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Some Results on Modular Hyperconvex Spaces

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Abstract

In recent years, many authors have focused on hyperconvex space and obtained a lot of valuable results (see [1, 2, 3, 5]). In this paper we develop some of those results for modular hyperconvex spaces. As a consequence we show that $A_\rho(X_\rho) \subseteq \varepsilon_\rho(X_\rho) \subseteq H_\rho(X_\rho)$ where, $A_\rho(X_\rho), \varepsilon_\rho(X_\rho)$, and $H_\rho(X_\rho)$ are modular admissible subsets, modular externally hyperconvex subsets and modular hyperconvex subsets in X_ρ, respectively.

Keywords : Hyperconvex space; Modular function; Modular hyperconvex space.

1 Introduction

Hyperconvex space, modular hyperconvex space and Fixed point theory play an important role in several subject of mathematics. For instance, it has been used in probability and mathematical statistics, boundary-value problems [3], the inverse function [9] , the existence of equilibria in economics [11, 12], and the existence of solutions of differential equations [6, 10].

For the discussion of the following sections, we state here some definitions, notations and known results. For convenience of readers, we suggest that one refer to [1, 2, 4, 5, 8] for details.

Let X be a vector space on $I\!R$, a function $\rho : X \to [0, +\infty]$ is called modular if for every x, y in X , (i) $\rho(x) = 0$ if and only if $x = 0$, (ii) $\rho(\alpha x) = \rho(x)$, for every $\alpha \in R$ where $\mid \alpha \mid = 1$, (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$, and ρ is called convex modular if , $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$. By a modular space we mean $X_\rho = \{x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$, where ρ is a modular function on X .
Following [4], for a modular space X_ρ, the sequence $\{x_n\}$ is called ρ-convergent to x if

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$\rho(x_n, x) \to x$, and it is called ρ-Cauchy if $\rho(x_n, x_m) \to 0$ as $n, m \to 0$. We will say that the modular function ρ satisfy the Fatou property if $\rho(x) \leq \liminf_n \rho(x_n)$ as $x_n \to x$,where $\{x_n\}$ is a sequence in X_ρ.

A modular function ρ is called complete if every ρ- Caushy sequence $\{x_n\}$ is ρ- convergent. A subset A of X_ρ is called ρ- closed if the ρ-limit of a ρ-convergent sequence of A always belong to A. By a ρ-ball $B_\rho(x, r)$, we mean $\{y \in X_\rho : \rho(x - y) \leq r\}$.

Finally, a subset A of X_ρ is called ρ-bounded if

$$\delta_\rho(A) = \{\rho(x - y) : x, y \in A\} < \infty$$

.

In general we note that ρ does not a metric because ρ does not satisfy the triangle inequality. For example ρ- convergent does not imply ρ- Caushy. However, ρ-balls are ρ-closed in a modular space X_ρ if and only if they have Fatou property, [5].

2 Main Results

In this section, we begin with basic definitions and notation. Then we discuss with more general properties on modular hyperconvixity , say, completeness of modular hyperconvex space and then we prove some technical results in modular hyperconvex spaces .

Definition 2.1. *A modular space X_ρ is called modular hyperconvex space if, for any collection of points $\{x_\alpha\}_{\alpha\in\Gamma}$ of X and for any collection $\{r_\alpha\}$ of non-negative real such that $\rho(1/2(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$ $(\alpha, \beta \in \Gamma)$, it follows that $\bigcap_{\alpha\in\Gamma} B_\rho(x_\alpha, r_\alpha) \neq \emptyset$*

Theorem 2.1. *Any modular hyperconvex space is complete.*

Proof. Let X_ρ be modular hyperconvex space and $\{x_n\}_{n\geq1}$ be a ρ-Cauchy sequence in X_ρ. For any $n \geq 1$, set $r_n = \sup_{m\geq n} \rho(x_n - x_m)$. Consider the collection of balls $\{B_\rho(x_n, r_n)\}_{n\geq1}$. Then

$$x_{n_k} \in B_\rho(x_{n_1}, r_{n_1}) \cap B_\rho(x_{n_2}, r_{n_2}) \cap ... \cap B_\rho(x_{n_k}, r_{n_k})$$

where $n_1 < n_2 < ... < n_k$. So

$$\begin{aligned} \rho(1/2(x_{n_i} - x_{n_j})) &= \rho(1/2x_{n_i} - 1/2x_{n_k} + 1/2x_{n_k} - 1/2x_{n_j}) \\ &= \rho(1/2(x_{n_i} - x_{n_k}) + 1/2(x_{n_k} - x_{n_j})) \\ &\leq \rho(x_{n_i} - x_{n_k}) + \rho(x_{n_k}, x_{n_j}) \\ &\leq r_{n_i} + r_{n_j} \end{aligned}$$

