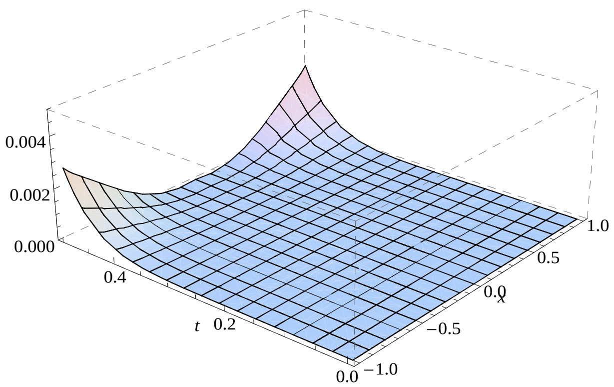

Figure 6: Plot of $Error(x,t)$ in Example 4.3.

Example 4.4 *We consider the second hyperbolic equation as the following*

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - bu + f(x,t), \tag{4.24}$$

where c, a and b are constant. If $c > 0$, $b < 0$ and $f(x,t) = 0$, then this equation is called the telegraph equation where u can be voltage or current through the wire. From (3.7), we have

$$N = 2,\ M = 2,\ a_0(x,t) = b,\ a_1(x,t) = c,$$
$$b_1(x,t) = 0,\ \ b_2(x,t) = a^2,\ \ f(x,t) = e^{xt^2},$$

and consider the initial conditions as follows

$$u(x,0) = \cos x^2,$$
$$u_t(x,0) = x \sin x^2.$$

Therefore, from (3.16) we obtain

$$\begin{aligned} &U_0(x) = \cos x^2, \\ &U_1(x) = x \sin x^2, \\ &U_{k+2}(x) = \frac{1}{(k+1)(k+2)} F_k(x) \\ &\quad + \sum_{r=0}^{k} \frac{1}{(k+1)(k+2)} B_{2,r}(x) \frac{\partial^2}{\partial x^2} U_{k-r}(x) \\ &\quad - \sum_{r=0}^{k} \frac{1}{(k+1)(k+2)} A_{0,r}(x) U_{k-r}(x) \\ &\quad - \sum_{r=0}^{k} \frac{k-r+1}{(k+1)(k+2)} A_{1,r}(x) \frac{\partial}{\partial t} U_{k-r+1}(x), \\ &k = 0, 1, 2, \ldots, \end{aligned} \tag{4.25}$$

where $F_k(x)$, $B_{2,r}(x)$, $A_{0,r}(x)$ and $A_{1,r}(x)$ are determined by (3.11). We consider

$$a = 0.5,\ \ b = -1,\ \ c = 0.8,\ \ p = 10,$$

and obtain the approximate solution by (4.25). The approximate solution and error functional have been shown in figures (7) and (8), respectively.

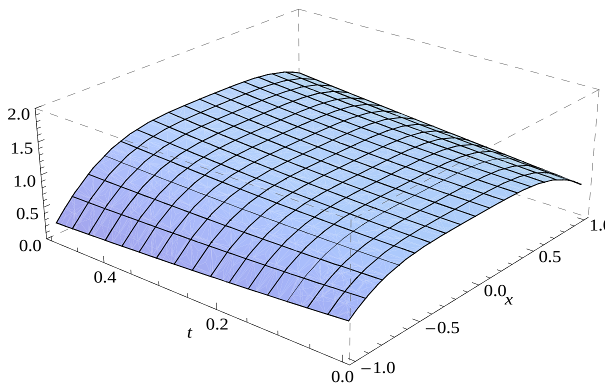

Figure 7: Plot of $u(x,t)$ in Example 4.4.

5 Conclusion

We introduced the RDTM for a rather wide class of the LPDEs with initial conditions in a general

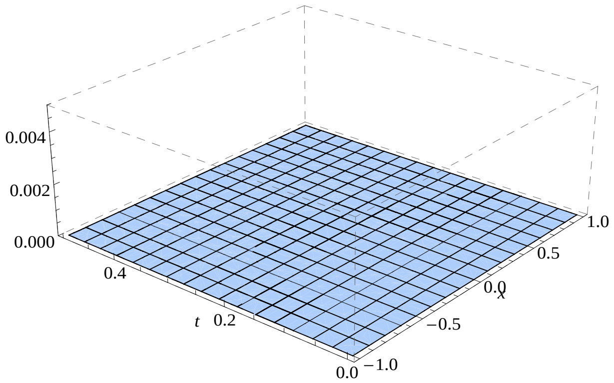

Figure 8: Plot of $Error(x,t)$ in Example 4.4.

form and obtained a recursive formula, i.e. (3.16) with (3.15), that can be used in another science and engineering by a software code of the Mathematica or Matlab software. The recursive formula is a rapidly method because it uses of differentiation that this operator consume the little time of computer at computations. Also, by attention to examples, is saw where the method has much carefulness.

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