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Some Properties of A New Fuzzy Distance Measure

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Abstract

The propose of this paper is to introduce a new fuzzy distance measure for fuzzy numbers. It computes the fuzzy distance between two fuzzy numbers. The metric properties of the proposed measure are also discussed in detail. Some numerical examples for computational implementation of the proposed method are also given.

Keywords : Fuzzy number; Fuzzy distance measure; Interval distance.

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1 Introduction

Fuzzy numbers are used to represent uncertain and incomplete information in decision making, linguistic controllers, biotechnological systems, expert systems, data mining, pattern recognition, etc. Recently, several authors have attempted to compute the fuzzy distance between fuzzy numbers. In [17], Voxman has proposed fuzzy distance by using extension principle of absolute distance between fuzzy numbers. Also, in [7] a fuzzy distance measure is proposed based on the interval difference and metric properties are also studied. However, there are some essential defects in the structure of fuzzy distance.

The rest of this paper is organized as follows: Section 2 contains preliminaries on fuzzy concepts. A metric distance measure for interval numbers with its properties is introduced in Section 3. Then, in Section 4, $\alpha-$distance for fuzzy numbers is defined and its properties are discussed in detail. In the Section 5, we use a procedure for ranking fuzzy numbers based on the α-distance. For comparing the proposed ranking method with some other

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approaches, some numerical examples are provided in Section 6. Finally, the paper ends with conclusions in Section 7.

2 Preliminaries

A fuzzy number A is a fuzzy subset of the real line R with the membership function μ_A which is (see [9]): normal (i.e. there exists an element x_0 such that $\mu_A(x_0) = 1$), fuzzy convex (i.e. $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_A(x_1), \mu_A(x_2))$, $\forall x_1, x_2 \in R$, $\forall \lambda \in [0,1]$), upper semicontinuous, suppAis bounded, where $supp\ A = cl\{x \in R : \mu_A(x) > 0\}$ and cl is the closure operator.
The $\alpha-$cut, $\alpha \in]0,1]$, of a fuzzy number A is a crisp set defined as

$$A_\alpha = \{x \in R : \mu_A(x) \geq \alpha\}.$$

Every $\alpha-$cut of a fuzzy number A is a closed interval $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where

$$A_L(\alpha) = inf\{x \in R : \mu_A(x) \geq \alpha\}, \qquad A_U(\alpha) = sup\{x \in R : \mu_A(x) \geq \alpha\}.$$

We denote

$$A_0 = [A_L(0), A_U(0)] = supp\ A.$$

The pair of functions (A_L, A_U) gives a parametric representation of the fuzzy number A. For two arbitrary fuzzy numbers A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ and B, $B_\alpha = [B_L(\alpha), B_U(\alpha)]$ the quantity

$$D(A,B) = \sqrt{\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha}$$

gives a distance between A and B (see, e.g., [11]). The expected interval $EI(A)$ of a fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, is defined by (see [10, 15])

$$EI(A) = [E_*(A), E^*(A)] = \left[\int_0^1 A_L(\alpha) a\alpha, \int_0^1 A_U(\alpha) d\alpha\right]$$

and the middle of the expected interval is called the expected value of a fuzzy number A, i.e.,

$$EV(A) = \frac{1}{2}\left(\int_0^1 A_L(\alpha) a\alpha + \int_0^1 A_U(\alpha) d\alpha\right)$$