Now, X_ρ is a modular hyperconvex space, so $\bigcap_{n\geq1} B_\rho(x_n, r_n) \neq \emptyset$. Since $\{x_n\}_{n\geq1}$ is a ρ-Cauchy sequence, $\lim_{n\to\infty} r_n = 0$, and so the intersection $\bigcap_{n\geq1} B_\rho(x_n, r_n)$ is reduced to one point x which is the ρ-limit of the sequence $\{x_n\}_{n\geq1}$. ◻

Now we introduce some notation which will be used throughout the next Lemma.

Definition 2.2. *Let A be a subset of a modular hyperconvex space X_ρ, set*

$$\begin{array}{lcl} r_x(A) & = & \{\sup\{d_\rho(x,y) \ : \ y \in A\}, \quad x \in X_\rho; \\ r(A) & = & \inf\{r_x(A) \ : \ x \in X_\rho\}; \\ R(A) & = & \inf\{r_x(A) \ : \ x \in A\}; \\ diam(A) & = & \{\sup\{d_\rho(x,y) \ : \ x, y \in A\}; \\ C(A) & = & \{x \in X_\rho \ : \ r_x(A) = r(A)\}; \\ C_A(A) & = & \{x \in A : \ r_x(A) = r(A)\}; \\ cov_\rho(A) & = & \bigcap\{B : B \ is \ a \ \rho - ball \ and \ B \supseteq A\}; \end{array}$$

$r(A)$ is called the reduce of A (relative to X_ρ), $diam(A)$ is called the diameter of A, $R(A)$ is called Chebyshev radius of A, $C(A)$ is called the Chebyshev center of A, and $cov_\rho(A)$ is called the cover of A.

Lemma 2.1. *Let A be a ρ-bounded subset of modular hyperconvex space X_ρ, then:*
1) $cov_\rho(A) = \bigcap\{B_\rho(x, r_x(A)) : x \in X_\rho\}$.
2) $r_x(cov_\rho(A)) = r_x(A)$, for any $x \in X_\rho$.
3) $r(cov_\rho(A))) = r(A)$.
4) $r(A) = 1/2(diam(A))$.
5) $diam(cov_\rho(A)) = diam(A)$.
6) If $A = cov_\rho(A)$, then $r(A) = R(A)$. In particular we have $R(A) = 1/2(diam(A))$.

Proof. 1) We note that $A \subseteq B_\rho(x, r_x(A))$ for each $x \in X_\rho$, so $cov_\rho(A) \subseteq \bigcap\{B_\rho(x, r_x(A)) : x \in X_\rho\}$. On the other hand, if $A \subseteq B_\rho(x, r)$ then $r_x(A) \le r$, so $B_\rho(x, r_x(A)) \subseteq B_\rho(x, r)$. Thus

$$\bigcap\{B_\rho(x, r_x(A)) : x \in X_\rho\} \subseteq B_\rho(x, r)$$

This implies that $cov_\rho(A) = \bigcap\{B_\rho(x, r_x(A)) : x \in X_\rho\}$.

2) By (1), $r_x(cov_\rho(A)) = \sup\{\rho(x-y) : y \in \bigcap_{x \in X_\rho} B_\rho(x, r_x(A))\}$. Now if $y \in cov_\rho(A)$ implies $y \in B_\rho(x, r_x(A))$ for any $x \in X_\rho$. Thus $r_x(cov_\rho(A)) \le r_x(A)$.

On the other hand $A \subseteq cov(A)$ so, $r_x(A) \le r_x(cov_\rho(A)$. Thus $r_x(cov_\rho(A)) = r_x(A)$. On the other hand, $A \subseteq cov_\rho(A)$ so, $r_z(A) \le r_z(cov_\rho(A))$. Thus $r_z(cov_\rho(A)) = r_z(A)$ for each $z \in X_\rho$.

3) By (2) and definition of r, we have $r(A) = \inf\{r_x(A) : x \in X_\rho\} = \inf\{r_x(cov_\rho(A)) : x \in X_\rho\} = r(cov_\rho(A))$.

