Solving Multi-objective Linear Bilevel Multi-follower Programming Problem

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Abstract

In sciences and industries such as signal optimization, traffic assignment, economic market and etc, many problems have been modeled by bilevel programming (BLP) problems, where in each level one must optimize some objective functions. There are so many algorithms in order to find the global optimum of the linear version of BLP problems.
This paper addresses multi-objective linear bilevel multi-follower programming (MOLBMFP) problems in which there is no sharing information among followers. It presents a new method for solving these problems.
Keywords : Bilvel programming; Multi-objective linear bilevel multi-follower programming; Fuzzy set theory; Fuzzy programming; Kth-best algorithm.

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1   Introduction

A bilevel programming (BLP) problem can be considered as the noncooperative, two-player game, which was first presented by Von Stackelberg [15]. In a basic BLP model, the control for the decision variables is partitioned amongst the players who seek to optimize their individual objective functions. The upper level is called the leader and the lower level is termed as the follower. The game which is said to be "static" implies that each player has only one move. The leader goes first and attempts to optimize his objective function, then the follower reacts in a way that is personally without regard to extramural effects by observing the leader's decision.
There are some methods for finding the global optimum of a bilevel programming problem in which only one leader and a follower are involved and each level has just one objective function to be optimized. The majority of researches on bilevel programming problems have been centered on the linear version of the problems, and there have been nearly two dozen algorithms proposed for solving BLP problems [1, 2, 5, 6, 7]. Some of them are:

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Penalty function approach [12, 16], Genetic algorithm [10, 11], Grid-search algorithm [2] and the Kth-best algorithm [3, 4, 13].

If the feasible region of every linear BLP problem is nonempty and compact, it shares an important property that at least one optimal (global) solution is attained at an extreme point of the constraint region [12]. The Kth-best method computes the global solution of linear BLP problems by enumerating the extreme points of the constraint region.

Our previous work presented a new approach to solve multi-objective linear bilevel programming problems [9]. this paper is aimed at presenting a method for solving multi-objective linear bilevel multi-follower programming problems. in section 2, we apply Fuzzy set theory and Fuzzy programming which was introduced by Zimmerman [17, 18] to convert our problem to a linear bilevel multi-follower programming (LBLMFP) problem. we express theoretical properties of LBLMFP problem in section 3, and it develops the Kth-best method, to find the global optimum solution of the achieved LBLMFP problem in section 4. A numerical example is illustrated in section 5 to show the efficiency of the new approach. We the conclusion is presented in section 6.

2 Converting the multi-objective linear bilevel programming problem to LBLMFP problem

Consider the model of the MOLBLMFP problem in general as it follows, in which $k\ (k \geq 2)$ followers are involved and there is no sharing information among them except the leader's.

$$\max_{x \in X} \quad \{F_1(x, y_1, y_2, \ldots, y_k), F_2(x, y_1, y_2, \ldots, y_k), \ldots, F_p(x, y_1, y_2, \ldots, y_k)\}$$

$$s.t \quad Ax + \sum_{t=1}^{k} B_t y_t \leq b \tag{2.1}$$

$$\max_{y_i \in Y_i} \quad \{f_{1i}(x, y_i), f_{2i}(x, y_i), \ldots, f_{ji}(x, y_i)\}$$

$$s.t \quad A_{li}x + C_{li}y_i \leq b_{li}$$

where $x \in X \subset \Re^n, y_i \in Y_i \subset \Re^{m_i}$, $F_r : X \times Y_1 \times \ldots \times Y_k \to \Re^1$, $f_{li} : X \times Y_i \to \Re^1$, $i = 1, \ldots, k,\ r = 1, \ldots, p,\ l = 1, \ldots, j$ (p, j are the number of the objective functions which must be optimized respectively by the leader and the followers.)and A, B_t, A_{li}, C_{li} are appropriate technology matrices. As there is no sharing variables among the followers, all followers have individual objective functions and constraints.