An often used fuzzy number is the trapezoidal fuzzy number, completely characterized by four real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$, denoted by $T = (t_1, t_2, t_3, t_4)$, with the parametric representation (T_L, T_U),

$$T_L(\alpha) = t_1 + (t_2 - t_1)\alpha, \quad T_U(\alpha) = t_4 - (t_4 - t_3)\alpha, \quad \alpha \in [0,1]$$

and the expected interval

$$EI(T) = \left[\frac{t_1 + t_2}{2}, \frac{t_3 + t_4}{2}\right]$$

Another important kind of fuzzy number was introduced in [5] as follows. Let $a_1, a_2, a_3, a_4 \in R$ such that $a_1 < a_2 \leq a_3 < a_4$. A fuzzy number A defined as

$$\mu_A(x) = \begin{cases} 0, & x \leq a_1 \\ \left(\frac{x-a_1}{a_2-a_1}\right)^r, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \left(\frac{a_4-x}{a_4-a_3}\right)^r, & a_3 \leq x \leq a_4 \\ 0, & x \geq a_4 \end{cases}$$

where $r > 0$, is denoted by $A = (a_1, a_2, a_3, a_4)_r$. If $A = (a_1, a_2, a_3, a_4)_r$ then

$$A_\alpha = [A_L(\alpha), A_U(\alpha)] = \left[a_1 + \alpha^{1/r}(a_2 - a_1), a_4 - \alpha^{1/r}(a_4 - a_3)\right], \quad \alpha \in [0, 1]$$

Theorem 2.1. *[4], Let $A = (a_1, a_2, a_t, a_4)_r$,*

(i) If

$$(5r + 1)a_1 + 2r(r - 1)a_2 - 2r(r + 2)a_3 + (r - 1)a_4 > 0 \tag{2.1}$$

then

$$T((a_1, a_2, a_3, a_4)_r) = \left(\frac{a_1 + ra_2}{1 + r}, \frac{a_1 + ra_2}{1 + r}, \frac{a_1 + ra_2}{1 + r}, \frac{-a_1 - ra_2 + 2ra_3 + 2a_4}{1 + r}\right) \tag{2.2}$$

(ii) If

$$(1 - r)a_1 + 2r(r + 2)a_2 - 2r(r - 1)a_3 - (5r + 1)a_4 > 0 \tag{2.3}$$

then

$$T((a_1, a_2, a_3, a_4)_r) = \left(\frac{2a_1 + 2ra_2 - ra_3 - a_4}{1 + r}, \frac{a_4 + ra_3}{1 + r}, \frac{a_4 + ra_3}{1 + r}, \frac{a_4 + ra_3}{1 + r}\right) \tag{2.4}$$

(iii) If

$$\begin{aligned} &(5r + 1)a_1 + 2r(r - 1)a_2 - 2r(r + 2)a_3 + (r - 1)a_4 \leq 0, \\ &(1 - r)a_1 + 2r(r + 2)a_2 - 2r(r - 1)a_3 - (5r + 1)a_4 \leq 0, \\ &(1 - r)a_1 + 2r(r + 2)a_2 - 2r(r + 2)a_3 + (r - 1)a_4 > 0 \end{aligned} \tag{2.5}$$

then

$$T((a_1, a_2, a_3, a_4)_r) = (t_1, t_2, t_3, t_4),$$

where

$$\begin{aligned} t_1 &= \tfrac{(9r+3)a_1+6r^2a_2-2r(r+2)a_3+(r-1)a_4}{2(1+r)(1+2r)} \\ t_2 &= \tfrac{(1-r)a_1+2r(r+2)a_2+2r(r+2)a_3+(1-r)a_4}{2(1+r)(1+2r)} \\ t_3 &= t_2 \\ t_4 &= \tfrac{(r-1)a_1-2r(r+2)a_2+6r^2a_3+(9r+3)a_4}{2(1+r)(1+2r)} \end{aligned} \tag{2.6}$$

(iv) If

$$(1-r)a_1 + 2r(r+2)a_2 - 2r(r+2)a_3 + (r-1)a_4 \leq 0 \tag{2.7}$$

then

$$T((a_1, a_2, a_3, a_4)_r) = (t_1, t_2, t_3, t_4),$$

where

$$\begin{aligned} t_1 &= \tfrac{(5r+1)a_1+2r(r-1)a_2}{(1+r)(1+2r)}, \\ t_2 &= \tfrac{(1-r)a_1+2r(r+2)a_2}{(1+r)(1+2r)}, \\ t_3 &= \tfrac{2r(r+2)a_3+(1-r)a_4}{(1+r)(1+2r)}, \\ t_4 &= \tfrac{2r(r-1)a_3+(5r+1)a_4}{(1+r)(1+2r)} \end{aligned} \tag{2.8}$$