4) Consider the collection $\{B_\rho(a, \delta/2) : a \in A\}$ where $\delta = diam(A)$. If $a, b \in A$ then $\rho(a-b) \le \delta = (\delta/2) + (\delta/2)$ so by modular hyperconvexity,

$$\bigcap_{a \in A} B_\rho(a, \delta/2) \neq \emptyset$$

If x is a point in this intersection then $\rho(x-a) \le \delta/2$ so, $r_x(A) \le \delta/2$.

On the other hand for each $a, b \in A$, $z \in X_\rho$ we have

$$\rho(a-b) \le \rho(a-z) + \rho(z-b)$$

so, $\delta \le 2r_x(A)$ imply $\delta \le 2r(A)$. Thus $\delta \le 2r(A) \le 2r_z(A) \le \delta$. Therefore $r(A) = \delta/2$.

5) By (3), (4) we have

$$diam(A) = 2r(A) = 2r(cov_\rho(A)) = diam(cov_\rho(A))$$

6) Since $1/2diam(A) \leq r(A) \leq R(A)$ and $A = cov_\rho(A)$, so we can write $A = \bigcap_{i\in I} B_{\rho_i}$ where B_{ρ_i} is ρ-balls in X_ρ (for each $i \in I$). Now, by (4), $\bigcap_{a\in A} B_{\rho_i}(a, \delta/2) \neq \emptyset$ where $\delta = diam(A)$. Thus any two ρ-ball drown from the collection $\{B_{\rho_i} : i \in I\} \cup \{B_\rho(a, \delta/2) : a \in A\}$ have nonempty intersection, so by hyperconvexity of X_ρ,
$C = A \cap \{B_\rho(a, \delta/2) : \; a \in A\} = \{B_{\rho_i} : i \in I\} \cap \{B_\rho(a, \delta/2) : a \in A\} \neq \emptyset$.
Now, if $x \in C$ then, $r_x(A) \leq \delta/2$ and therefore $\delta/2 \leq r(A) \leq R(A) \leq r_x(A) \leq \delta/2$. Hence

$$r(A) = R(A) = 1/2(diam(A))$$

. □

Definition 2.3. *Let X_ρ be a modular space such that has Fatou property. A subset A of X_ρ is called modular admissible set if A is an intersection of ρ-closed balls in X_ρ.*
The collection of all modular admissible sets in X_ρ is denoted by $A_\rho(X_\rho)$

Definition 2.4. *Let X_ρ be a modular space. A subset C of X_ρ is called modular proximal if $C \cap B_\rho(x, dist_\rho(x, c)) \neq \emptyset$ where $x \in X_\rho$ and*

$$dist_\rho(x, c) = \inf\{\rho(x - y) : y \in C\}.$$

Definition 2.5. *A subset E of modular space X_ρ is called modular externally hyperconvex (relative to X_ρ) if given any family $\{x_\alpha\}$ of point in X_ρ and any family $\{r_\alpha\}$ of real positive numbers satisfying $\rho(1/2(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$*
(for all $\alpha, \beta \in \Gamma$) and $dist_\rho(x_\alpha, E) \leq r_\alpha$ then it follows

$$\bigcap_{\alpha\in\Gamma} B_\rho(x_\alpha, r_\alpha) \cap E \neq \emptyset$$

The class of all modular externally hyperconvex subsets of X_ρ is denoted by $\varepsilon_\rho(X_\rho)$ and the class of all modular hyperconvex of X_ρ is denoted by $H_\rho(X_\rho)$.

Lemma 2.2. *If E is either a modular admissible or modular externally hyperconvex of a modular hyperconvex X_ρ. Then E is modular proximal in X_ρ.*

Proof. We write the proof for the case E is a modular admissible subset. Other case is similar. Let $A = \bigcap_{i\in I} B_{\rho_i}$, then for any $\epsilon > 0$, there exists $a_\epsilon \in E$ such that $\rho(x - a_\epsilon) \leq dist_\rho(x, A) + \epsilon$.
Clearly this implies

$$\cap_{i\in I} B_{\rho_i} \cap B_\rho(x, dist_\rho(x, A) + \epsilon) \neq \emptyset$$

We note a_ϵ belong to the above intersection for any $\epsilon > 0$. Thus

$$A \cap B_\rho(x, dist_\rho(x, A)) = \bigcap_{i\in I} B_{\rho_i} \cap (\bigcap_{\epsilon>0} B_\rho(x, dist_\rho(x, A) + \epsilon)) \neq \emptyset$$