To change the MOLBLMFP problem (2.1) to a BLMFP problem, first we need to find upper and also lower bounds for any objective function in the leader's level and also the followers' levels of the MOLBLMFP problem (2.1). One may call Z_r^U and Z_r^L, respectively as upper and lower bounds of objective functions of the first level and Z_{li}^U and Z_{li}^L respectively as upper and the lower bounds of objective functions of the followers. with the aim of doing so, the following problems should be solved for $i = 1, 2, \ldots, k$ (based on the method explained in [18]). Note that the following problems have solutions if the feasible

region of the leader and the followers is nonempty and compact.

$$(LP)_r \qquad \max_{x \in X} \quad F_r(x, y_1, y_2, \ldots, y_k)$$

$$(r = 1, \ldots, p)$$

$$s.t \qquad Ax + \sum_{t=1}^{k} B_t y_t \leq b \tag{2.2}$$

$$(LP)_l \qquad \max_{y_i \in Y_i} \quad f_{li}(x, y_i)$$

$$(l = 1, \ldots, j)$$

$$s.t \qquad A_{li}x + C_{li}y_i \leq b_{li} \tag{2.3}$$

Let $(x^{[1]*}, y_1^{[1]*}, \ldots, y_k^{[1]*}), \ldots, (x^{[p]*}, y_1^{[p]*}, \ldots, y_k^{[p]*})$ be the solutions of (2.2). Matrix A is defined as follows:

$$A = \begin{bmatrix} F_1(x^{[1]*}, y_1^{[1]*}, \ldots, y_k^{[1]*}) & F_2(x^{[1]*}, y_1^{[1]*}, \ldots, y_k^{[1]*}) & \ldots & F_p(x^{[1]*}, y_1^{[1]*}, \ldots, y_k^{[1]*}) \\ F_1(x^{[2]*}, y_1^{[2]*}, \ldots, y_k^{[2]*}) & F_2(x^{[2]*}, y_1^{[2]*}, \ldots, y_k^{[2]*}) & \ldots & F_p(x^{[2]*}, y_1^{[2]*}, \ldots, y_k^{[2]*}) \\ \vdots & \vdots & \ddots & \vdots \\ F_1(x^{[p]*}, y_1^{[p]*}, \ldots, y_k^{[p]*}) & F_2(x^{[p]*}, y_1^{[p]*}, \ldots, y_k^{[p]*}) & \ldots & F_p(x^{[p]*}, y_1^{[p]*}, \ldots, y_k^{[p]*}) \end{bmatrix}$$

The maximum and minimum values in each column of A are shown by Z_r^U and Z_r^L, ($r = 1, \ldots, p$), respectively. The difference $Z_r^U - Z_r^L$ is the constants of admissible violations. Now we can define a membership function corresponding to each level (the leader and the followers) for any Fuzzy goal, as the following:
for $r = 1, \ldots, p$

$$\mu_r(F_r) = \begin{cases} 1 & F_r(x, y_1, \ldots, y_k) \geq Z_r^U \\ \frac{F_r(x, y_1, \ldots, y_k) - Z_r^L}{Z_r^U - Z_r^L} & Z_r^l \leq F_r(x, y_1, \ldots, y_k) - Z_r^L \leq Z_r^U \\ 0 & F_r(x, y_1, \ldots, y_k) - Z_r^L \leq Z_r^L \end{cases} \tag{2.4}$$

One must also follow the above process for the followers (by considering (2.3)) thus the upper and lower bounds of the followers' objective functions can be denoted by z_{li}^U and z_{li}^L for $i = 1, \ldots, k$ and $l = 1, \ldots, l$, also the membership functions of the followers are defined as above and are denoted by $\phi_i(f_{li}(x, y_i))$ for $i = , \ldots, k, l = 1, \ldots, j$. Since membership functions are the degrees of satisfaction, they must be maximized. So if we consider $\lambda = \min\{\mu_1, \ldots, \mu_p\}$ and $\lambda_i = \min\{\phi_1, \ldots, \phi_k\}$, where $i = 1, \ldots, k$ is the number of the followers, then the MOLBLMFP problem (2.1) changes to the LBLMFP problem (2.5) as follows:

$$\max_{x \in X, \lambda} \quad \lambda$$

$$s.t \qquad Ax + \sum_{t=1}^{k} B_t y_t \leq b$$

$$\mu_r(F_r) \geq \lambda, \qquad r = 1, \ldots, p \tag{2.5}$$

$$0 \leq \lambda \leq 1$$

$$\max_{y_i \in Y_i, \lambda_i} \quad \lambda_i$$

$$s.t \qquad A_{li}x + C_{li}y_i \le b_{li}$$

$$\phi_i(f_{li}(x, y_i)) \ge \lambda_i, \quad l = 1, \ldots, j$$

$$0 \le \lambda_i \le 1$$

By substituting the membership functions from (2.4) to (2.5), the LBLMFP problem (2.5) changes to:

$$\max_{x \in X, \lambda} \quad \lambda$$

$$s.t \qquad Ax + \sum_{t=1}^{k} B_t y_t \le b$$

$$F_r(x, y_1, \ldots, y_k) - (Z_r^U - Z_r^L)\lambda \ge Z_r^L, \quad r = 1, \ldots, p$$

$$0 \le \lambda \le 1$$

$$\max_{y_i \in Y_i, \lambda_i} \quad \lambda_i \tag{2.6}$$

$$s.t \qquad A_{li}x + C_{li}y_i \le b_{li}$$

$$f_{li}(x, y_i) - (z_{li}^U - z_{li}^L)\lambda_i \ge z_{li}^L, \qquad\qquad l = 1, \ldots, j, i = 1, \ldots, k$$

$$0 \le \lambda_i \le 1$$

Now we have a LBLMFP problem, in which, $x \in X \subset \Re^n$ and $\lambda \in [0, 1]$ are the decision variables for the leader and $y_i \in Y_i \subset \Re^{m_i}$ and $\lambda_i \in [0, 1]$ for $i = 1, \ldots, k$ are the decision variables for the followers.

3   Theoretical properties of LBLMFP problem

The Kth-best algorithm is aimed at investigating the constraint region of the LBLP problem to find the optimal solution at a vertex of the constraint region. Now we explore the definitions expressed in [12, 13] for the LBLMFP problem (2.6).

Constraint region of the linear BLMFP problem (2.6) is defined as follows:

$$S = \{(x, y_1, \ldots, y_k, \lambda, \lambda_i) | Ax + \sum_{t=1}^{k} B_t y_t \le b, F_r(x, y_1, \ldots, y_k) - (Z_r^U - Z_r^L)\lambda \ge Z_r^L,$$

$$A_{li}x + C_{li}y_i \le b_{li}, f_{li}(x, y_i) - (z_{li}^U - z_{li}^L)\lambda_i \ge z_{li}^L, 0 \le \lambda \le 1,$$

$$0 \le, \lambda_i \le 1, i = 1, \ldots, k, l = 1, \ldots, j, r = 1, \ldots, p\}$$

The linear BLFMP problem constraint region refers to all possible combinations of choices that the leader and the followers may make.

Denote the *projection of S onto the leader's decision space* by $S(X)$ and the *feasible set* for each follower for all $(x, \lambda) \in S(X)$, these two sets are defined as follows:

$$S(X) = \{(x, \lambda) | x \in X, \lambda \in [0, 1] : \exists y_i \in Y_i, \exists \lambda_i \in [0, 1], (x, y_1, \ldots, y_k, \lambda, \lambda_i) \in S, i = 1, \ldots, k\}$$

(Constraints of the leader and the followers are satisfied.)

$$S_i(x) = \{y_i \in Y_i, \lambda_i \in [0, 1] : (x, y_1, \ldots, y_k, \lambda, \lambda_i) \in S, i = 1, \ldots, k\}$$

Now we define for each follower, the *rational reaction set* for any $(x, \lambda) \in S(X)$ as the following:

$$P_i(x) = \{y_i \in Y_i, \lambda_i \in [0, 1] : (y_i, \lambda_i) \in argmin[\lambda_i : (y_i, \lambda_i) \in S_i(x)], i = 1, \ldots, k\}$$

$P_i(x)$ is equivalent to the set of the solutions of the following problem for any $(x, \lambda) \in S(X)$ and $i = 1, 2, \ldots, k$:

$$\max_{y_i \in Y_i, \lambda_i} \quad \lambda_i$$

$$s.t \qquad Ax + \sum_{t=1}^{k} B_t y_t \leq b$$

$$F_r(x, y_1, \ldots, y_k) - (Z_r^U - Z_r^L)\lambda \geq Z_r^L, \quad r = 1, \ldots, p \tag{3.7}$$

$$0 \leq \lambda \leq 1$$

$$A_{li} x + C_{li} y_i \leq b_{li}$$

$$f_{li}(x, y_i) - (z_{li}^U - z_{li}^L)\lambda_i \geq z_{li}^L, \qquad l = 1, \ldots, j, i = 1, \ldots, k$$

$$0 \leq \lambda_i \leq 1$$

The followers select y_i and λ_i from their objective functions by considering the leader's action. Now we present the Inducible Region of the problem (2.6) by:

$$IR = \{(x, y_1, \ldots, y_k, \lambda, \lambda_i) : \in (x, y_1, \ldots, y_k, \lambda, \lambda_i) \in S, (y_i, \lambda_i) \in P_i(x), i = 1, \ldots, k\}$$

To ensure that (2.6) has an optimal solution, the following assumptions must be considered:

(i) S is nonempty and compact.