Remark 2.1. *[4], If A is a trapezoidal number, then $T(A) = A$.*

Definition 2.1. *[18], For two arbitrary fuzzy numbers A and B with $\alpha-$cuts sets $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, we call*

$$d(A,B) = \left(\int_0^1 f(\alpha) \left((A_L(\alpha) - B_L(\alpha))^2 + (A_U(\alpha) - B_U(\alpha))^2 \right) d\alpha \right)^{\frac{1}{2}} \tag{2.9}$$

the weighted distance between fuzzy numbers A and B, where $f(\alpha)$ is nonnegative and increasing on $[0,1]$ with $f(0) = 0$ and $\int_0^1 f(\alpha)d\alpha = \frac{1}{2}$.

In actual applications, function $f(\alpha)$ can be chosen according to the actual situation. In this paper, we use $f(\alpha) = \alpha$.

Definition 2.2. *For two arbitrary fuzzy numbers A and B with $\alpha-$cuts sets $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ and $B_\alpha = [B_L(\alpha), B_U(\alpha)]$, the Housdorff distance is defined to be:*

$$d_H(A,B) = \max \left\{ \sup_{x \in A_\alpha} \inf_{y \in B_\alpha} |x-y|, \sup_{y \in B_\alpha} \inf_{x \in A_\alpha} |x-y| \right\} \tag{2.10}$$

Definition 2.3. *[16], For two arbitrary interval numbers $U = [u_1, u_2]$ and $V = [v_1, v_2]$, we say*

- *$U = V$ if and only if $u_1 = v_1$ and $u_2 = v_2$.*
- *$U \preceq V$ if and only if $u_1 \leq v_1$ and $u_2 \leq v_2$.*
- *$U \prec V$ if and only if $U \preceq V$, but $U \neq V$.*

Definition 2.4. *[2], For arbitrary trapezoidal fuzzy number $V = (x_0, y_0, \sigma, \beta)$ with two defuzzifier x_0, y_0 and left fuzziness $\sigma > 0$ and right fuzziness $\beta > 0$ and parametric form $V_\alpha = (V_L(\alpha), V_U(\alpha))$, we define the magnitude of the trapezoidal fuzzy number V as*

$$Mag(V) = \frac{1}{2} \left(\int_0^1 (V_L(\alpha) + V_U(\alpha) + x_0 + y_0) f(\alpha) d\alpha \right) \tag{2.11}$$

where $f(\alpha)$ is nonnegative and increasing on $[0,1]$ with $f(0) = 0$ and $\int_0^1 f(\alpha)d\alpha = \frac{1}{2}$.

Remark 2.2. *[2], For two arbitrary trapezoidal fuzzy numbers U and V, we have*

$$Mag(U + V) = Mag(U) + Mag(V)$$

Definition 2.5. *[2], The ranking of two arbitrary trapezoidal fuzzy numbers U and V by the $Mag(.)$ as follows:*

- *$U \sim V$ if and only if $Mag(U) = Mag(V)$.*
- *$U \prec V$ if and only if $Mag(U) < Mag(V)$.*
- *$U \succ V$ if and only if $Mag(U) > Mag(V)$.*

Definition 2.6. *[8], If V is a fuzzy number with $\alpha-$cut representation $[V_L(\alpha), V_U(\alpha)]$, then the value of V is defined by*

$$Val(V) = \int_0^1 f(\alpha)(V_L(\alpha) + V_U(\alpha))d\alpha$$

and the ambiguity of V is defined by

$$Amb(V) = \int_0^1 f(\alpha)(V_U(\alpha) - V_L(\alpha))d\alpha$$

where $f(\alpha)$ is nonnegative and increasing on $[0, 1]$ with $f(0) = 0$ and $\int_0^1 f(\alpha)d\alpha = \frac{1}{2}$.