This implies that E is a modular proximal in X_ρ. □

Theorem 2.2. *If X_ρ is modular hyperconvex, then*

$$A_\rho(X_\rho) \subseteq \varepsilon_\rho(X_\rho) \subseteq H_\rho(X_\rho)$$

Proof. Let A be a modular admissible subset of X_ρ, $\{x_\alpha\}_{\alpha\in\Gamma}$ be a family of points of X_ρ and $\{r_\alpha\}_{\alpha\in\Gamma}$ be a family of positive real numbers that satisfies $dist_\rho(x_\alpha, A) \leq r_\alpha$, $\rho(1/2(x_\alpha - x_\beta)) \leq r_\alpha - r_\beta$ (for all $\alpha, \beta \in \Gamma$). By the Lemma 2.2, A is a modular proximal in X_ρ. Thus for any $\alpha \in \Gamma$, there exists $a_\alpha \in A$ such that

$$\rho(x_\alpha - r_\alpha) = dist_\rho(x_\alpha, A).$$

So

$$A \cap B_\rho(x_\alpha, r_\alpha) \neq \emptyset.$$

Furthermore X_ρ is a modular hyperconvex so, $\bigcap_{\alpha\in\Gamma} B_\rho(x_\alpha, r_\alpha) \neq \emptyset$.
On the other hand $A = \bigcap_{i\in I} B_{\rho_i}$. Clearly this implies

$$A \cap (\bigcap_{\alpha\in\Gamma} B_\rho(x_\alpha, r_\alpha) \neq \emptyset.$$

Thus A is a modular externally hyperconvex in X_ρ and $A_\rho(X_\rho) \subseteq \varepsilon_\rho(X_\rho)$. Other inclusion is trivial. $\square$

For next Theorem we need the following Lemma, that is similar to Lemma due to R. Sine, [7].

Lemma 2.3. *If X_ρ is a modular hyperconvex space and $D = \bigcap_\alpha B_\rho(x_\alpha, r_\alpha)$, then for any $\epsilon > 0$*

$$N_\epsilon(D) = \bigcap_\alpha B_\rho(x_\alpha, r_\alpha + \epsilon)$$

Theorem 2.3. *If X_ρ is a modular hyperconvex space and if A is a modular externally hyperconvex subset of X_ρ. Then $N_\epsilon(A)$ is a modular externally hyperconvex in χ_ρ for each $\epsilon > 0$.*

Proof. Let $\{x_\alpha\}$ be sequences in X_ρ and $\{r_\alpha\}$ be a sequence in $I\!R$ such that $\rho(x_\alpha - x_\beta) \leq r_\alpha + r_\beta$, $dist(x_\alpha, N_\epsilon(A)) \leq r_\alpha$. Therefore $dist_\rho(x_\alpha, A) \leq r_\alpha + \epsilon$. Since A is modular externally hyperconvex, this implies

$$A \cap (\bigcap_\alpha B_\rho(x_\alpha, r_\alpha + \epsilon)) \neq \emptyset$$

By Sine's Lemma

$$\bigcap_\alpha B_\rho(x_\alpha, r_\alpha + \epsilon) = N_\epsilon(\bigcap_\alpha B_\rho(x_\alpha, r_\alpha))$$

Thus

$$A \cap N_\epsilon(\bigcap_\alpha B_\rho(x_\alpha, r_\alpha)) \neq \emptyset$$

This implies that there exist $y \in A$ such that

$$dist_\rho(y, \bigcap_\alpha B_\rho(x_\alpha, r_\alpha)) \leq \epsilon$$

On the other hand, $\bigcap_\alpha B_\rho(x_\alpha, r_\alpha)$ is modular admissible and so is modular proximal. Thus there exist $b \in \bigcap_\alpha B_\rho(x_\alpha, r_\alpha)$ such that

$$dist_\rho(y, \bigcap_\alpha B_\rho(x_\alpha, r_\alpha)) = \rho(y - b) \leq \epsilon$$

Hence

$$dist_\rho(b - A) = \inf\{\rho(b - a) \ : \ a \in A\} \leq \rho(b - y) \leq \epsilon$$

So $b \in N_\epsilon(A) \cap (\bigcap_\alpha B_\rho(x_\alpha, r_\alpha))$. Thus

$$N_\epsilon(A) \cap (\bigcap_\alpha B_\rho(x_\alpha, r_\alpha)) \neq \emptyset$$

This means that $N_\epsilon(A)$ is a modular externally hyperconvex in X_ρ. □

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