(ii) $P_i(x)$ is nonempty,(i.e. $P_i(x) \neq \emptyset$).

(iii) $P_i(x)$, is a point to point map from $S(X)$ to K_i which $K_i \subseteq S_i(x)$(otherwise some difficulties may appear, which are explained in [2, 8]).

Thus in terms of the above definitions and notations, the LBLMFP problem (2.6) can be written as [14]:

$$\max\{\lambda : (x, y_1, \ldots, y_k, \lambda, \lambda_i) \in IR\} \tag{3.8}$$

Every theorem for LBLMFP problems satisfies MOLBLMFP problems as well. Since we made the LBLMFP problem (2.6) from MOLBLMFP problem (2.1), these two are equivalents. The most important theorems are:

Theorem 3.1. *If S is nonempty and compact, there exists an optimal solution for a linear BLMFP problem.*

Proof: Since S is nonempty and compact, there exists a point $(x^*, y_1^*, \ldots, y_k^*, \lambda^*, \lambda_i^*) \in S$. Then by the definition of the projection of S onto the leader's decision space, we have

$$(x^*, \lambda^*) \in S(X) \neq \emptyset$$

Consequently by the definition of the feasible set for each follower $S_i(x^*) \neq \emptyset$. Also by the definition of each follower's rational reaction set, we have $P_i(x^*) \neq \emptyset$. Hence there exists $(y_{0_i}, \lambda_{0_i}) \in P_i(x^*)$ for all $i = 1, 2, \ldots, k$ such that $(x^*, y_{0_1}^*, \ldots, y_{0_k}^*, \lambda^*, \lambda_{0_i}^*) \in S$ for all $i = 1, 2, \ldots, k$. Therefore, we have $IR \neq \emptyset$ by the definition of Inducible Region. Since we are optimizing a linear function over IR (by (3.8)) which is nonempty and bounded, an optimal solution to the linear BLMFP problem must exist.

Theorem 3.2. *[14], A solution for the linear BLMFP problem occurs at a vertex of IR.*

Theorem 3.3. *[14], The solution of the linear BLMFP problem occurs at a vertex of S.*

4   The Kth-best algorithm

Section 3 implies that we are efficiently able to find an optimal solution for a linear MOBLMFP problem, by searching the constraint region S. Thus, first we should arrange all the extreme points in S according to the leader's objective function (step 1 in the algorithm), then check the first extreme point, if it is on the Inducible Region IR. Therefore, if yes, according to (3.8), the current extreme point is the optimal solution (step 2 in the algorithm). Otherwise, the next extreme point is selected and examined (step 3 and step 4 in the algorithm). Now for solving (2.6) we can present the steps of the Kth-best algorithm as the following:

Step 1.   Put $q \leftarrow 1$. Solve the problem $\max\{\lambda : (x, y_1, \ldots, y_k, \lambda, \lambda_i) \in S, i = 1, 2, \ldots, k\}$ with the simplex method to obtain the optimal solution $(x^{[1]}, y_1^{[1]}, \ldots, y_k^{[1]}, \lambda^{[1]}, \lambda_i^{[1]})$. Let $W = \{(x^{[1]}, y_1^{[1]}, \ldots, y_k^{[1]}, \lambda^{[1]}, \lambda_i^{[1]})\}$ and $T = \emptyset$. Go to Step 2.

Step 2.   Solve (3.7) by the assumption $x = x^{[q]}, \lambda = \lambda^{[q]}$ with the simplex method to obtain the optimal solution $(\widetilde{y_i}, \widetilde{\lambda_i})$ for all $i = 1, 2, \ldots, k$ and check If $(\widetilde{y_i}, \widetilde{\lambda_i}) = (y_i^{[q]}, \lambda_i^{[q]})$. In case of having multiple optimal solution for (2.6) (like $\widetilde{y}_{i_1}, \ldots, \widetilde{y}_{i_t}$), it should be checked if $\exists t_0 : \widetilde{y}_{i_{t_0}} = y_i^q$, then $\widetilde{y}_{i_{t_0}}$ should be selected as the solution. $(x^{[q]}, y_1^{[q]}, \ldots, y_k^{[q]}, \lambda^{[q]}, \lambda_i^{[q]})$ is the global optimum of (2.1). Otherwise go to Step 3.