Definition 2.7. *[6], The width of the fuzzy number V is defined by*

$$w(V) = \int_0^1 (V_U(\alpha) - V_L(\alpha))d\alpha$$

is an useful parameter characterizing the nonspecificity of a fuzzy number.

3 Interval distance measure

let us consider two fuzzy numbers as $A = (a_1, a_2, a_3, a_4)_r$ and $B = (b_1, b_2, b_3, b_4)_r$. Then if, $T(A)$ and $T(B)$ are trapezoidal approximation of A and B, respectively, which are obtained from Grzegorzewski and Mrowka [14] and Ban [4] method, we define the interval distance

$$D^*(A, B) = [d(T(A), T(B)), d_H(T(A), T(B))] \tag{3.12}$$

where $d(T(A), T(B))$ and $d_H(T(A), T(B))$ be introduced in definitions (2.1) and (2.2), respectively.

Theorem 3.1. *Let A, B and C are arbitrary fuzzy numbers. Then*

(i) $d(T(A), T(B)) \leq d_H(T(A), T(B))$.

(ii) $D^*(A, A) = \tilde{0}$.

(iii) $D^*(A, B) = D^*(B, A)$.

(iv) $D^*(A,C) \preceq D^*(A,B) + D^*(B,C)$

Proof: From definitions (2.1), (2.2) and (2.3), the cases $(ii) - (iv)$ are obvious. Now, we prove case (i):
Consider $T_\alpha(A) = [T_{AL}(\alpha), T_{AU}(\alpha)]$ and $T_\alpha(B) = [T_{BL}(\alpha), T_{BU}(\alpha)]$ and

$$d_H(T(A), T(B)) = \sup_{\alpha \in [0,1]} \{\max\{|T_{AL}(\alpha) - T_{BL}(\alpha)|, |T_{AU}(\alpha) - T_{BU}(\alpha)|\}\} = \Omega \geq 0$$

Then for all $\alpha \in [0,1]$, we have:

$$|T_{AL}(\alpha) - T_{BL}(\alpha)| \leq \Omega$$

$$|T_{AU}(\alpha) - T_{BU}(\alpha)| \leq \Omega$$

Therefore,

$$\begin{aligned} d^2(T(A), T(B)) &= \int_0^1 \left((T_{AL}(\alpha) - T_{BL}(\alpha))^2 + (T_{AU}(\alpha) - T_{BU}(\alpha))^2\right) \alpha d\alpha \\ &\leq 2\Omega^2 \int_0^1 \alpha d\alpha \\ &= \Omega^2 \end{aligned}$$

So,

$$d(T(A), T(B)) \leq \Omega = d_H(T(A), T(B))$$

and the proof is complete.

4 Fuzzy distance measure

Now, we convert the interval distance measure (3.12) to the fuzzy distance measure as follows:

Definition 4.1. *For two arbitrary fuzzy numbers A and B, the fuzzy distance measure is*

$$D_F(A,B) = [d(T(A),T(B)),\ \lambda d_H(T(A),T(B)) + (1-\lambda) d(T(A),T(B)),\ d_H(T(A),T(B))] \quad \lambda \in [0,1] \tag{4.13}$$

Let λ represent the decision maker‘s preference. λ gives deferent fuzzy numbers to the possible values of the fuzzy distance measure. The quality of a fuzzy number is the main factor considered in the existing approaches, such as those based on area measurement, for ranking fuzzy numbers. $\lambda \in [0,1]$ is the index of optimism that reflects a decision maker‘s degree of optimism. The large index of optimism implies that the decision maker is more optimistic, and only the maximum value of the optimism is considered when $\lambda = 1$. On the other hand, a more pessimistic decision maker will take a smaller value of the index. With the optimistic measure changing from 0 to 1, the preference valuation of fuzzy number changes monotonically and continuously from the minimum to the maximum of the fuzzy number support, which is consistent with our common sense of decision making.

