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Research Article

# Study on usage of Elzaki transform for the ordinary differential equations with non-constant coefficients

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**Abstract**

Although Elzaki transform is stronger than Sumudu and Laplace transforms to solve the ordinary differential equations with non-constant coefficients, but this method does not lead to finding the answer of some differential equations. In this paper, a method is introduced to find that a differential equation by Elzaki transform can be solved?

*Keywords* : Elzaki transform; Sumudu transform; Laplace transform; Differential equation.

## 1 Introduction

Elzaki transform is the revised form of Laplace and Sumudu transforms. This transform is an integral equation that is defined by T. Elzaki in following form

$$E[f(t)] = u^2 \int_0^\infty f(ut)e^{-t}dt = T(u), \quad u \in (K_1, K_2), \qquad K_1, K_2 > 0. \quad (1.1)$$

By changing the variable, relation (1.1) turns into the following form

$$E[f(t)] = u \int_0^\infty f(t)e^{-t/u}dt = T(u). \quad (1.2)$$

In this paper, we have obtained some relations between numerical coefficients of the variables of the ordinary differential equation with the initial value. So if these relations govern the supposed differential equation, then Elzaki transform will be suitable method for solving the supposed differential equation.

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## 2 Elzaki transform

Using the definition of Elzaki transform, Elzaki transform of derived functions can be obtained in the following form:

1. $E[f'(t)] = \dfrac{T(u)}{u} - uf(0),$

2. $E[f''(t)] = \dfrac{T(u)}{u^2} - f(0) - uf'(0),$

3. $E[f^{(m)}(t)] = \dfrac{T(u)}{u^m} - \sum_{k=0}^{m-1} u^{2-m+k} f^{(k)}(0).$

And also there are relations between alternative multiples of Elzaki transform in the following form:

1. $E[tf(t)] = u^2 \dfrac{d}{du}T(u) - uT(u),$

2. $E[t^2 f(t)] = u^4 \dfrac{d^2}{du^2}T(u),$

3. $E[t^3 f(t)] = u^6 \dfrac{d^3}{du^3}T(u) + 3u^5 \dfrac{d^2}{du^2}T(u),$

4. $E[tf'(t)] = u^2\frac{d}{du}\left[\frac{T(u)}{u} - f(0)\right] - u\left[\frac{T(u)}{u} - f(0)\right]$,

5. $E[t^2f'(t)] = u^4\frac{d^2}{du^2}\left[\frac{T(u)}{u} - f(0)\right]$,

6. $E[t^3f'(t)] = u^6\frac{d^3}{du^3}\left[\frac{T(u)}{u} - f(0)\right] + 3u^5\frac{d^2}{du^2}\left[\frac{T(u)}{u} - f(0)\right]$.

**Theorem 2.1** *If $E[f(t)] = T(u)$, then*

1. $E[tf^{(m)}(t)] = u^2\frac{d}{du}E[f^{(m)}(t)] - uE[f^{(m)}(t)]$,
2. $E[t^2f^{(m)}(t)] = u^4\frac{d^2}{du^2}E[f^{(m)}(t)]$,
3. $E[t^3f^{(m)}(t)] = u^6\frac{d^3}{du^3}E[f^{(m)}(t)] + 3u^5E[f^{(m)}(t)]$.

**Proof.** (1) By using induction on $m$, if $m = 1$, then

$$E[tf'(t)] = u^2\frac{d}{du}\left[\frac{T(u)}{u} - f(0)\right] - u\left[\frac{T(u)}{u} - f(0)\right] = u^2\frac{d}{du}E[f'(t)] - uE[f'(t)]$$

that will be a true relation. And also for $m = n$ we have

$$E[tf^{(n)}(t)] = u^2\frac{d}{du}E[f^{(n)}(t)] - uE[f^{(n)}(t)].$$

If $m = n + 1$, we should show that the relation

$$E[tf^{(n+1)}(t)] = u^2\frac{d}{du}E[f^{(n+1)}(t)] - uE[f^{(n+1)}(t)]$$

is true. To prove this relation, it is enough to put $f^{(n)}(t) = g(t)$, so we have

$$E[tf^{(n+1)}(t)] = E[tg'(t)] = u^2\frac{d}{du}\left[\frac{E[g(t)]}{u} - g(0)\right] - u\left[\frac{E[g(t)]}{u} - g(0)\right],$$

and so by replacing $E[g(t)]$, the above relation is obtained. The two other relations will also be proved in the similar way.