Step 3.   Let $W_{[q]}$ denote the set of adjacent extreme points of $(x^{[q]}, y_1^{[q]}, \ldots, y_k^{[q]}, \lambda^{[q]}, \lambda_i^{[q]})$ such that $(x, y_1, \ldots, y_k, \lambda, \lambda_i)$ implies $\lambda \leq \lambda^{[q]}$. Let $T = T \cup \{(x^{[q]}, y_1^{[q]}, \ldots, y_k^{[q]}, \lambda^{[q]}, \lambda_i^{[q]})\}$ and $W = (W \cup W_{[q]}) \setminus T$. Go to Step 4.

Step 4.   Set $q \leftarrow q + 1$ and choose $(x^{[q]}, y_1^{[q]}, \ldots, y_k^{[q]}, \lambda^{[q]}, \lambda_i^{[q]})$ so that

$$\lambda^{[q]} = \max\{\lambda : (x, y_1, \ldots, y_k, \lambda, \lambda_i) \in W\}$$

Go to Step 2.

5 Numerical Example

In this section, we apply the mentioned method to show its efficiency. So Consider the following MOLBLMFP problem.

Example 5.1. *Consider the following MOLBLMFP problem with $x \in X \subset \Re^1, y_1 \in Y_1 \subset \Re^1, y_2 \in Y_2 \subset \Re^1$ and $X = \{x \geq 0\}, Y_1 = \{y_1 \geq 0\}, Y_2 = \{y_2 \geq 0\}$*

$$\max_{x \in X} \quad (x + 2y_1 + 3y_2, y_1 - y_2)$$

$$s.t \quad x + 2y_1 + 3y_2 \leq 6$$

$$y_1 \leq 2$$

$$\max_{y_1 \in Y_1} \quad (x + y_1, y_1)$$

$$s.t \quad x + y_1 \leq 3 \tag{5.9}$$

$$y_1 \leq 1$$

$$\max_{y_2 \in Y_2} \quad (x + y_2, y_2)$$

$$s.t \quad x + y_2 \leq 4$$

$$y_2 \leq 2$$

The leader's constraint region can be shown in the fig. 1 and the constraint regions of the first follower and the second follower are respectively presented at the Fig.2 and Fig.3.
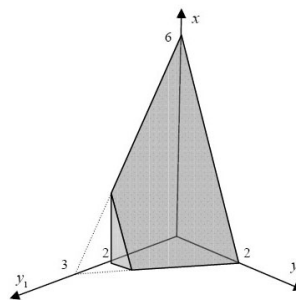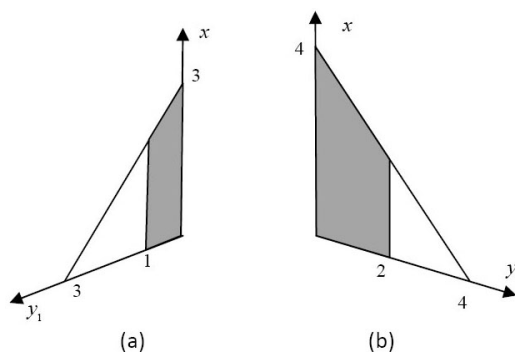

Fig. 1. The leader's constraint region.

Fig. 2. (a) The first follower's constraint region, (b) The second follower's constraint region

As it is viewed in the figures, The leader's and each followers' constraint region are nonempty and compact, thus first we are able to convert (5.9) to a linear BLMFP problem by using the method in section 2. So, the following problem can be set easily.