It is obvious that $D_F(A,B)$ is a fuzzy number where

$$supp\ D_F(A,B) = [d(T(A),T(B)),\ d_H(T(A),T(B))] = D^*(A,B)$$

and

$$core\ D_F(A,B) = \lambda d_H(T(A),T(B)) + (1-\lambda)d(T(A),T(B)), \quad \lambda \in [0,1]$$

Theorem 4.1. *For arbitrary fuzzy numbers U, V and W, we have*

(i) $D_F(U,U) = \tilde{0}$

(ii) $D_F(U,V) = D_F(V,U)$

(iii) $D_F(U,W) \preceq D_F(U,V) + D_F(V,W)$

Proof: The cases $(i), (ii)$ are obvious. Now, we prove case (iii):
Suppose

$$\begin{aligned}
D_F(U,W) &= [d(T(U),T(W)),\ \lambda d_H(T(U),T(W)) + (1-\lambda)d(T(U),T(W)),\ d_H(T(U),T(W))] \\
D_F(U,V) &= [d(T(U),T(V)),\ \lambda d_H(T(U),T(V)) + (1-\lambda)d(T(U),T(V)),\ d_H(T(U),T(V))] \\
D_F(V,W) &= [d(T(V),T(W)),\ \lambda d_H(T(V),T(W)) + (1-\lambda)d(T(V),T(W)),\ d_H(T(V),T(W))]
\end{aligned} \tag{4.14}$$

From definitions (2.1) and (2.2), we know

$$\begin{aligned}
d(T(U),T(W)) &\le d(T(U),T(V)) + d(T(V),T(W)) \\
d_H(T(U),T(W)) &\le d_H(T(U),T(V)) + d_H(T(V),T(W))
\end{aligned} \tag{4.15}$$

Then, for all $\alpha \in [0,1]$,

$$\begin{aligned}
(1-\alpha)d(T(U),T(W)) + \alpha d_H(T(U),T(W)) &\le (1-\alpha)[d(T(U),T(V)) + d(T(V),T(W))] \\
&+ \alpha[d_H(T(U),T(V)) + d_H(T(V),T(W))]
\end{aligned} \tag{4.16}$$

So,

$$\begin{aligned}
D_{FL}(U,W)(\alpha) &= (1-\alpha)d(T(U),T(W)) + \alpha[(1-\lambda)d(T(U),T(W)) + \lambda d_H(T(U),T(W))] \\
&\le (1-\alpha)[d(T(U),T(V)) + d(T(V),T(W))] \\
&+ \alpha(1-\lambda)[d(T(U),T(V)) + d(T(V),T(W))] \\
&+ \alpha\lambda[d_H(T(U),T(V)) + d_H(T(V),T(W))] \\
&= (1-\alpha)[d(T(U),T(V))] + \alpha[(1-\lambda)(d(T(U),T(V))) + \lambda(d_H(T(U),T(V)))] \\
&+ (1-\alpha)[d(T(V),T(W))] + \alpha[(1-\lambda)(d(T(V),T(W))) + \lambda(d_H(T(V),T(W)))] \\
&= D_{FL}(U,V)(\alpha) + D_{FL}(V,W)(\alpha)
\end{aligned} \tag{4.17}$$

Therefore,

$$D_{FL}(U,W)(\alpha) \le D_{FL}(U,V)(\alpha) + D_{FL}(V,W)(\alpha) \tag{4.18}$$

Also in a similar way we obtain

$$D_{FU}(U,W)(\alpha) \le D_{FU}(U,V)(\alpha) + D_{FU}(V,W)(\alpha) \tag{4.19}$$

By using Eqs. (4.16), (4.18) and (4.19) and definition (2.4), we get

$$\begin{aligned}
Mag(D_F(U,W)) &\le Mag(D_F(U,V) + D_F(V,W)) \\
&= Mag(D_F(U,V)) + Mag(D_F(V,W))
\end{aligned} \tag{4.20}$$