**Remark 2.1** *By using other form of $E[f^{(m)}(t)]$, like Theorem 2.1, we have*

1.

$$E[tf^{(m)}(t)] = \frac{T'(u)}{u^{m-2}} - (m+1)\frac{T(u)}{u^{m-1}} - \sum_{k=0}^{m-1}(1+k-m)u^{3+k-m}f^{(k)}(0), \quad (2.3)$$

2.

$$E[t^2f^{(m)}(t)] = \frac{T''(u)}{u^{m-4}} - 2m\frac{T'(u)}{u^{m-3}} + m(m+1)\frac{T(u)}{u^{m-2}} - \sum_{k=0}^{m-1}(2+k-m)(1+k-m)u^{k-m+4}f^{(k)}(0), \quad (2.4)$$

3.

$$E[t^3f^{(m)}(t)] = \frac{T'''(u)}{u^{m-6}} - 3(m-1)\frac{T''(u)}{u^{m-5}} + 3m(m-1)\frac{T'(u)}{u^{m-4}} - m(m+1)(m-1)\frac{T(u)}{u^{m-3}} - \sum_{k=0}^{m-1}s_{k,m}u^{k-m+5}f^{(k)}(0), \quad (2.5)$$

*where $s_{k,m} = (2+k-m)(k+1-m)(3+k-m)$.*

**Theorem 2.2** *In the following differential equation*

$$(a_mt^2 + b_mt + c_m)f^{(m)}(t) + (u_{m-1}t^2 + b_{m-1}t + c_{m-1})f^{(m-1)}(t) + \cdots + (a_0t^2 + b_0t + c_0)f(t) = g(t), \quad (2.6)$$

*the Elzaki transform is a suitable method for using, if*

$$c_m = 0, \quad c_{m-1} = (m+1)b_m, \quad 2a_1 = b_0,$$

*and $i(i+1)a_0 - ib_{i-1} + c_{i-2} = 0$ for $i = 2, \ldots, m$.*

**Proof.** If we take Elzaki transform of both sides of (2.6) and by using Remark 2.1, we have

$$T''(u)\left[\frac{a_m}{u^{m-4}}+\frac{a_{m-1}}{u^{m-5}}+\frac{a_{m-2}}{u^{m-6}}+\cdots\right]+$$
$$T'(u)\Big[-2m\frac{a_m}{u^{m-3}}-2(m-1)\frac{a_m}{u^{m-4}}-$$
$$2(m-2)\frac{a_{m-2}}{u^{m-5}}-\cdots-2\times 2\frac{a_2}{u^{-1}}-$$
$$2\frac{a_1}{u^{-2}}+\frac{b_m}{u^{m-2}}+\cdots+\frac{b_0}{u^{-2}}\Big]$$
$$+T(u)\Big[a_m\times\frac{m(m+1)}{u^{m-2}}+$$
$$a_{m-1}\frac{(m-1)m}{u^{m-3}}+a_{m-2}\frac{(m-2)(m-1)}{u^{m-4}}+$$
$$\cdots+a_2\times\frac{2\times 3}{u^0}+a_1\times\frac{1\times 2}{u^1}-$$
$$(m+1)\frac{b_m}{u^{m-1}}-m\frac{b_{m-1}}{u^{m-2}}-(m-1)\frac{b_{m-2}}{u^{m-3}}-$$
$$\cdots-2\frac{b_1}{u^0}-\frac{b_0}{u^{-1}}+\frac{c_m}{u^m}+\frac{c_{m-1}}{u^{m-1}}+\cdots+\frac{c_0}{u^0}\Big]$$
$$=E[g(t)]-p(u),$$

where $p(u)$ consists of some expressions that are started by $\sum$ and do not influence the proof steps.