$$\max_{x \in X, \lambda} \quad \lambda$$

$$s.t \qquad x + 2y_1 + 3y_2 \leq 6$$

$$y_1 \leq 2$$

$$x + 2y_1 + 3y_2 - 2\lambda \geq 4$$

$$y_1 - y_2 - 2\lambda \geq 0$$

$$0 \leq \lambda \leq 1$$

$$\max_{y_1 \in Y_1, \lambda_1} \quad \lambda_1$$

$$s.t \qquad x + y_1 \leq 3$$

$$y_1 \leq 1$$

$$x + y_1 - 2\lambda_1 \geq 1 \qquad\qquad (5.10)$$

$$y_1 - \lambda_1 \geq 0$$

$$0 \leq \lambda_1 \leq 1$$

$$\max_{y_2 \in Y_2, \lambda_2} \quad \lambda_2$$

$$s.t \qquad x + y_2 \leq 4$$

$$y_2 \leq 2$$

$$x + y_2 - 2\lambda_2 \geq 2$$

$$y_2 - 2\lambda_2 \geq 0$$

$$0 \leq \lambda_2 \leq 1$$

Note that (5.10) is a linear BLMFP problem. Now according to the theorems in section 3, since the constraint region of the linear BLMFP problem (5.10) is nonempty and compact, there exists an optimal solution (5.10) at a vertex of S. Now for finding the optimal solution of (5.10) we apply the Kth-best algorithm, expressed in section 4. According to

the Kth-best approach, the example can be rewritten as follows:

$$\max_{x \in X, \lambda} \quad \lambda$$

$$s.t \quad x + 2y_1 + 3y_2 \leq 6$$

$$y_1 \leq 2$$

$$x + 2y_1 + 3y_2 - 2\lambda \geq 4$$

$$y_1 - y_2 - 2\lambda \geq 0$$

$$0 \leq \lambda \leq 1$$

$$x + y_1 \leq 3$$

$$y_1 \leq 1$$

$$x + y_1 - 2\lambda_1 \geq 1$$

$$y_1 - \lambda_1 \geq 0$$

$$0 \leq \lambda_1 \leq 1$$

$$x + y_2 \leq 4$$

$$y_2 \leq 2$$

$$x + y_2 - 2\lambda_2 \geq 2$$

$$y_2 - 2\lambda_2 \geq 0$$

$$0 \leq \lambda_2 \leq 1$$

Step 1, set $q = 1$ and solve the above problem with the simplex method to obtain the optimal solution

$$(x^{[1]}, y_1^{[1]}, y_2^{[1]}, \lambda^{[1]}, \lambda_1^{[1]}, \lambda_2^{[1]}) = (2, 1, 0.25, 0.375, 0, 0)$$

Let $W = (2, 1, 0.25, 0.375, 0, 0)$ and $T = \emptyset$. Go to Step 2.

Iteration 1:

Setting $i \leftarrow 1$ and by (3.7), we have

$$\max \quad \lambda_1$$

$$s.t \quad x + 2y_1 + 3y_2 \le 6$$

$$y_1 \le 2$$

$$x + 2y_1 + 3y_2 - 2\lambda \ge 4$$

$$y_1 - y_2 - 2\lambda \ge 0$$

$$0 \le \lambda \le 1$$

$$x + y_1 \le 3$$

$$y_1 \le 1$$

$$x + y_1 - 2\lambda_1 \ge 1$$

$$y_1 - \lambda_1 \ge 0$$

$$0 \le \lambda_1 \le 1$$

$$x + y_2 \le 4$$

$$y_2 \le 2$$

$$x + y_2 - 2\lambda_2 \ge 2$$

$$y_2 - 2\lambda_2 \ge 0$$

$$0 \le \lambda_2 \le 1$$

$$x = 2$$

$$\lambda = 0.375$$

Using the simplex method we have $(\widetilde{y}_1, \widetilde{\lambda}_1) = (1, 1)$. Because $(\widetilde{y}_1, \widetilde{\lambda}_1) \ne (y_1^{[1]}, \lambda_1^{[1]})$ we go to Step 3. we have

$$W_{[1]} \quad = \{(2, 1, 0.25, 0.375, 0, 0), (2, 1, 0.25, 0.375, 1, 0), (2, 1, 0.25, 0.375, 0, 0.125)$$

$$, (2, 1, 0.667, 0.167, 0, 0), (2, 1, 0, 0, 0, 0), (1.667, 1, 0.333, 0.333, 0, 0), (2.75, 0.25, 0.25, 0, 0, 0)\}$$

and $T = \{(2, 1, 0.25, 0.375, 0, 0)\}$ thus

$$W \quad = \{(2, 1, 0.25, 0.375, 1, 0), (2, 1, 0.25, 0.375, 0, 0.125), (2, 1, 0.667, 0.167, 0, 0), (2, 1, 0, 0, 0, 0)$$