Finally, from definition (2.5), we have

$$D_F(U,W) \leq D_F(U,V) + D_F(V,W)$$

Theorem 4.2. *Let us consider two arbitrary fuzzy numbers U and V, then if $\{\lambda_k\}$ be an increasing real sequence with starting point $\lambda_0 = 0$ and $\lambda_k \longrightarrow 1$ when $k \longrightarrow \infty$, we have the following:*

(i)

$$Mag(D_F(U,V,\lambda_j)) \leq Mag(D_F(U,V,\lambda_{j+1})), \quad for\ all\ \lambda_j \leq \lambda_{j+1},\ j = 1,2,3,\ldots \tag{4.21}$$

or equivalently, we have

$$D_F(U,V,\lambda_j) \preceq D_F(U,V,\lambda_{j+1}),\ for\ all\ \lambda_j \leq \lambda_{j+1},\ j = 1,2,3,\ldots \tag{4.22}$$

(ii)

$$Val(D_F(U,V,\lambda_j)) \leq Val(D_F(U,V,\lambda_{j+1})), \quad for\ all\ \lambda_j \leq \lambda_{j+1},\ j = 1,2,3,\ldots \tag{4.23}$$

(iii)

$$Amb(D_F(U,V,\lambda_j)) = Amb(D_F(U,V,\lambda_{j+1})), \quad for\ all\ \lambda_j \leq \lambda_{j+1},\ j = 1,2,3,\ldots \tag{4.24}$$

(iv)

$$EV(D_F(U,V,\lambda_j)) \leq EV(D_F(U,V,\lambda_{j+1})), \quad for\ all\ \lambda_j \leq \lambda_{j+1},\ j = 1,2,3,\ldots \tag{4.25}$$

(v)

$$w(D_F(U,V,\lambda_j)) = w(D_F(U,V,\lambda_{j+1})), \quad for\ all\ \lambda_j \leq \lambda_{j+1},\ j = 1,2,3,\ldots \tag{4.26}$$

Proof:

(i) From case (iv) Theorem (3.1), we know

$$d(T(U),T(V)) \leq d_H(T(U),T(V))$$

then, for $0 \leq \lambda_j \leq \lambda_{j+1} \leq 1$, $j = 1,2,3,\ldots$, we have

$$\lambda_j(d_H(T(U),T(V)) - d(T(U),T(V))) \leq \lambda_{j+1}(d_H(T(U),T(V)) - d(T(U),T(V)))$$

So,

$$\begin{aligned} d(T(U),T(V)) \quad &+\lambda_j(d_H(T(U),T(V)) - d(T(U),T(V))) \\ &\leq d(T(U),T(V)) + \lambda_{j+1}(d_H(T(U),T(V)) - d(T(U),T(V))) \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\lambda_j\ d_H(T(U),T(V)) \quad &+(1-\lambda_j)d(T(U),T(V)) \\ &\leq \lambda_{j+1}\ d_H(T(U),T(V))+(1-\lambda_{j+1})d(T(U),T(V))\end{aligned} \tag{4.27}$$

Then,

$$\begin{aligned}&(1-\alpha)d(T(U),T(V))+\alpha[\lambda_j\ d_H(T(U),T(V))+(1-\lambda_j)d(T(U),T(V))] \\ &\leq (1-\alpha)d(T(U),T(V))+\alpha[\lambda_{j+1}\ d_H(T(U),T(V))+(1-\lambda_{j+1})d(T(U),T(V))]\end{aligned} \tag{4.28}$$