Now for solving the equation by Elzaki method the coefficient of $T(u)$ must equal to zero. Thus by considering $u^m$, we should have $c_m=0$. By considering $u^{m-1}$, we should have $-(m+1)b_m+c_{m-1}=0$ and hence $c_{m-1}=(m+1)b_m$. By considering $u^{m-2}$, we should have $a_m\times m(m+1)-mb_{m+1}+c_{m-2}=0$, and for $u^{m-3}$, we should have $a_{m-1}(m-1)m-(m-1)b_{m-2}+c_{m-3}=0$. In the same way, we will continue until for $u^0$, well should have $6a_2-2b_1+c_0=0$. By considering $u^{-1}$, we should have

$$2a_1-b_0=0,\qquad 2a_1=b_0.$$

Therefore, in general we can say that

$$c_m=0,\quad c_{m-1}=(m+1)b_m,\quad 2a_1=b_0,$$

and $i(i+1)a_i-ib_{i-1}+c_{i-2}=0$ for $i=2,\dots,m$.

**Theorem 2.3** *In the following differential equation*

$$(d_mt^3+a_mt^2+b_mt+c_m)f^{(m)}(t)+$$
$$(d_{m-1}t^3+a_{m-1}t^2+b_{m-1}t+c_{m-1})f^{(m-1)}(t)+$$
$$\cdots+(d_0t^3+a_0t^2+b_0t+c_0)f(t)=g(t),\quad (2.7)$$

*the Elzaki transform is a suitable way to obtain the answer, when*

$$c_m=0,\quad c_{m-1}=0,\quad b_m=0,$$
$$b_{m-1}=2ma_m,$$
$$m(m+1)a_m-mb_{m-1}+c_{m-2}=0,\quad (2.8)$$

*and* $3i(i-1)d_i-2(i-1)a_i+b_{i-2}=0$ *for* $i=2,\dots,m$, *and* $-(i-1)i(i+1)d_i+i(i-1)a_{i-1}-(i-1)b_{i-2}+c_{i-3}=0$ *for* $i=3,\dots,m$.

**Proof.** After taking Elzaki transform from both sides of (2.7) and by using the pervious results, like the Theorem 2.2, by knowing that the coefficients of $T(u)$ and $T'(u)$ should be to zero, the theorem is proved very easily.

## 3 Examples

In this section some examples show the usage of Elzaki transform for solving the ordinary differential equations with non-constant coefficients.

**Example 3.1** *Consider the following differential equation with non-constant coefficients,*

$$t^2y''+4ty'+2y=12t^2,$$
$$y(0)=y'(0)=0.\quad (3.9)$$

Since the conditions of Theorem 2.2 are satisfied, so using the Elzaki transform leads to finding the answer. By applying the Elzaki transform to (3.9), we have

$$E[t^2y'']+4E[ty']+2E[y]=12E[t^2].\quad (3.10)$$

After simplifying equation (3.10), we have $T''(u)=24u^3$. And after two times integration, $T(u)=2u^4+c_1u+c_0$. If $c_1=c_2=0$, we have $T(u)=2u^4$. And if we use the inverse Elzaki transform we have $y(t)=t^2$.