$$, (1.667, 1, 0.333, 0.333, 0, 0), (2.75, 0.25, 0.25, 0, 0, 0)\}$$

Then go to Step 4. Update $q = 2$ and choose

$$(x^{[2]}, y_1^{[2]}, y_2^{[2]}, \lambda^{[2]}, \lambda_1^{[2]}, \lambda_2^{[2]}) = (2, 1, 0.25, 0.375, 1, 0),$$

then go to Step 2.
Iteration 2:
Setting $i \leftarrow 1$ and by (3.7), we have

$$\max \quad \lambda_1$$

$$s.t \quad x + 2y_1 + 3y_2 \leq 6$$

$$y_1 \leq 2$$

$$x + 2y_1 + 3y_2 - 2\lambda \geq 4$$

$$y_1 - y_2 - 2\lambda \geq 0$$

$$0 \leq \lambda \leq 1$$

$$x + y_1 \leq 3$$

$$y_1 \leq 1$$

$$x + y_1 - 2\lambda_1 \geq 1$$

$$y_1 - \lambda_1 \geq 0$$

$$0 \leq \lambda_1 \leq 1$$

$$x + y_2 \leq 4$$

$$y_2 \leq 2$$

$$x + y_2 - 2\lambda_2 \geq 2$$

$$y_2 - 2\lambda_2 \geq 0$$

$$0 \leq \lambda_2 \leq 1$$

$$x = 2$$

$$\lambda = 0.375$$

Using the simplex method we have $(\widetilde{y}_1, \widetilde{\lambda}_1) = (1,1)$. Because $(\widetilde{y}_1, \widetilde{\lambda}_1) = (y_1^{[1]}, \lambda_1^{[1]})$ setting $i \leftarrow i+1$ and by (3.7) we have

$$\max \quad \lambda_2$$
$$s.t \quad x + 2y_1 + 3y_2 \leq 6$$
$$y_1 \leq 2$$
$$x + 2y_1 + 3y_2 - 2\lambda \geq 4$$
$$y_1 - y_2 - 2\lambda \geq 0$$
$$0 \leq \lambda \leq 1$$
$$x + y_1 \leq 3$$
$$y_1 \leq 1$$
$$x + y_1 - 2\lambda_1 \geq 1$$
$$y_1 - \lambda_1 \geq 0$$
$$0 \leq \lambda_1 \leq 1$$
$$x + y_2 \leq 4$$
$$y_2 \leq 2$$
$$x + y_2 - 2\lambda_2 \geq 2$$
$$y_2 - 2\lambda_2 \geq 0$$
$$0 \leq \lambda_2 \leq 1$$
$$x = 2$$
$$\lambda = 0.375$$

Using the simplex method we have $(\widetilde{y}_2, \widetilde{\lambda}_2) = (0.25, 0.125)$. Because $(\widetilde{y}_2, \widetilde{\lambda}_2) \neq (y_1^{[2]}, \lambda_1^{[2]})$ we should continue the algorithm by Step 3. If we follow the Steps of the algorithm as as before, the optimal solution is obtained at the point $(x^*, y_1^*, y_2^*, \lambda^*, \lambda_1^*, \lambda_2^*) = (2, 1, 0.25, 0.375, 1, 0.125)$ and therefore we get the optimal objective values as follows:
$x + 2y_1 + 3y_2 = 4.75$
$y_1 - y_2 = 0.75$
$x + y_1 = 3$
$y_1 = 1$
$x + y_2 = 2.25$
$y_2 = 0.25$
The solution shows that the leader gains 37.5 percent of his Fuzzy goal, the first follower gains 100 percent and the seccond follower gains only 12.5 percent, but it must be noted that this is the maximum degree of satisfaction that they can obtain.

6   Conclusion

This paper presented a method to find the global optimal solution of the multi-objective linear bilevel multi-follower programming problems in which there are no sharing variables except the leader's, by using Fuzzy programming and Kth-best algorithm and a numerical example was presented at the end. If the constraint regions of the leader and all the followers are non-empty and compact we are able to solve any linear multi-objective bilevel programming problems.

It might suggest that this approach can be adopted to solve non-linear problems. Also we can use this method for solving MOLBLMFP problems in which there are sharing variables among followers.

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