Thus,

$$D_{FL}(U,V,\lambda_j) \leq D_{FL}(U,V,\lambda_{j+1}) \tag{4.29}$$

Similarly, we get

$$D_{FU}(U,V,\lambda_j) \leq D_{FU}(U,V,\lambda_{j+1}) \tag{4.30}$$

By using Eqs. (4.27), (4.29) and (4.30), we can obtain

$$Mag(D_F(U,V,\lambda_j)) \leq Mag(D_F(U,V,\lambda_{j+1}))$$

So, equivalently, we have

$$D_F(U,V,\lambda_j) \preceq D_F(U,V,\lambda_{j+1}), \quad for\ all\ \lambda_j \leq \lambda_{j+1},\ j=1,2,3,\ldots$$

and the proof is complete.

(ii) From Eqs. (4.29) and (4.30) the proof is clear.

(iii) For all $\lambda \in [0,1]$, we have

$$D_{FU}(U,V,\lambda)-D_{FL}(U,V,\lambda)=(1-\alpha)(d_H(T(U),T(V))-d(T(U),T(V))) \tag{4.31}$$

then,

$$Amb(D_F(U,V,\lambda_j))=Amb(D_F(U,V,\lambda_{j+1})), \quad for\ all\ \lambda_j \leq \lambda_{j+1},\ j=1,2,3,\ldots$$

(iv) From Eqs. (4.29) and (4.30) the proof is clear.

(v) By using Eq. (4.31), we have

$$w(D_F(U,V,\lambda_j))=w(D_F(U,V,\lambda_{j+1})), \quad for\ all\ \lambda_j \leq \lambda_{j+1},\ j=1,2,3,\ldots$$

5 Numerical example

Example 5.1. *Allahviranloo and Adabitabar Firozja [3] considered the fuzzy numbers A and B, given by* $\alpha-$*cuts set* $A_\alpha=[2\alpha-2,1-\sqrt{\alpha}]$ *and* $B_\alpha=[\sqrt{\alpha}-1,1-\sqrt{\alpha}]$, $\alpha\in[0,1]$. *Ban [4] obtained trapezoidal approximation of these fuzzy numbers as follows:*

$$T(A)=\left(-\tfrac{59}{30},-\tfrac{1}{30},-\tfrac{1}{30},\tfrac{7}{10}\right)$$

$$T(B)=\left(-\tfrac{2}{3},0,0,\tfrac{2}{3}\right)$$

The fuzzy distance computed by proposed method is:

Table 1

λ	$D_F(A,B)$
0	(0.5378, 0.5378, 1.3000)
0.25	(0.5378, 0.7284, 1.3000)
0.5	(0.5378, 0.9189, 1.3000)
0.75	(0.5378, 1.1095, 1.3000)
1	(0.5378, 1.3000, 1.3000)

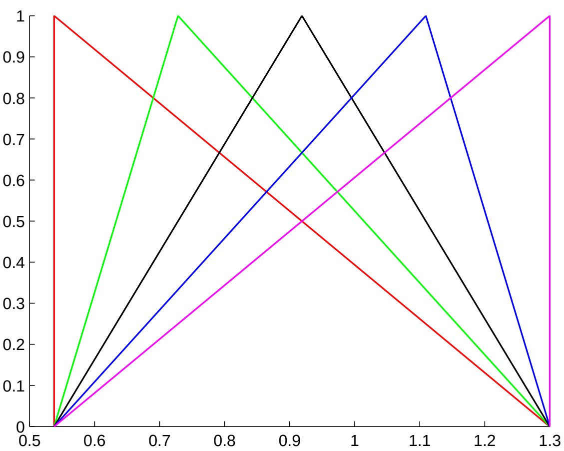

Fig. 1. The obtained fuzzy distances for $\lambda = 0, 0.25, 0.5, 0.75, 1$

Example 5.2. *[7], Let* $A = (2, 3, 5, 7)$ *and* $B = (5, 6, 9, 10)$. *The fuzzy distance computed by proposed method is:*