**Example 3.2** *Consider Legendre differential equation*

$$(1-t^2)y''-2ty'+2y=0,$$
$$y(0)=0,\ y'(0)=1.\quad (3.11)$$

From equation (3.11), we have

$$a_0=b_0=0\ ,\ c_0=2\ ,\ c_1=0\ ,$$
$$b_1=-2\ ,\ a_2=-1\ ,\ b_2=0\ ,\ c_2=1,$$

so with respect to the conditions of Theorem 2.2, $c_m$ should be equal to 0, while $c_2$ is equal to 1. So the conditions of Theorem 2.2 are not satisfied. Now if we take Elzaki transform form both sides of (3.11), we have

$$E[y''] - E[t^2y''] - 2E[ty'] + 2E[y] = 0. \quad (3.12)$$

After simplifying (3.12), we have

$$-u^2T''(u) - 6uT'(u) + T(u)\left(\frac{1}{u^2} + 6\right) = 0,$$

therefore the differential equation (3.11) changed into a second differential equation with non-constant coefficients. So using Elzaki transform did not lead to finding the answer.

**Example 3.3** *Consider the following differential equation*

$$t^4y^{(4)} + 4t^3y''' - 2t^2y'' - 4ty' = 0. \quad (3.13)$$

*To Solve this equation, we consider*

$$t^3y^{(4)} + 4t^2y''' - 2ty'' - 4y' = 0, \quad (3.14)$$

*and by getting $y'(t) = g(t)$, (3.14) changes into*

$$t^3g''' + 4t^2g'' - 2tg' - 4g = 0. \quad (3.15)$$

Now, since the conditions of Theorem 2.3 are satisfied, we take Elzaki transform from both sides of (3.15), as

$$E[t^3g'''] + 4E[t^2g''] - 2E[tg'] - 4E[g] = 0,$$

or

$$u^3T'''(u) - 6u^2T''(u) + 18uT'(u) - 24T(u) + 4u^2T''(u) - 16uT'(u) + 24T(u) - 2uT'(u) + 4T(u) - 4T(u) = 0.$$

Thus, after simplifying, we have $\frac{T'''(u)}{T''(u)} = \frac{2}{u}$, after integration from both sides, so we have

$$\ln T''(u) = \ln c_1u^2, \quad T''(u) = c_1u^2,$$

and hence $T(u) = \frac{c_1}{12}u^4 + c_2u + c_3$. If $c_2 = c_3 = 0$, thus $T(u) = \frac{c_1}{12}u^4$. By using the inverse Elzaki transform well have $g(t) = \frac{c_1}{24}t^2$. On the other hand, because $y'(t) = g(t)$, so we have $y'(t) = \frac{c_1}{24}t^2$, and hence $y(t) = \frac{c_1}{72}t^3 + c_4$.

**Example 3.4** *Consider the following differential equation*

$$(t^4 + 5t^3)y''' + 7t^2y'' + 8ty' = t^3 - 2t. \quad (3.16)$$

*First, we should simplify (3.16) as*

$$(t^3 + 5t^2)y''' + 7ty'' + 8y' = t^2 - 2. \quad (3.17)$$

Now, we investigate the conditions of Theorem 2.3 for the equation (3.17). It is easy to see that, the conditions of Theorem 2.3 are not satisfied. Therefore, using Elzaki transform does not lead to finding the answer of differential equation (3.17). If we use Elzaki transform to solve equation (3.17), we have

$$E[t^3y'''] + 5E[t^2y'''] + 7E[ty''] + 8E[y'] = E[t^2] - E[2]. \quad (3.18)$$

After using Elzaki transform rules and simplifying the equation, we have

$$u^3T'''(u) + (5u - 6u^2)T''(u) + (8u - 23)T'(u) + \left(\frac{47}{u} - 24\right)T(u) = 2u^4 - 2u^2 + 11uf(0).$$

So (3.17) changed into a new differential equation with non-constant coefficients. Consequently, we observed that Elzaki transform is not a suitable method to obtain the answer of this differential equation.

## 4 Conclusion

Elzaki transform is more suitable than Sumudu and Laplace transforms to solve the ordinary differential equations with non-constant coefficients, but it is better that before using this method, we assured that if using this method leads to finding the answer or not? Even sometimes it is possible that the differential equations change into a higher order differential equation with non-constant(variable) coefficients.

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