Table 2

λ	$D_F(A,B)$
0	(3.2660 3.2660 4.0000)
0.25	(3.2660 3.4495 4.0000)
0.5	(3.2660 3.6330 4.0000)
0.75	(3.2660 3.8165 4.0000)
1	(3.2660 4.0000 4.0000)

Also, the fuzzy distance between A, B from Voxman's measure and Chakraborty's measure are computed as follows:

$$\tilde{d}_{Voxman} = (0, 1, 6, 8), \quad \tilde{d}_{Chakraborty} = (0, 1, 6, 7)$$

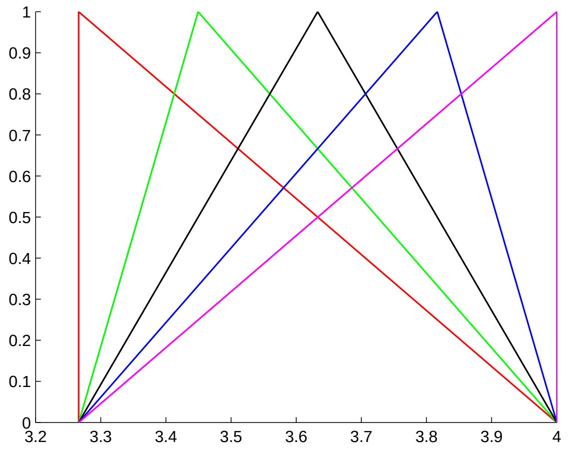

Fig. 2. The obtained fuzzy distances for $\lambda = 0, 0.25, 0.5, 0.75, 1$

Example 5.3. *[7], Let two triangular fuzzy numbers* $A = (0.3, 0.5, 0.7)$ *and* $B = (0.4, 0.6, 0.9)$. *The fuzzy distance computed by our proposed method is:*

Table 3

λ	$D_F(A, B)$
0	*(0.1291 0.1291 0.2000)*
0.25	*(0.1291 0.1465 0.2000)*
0.5	*(0.1291 0.1645 0.2000)*
0.75	*(0.1291 0.1823 0.2000)*
1	*(0.1291 0.2000 0.2000)*

Also, the fuzzy distance between A, B from Voxman's measure and Chakraborty's measure are computed as follows:

$$\tilde{d}_{Voxman} = (0, 0.1, 0.6), \quad \tilde{d}_{Chakraborty} = (0, 0.1, 0.35)$$

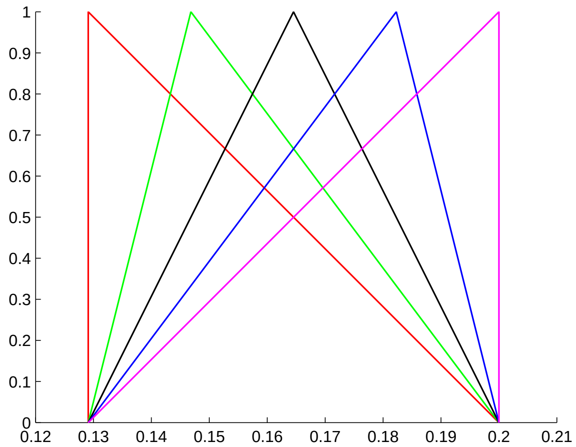

Fig. 3. The obtained fuzzy distances for $\lambda = 0, 0.25, 0.5, 0.75, 1$

6 Conclusions

In this paper, an interval distance measure on fuzzy numbers was introduced. Subsequently, it was extended to the fuzzy distance measure for fuzzy numbers and the metric properties were discussed in detail. Then, we proved that our proposed method preserve the Ambiguity and the width of the fuzzy distance measure and the value, expected value and the magnitude of the fuzzy distance measure increase for all $\lambda_j \leq \lambda_{j+1}$, $j = 1, 2, \ldots$. The proposed method is illustrated by some numerical examples.

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