



## A Note on Modular Hyperconvex Space

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### Abstract

Recently, it has been obtained a lot of results on hyperconvex space (see [1, 2, 3, 5]). In this paper we develop some of those results for modular hyperconvex spaces.

*Keywords* : Hyperconvex space, Modular function, Modular hyperconvex space.

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## 1 Introduction

The theory of hyperconvex space was initiated by Aronzajn and Panitchpakdi [1]. It was proved that a hyperconvex space is a nonexpansive absolute retract ,i.e. it is a nonexpansive retract of any metric space in which it is isometrically embedded. Many interesting results and applications of the theory of hyperconvex spaces in another branch of mathematics are the base of our motivation to study this subject. For example ,it has been used in probability and mathematical statistics, boundary-value problems [3], the inverse function [10], and the existence of solutions of differential equations [9, 12]. The theory of modular function space was initiated by Nakano in 1950 in connection with the theory of order spaces and redefined and generalized by Luxemburg and Orlicz in 1959.

The organization of this paper is the following: We start with introducing the definitions and notations which will be used later. For convenience of readers, we suggest that one refers to [1, 2, 4, 5, 13]. Section two startes with the proof of the existence of the selection of Lipschitz set valued mappings  $T^*$  from a modular hyperconvex space  $\mathcal{H}_\rho$  that choose their values from the space of the external modular hyperconvex subset  $\mathcal{H}_\rho$  i.e  $\varepsilon_\rho(\mathcal{H}_\rho)$ . Also we get interesting results by considering intersection of the sets in modular hyperconvex spaces such as the intersection of a modular admissible subset and

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a externally modular hyperconvex subset with respect to the hyperconvex modular space  $\mathcal{H}_\rho$  is externally modular hyperconvex with respect to  $\mathcal{H}_\rho$ .

Also we show that when the communion of externally modular hyperconvex subsets of a modular hyperconvex space is nonempty.

## 2 Preliminaries

Let  $X$  be a vector space on  $\mathbb{R}$ , a function  $\rho: X \rightarrow [0, +\infty]$  is called modular if for every  $x, y$  in  $X$ , (i)  $\rho(x) = 0$  if and only if  $x = 0$ , (ii)  $\rho(\alpha x) = \rho(x)$ , for every  $\alpha \in \mathbb{R}$  where  $|\alpha| = 1$ , (iii)  $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$  if  $\alpha + \beta = 1$  and  $\alpha \geq 0, \beta \geq 0$ , and  $\rho$  is called convex modular if,  $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$  if  $\alpha + \beta = 1$  and  $\alpha \geq 0, \beta \geq 0$ . By a modular space we mean  $X_\rho = \{x \in X : \lim_{\lambda \rightarrow 0} \rho(\lambda x) = 0\}$ , where  $\rho$  is a modular function on  $X$ . Following Khamsi [3], for a modular space  $X_\rho$ , the sequence  $\{x_n\}$  is called  $\rho$ -convergent to  $x$  if  $\rho(x_n, x) \rightarrow 0$ , and it is called  $\rho$ -Cauchy if  $\rho(x_n, x_m) \rightarrow 0$  as  $n, m \rightarrow \infty$ . We will say that the modular function  $\rho$  satisfies the Fatou property if  $\rho(x) \leq \liminf_n \rho(x_n)$  as  $x_n \rightarrow x$ , where  $\{x_n\}$  is a sequence in  $X_\rho$ . A modular function  $\rho$  is called complete if every  $\rho$ -Cauchy sequence  $\{x_n\}$  is  $\rho$ -convergent. A subset  $A$  of  $X_\rho$  is called  $\rho$ -closed if the  $\rho$ -limit of a  $\rho$ -convergent sequence of  $A$  always belongs to  $A$ . By a  $\rho$ -ball  $B_\rho(x, r)$ , we mean  $\{y \in X_\rho : \rho(x - y) \leq r\}$ .

Finally, a subset  $A$  of  $X_\rho$  is called  $\rho$ -bounded if

$$\delta_\rho(A) = \sup\{\rho(x - y) : x, y \in A\} < \infty.$$

We note that  $\rho$  does not behave in general as a metric because  $\rho$  does not satisfy the triangle inequality. For example  $\rho$ -convergent does not imply  $\rho$ -Cauchy. However,  $\rho$ -balls are  $\rho$ -closed in a modular space  $X_\rho$  if and only if they have Fatou property [5].

**Definition 2.1.** A modular space  $X_\rho$  is called modular hyperconvex space if, for any collection of points  $\{x_\alpha\}_{\alpha \in \Gamma}$  of  $X$  and for any collection  $\{r_\alpha\}$  of non-negative real numbers such that  $\rho(1/2(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$  ( $\alpha, \beta \in \Gamma$ ), it follows that  $\bigcap_{\alpha \in \Gamma} B_\rho(x_\alpha, r_\alpha) \neq \emptyset$ .

If  $X_\rho$  is a modular space we show the family of all the nonempty and bounded subset  $X_\rho$  by  $B_\rho(X_\rho)$ .

**Definition 2.2.** Let  $X_\rho$  be a modular space such that has Fatou property. A subset  $A$  of  $X_\rho$  is called modular admissible subset if  $A$  is an intersection of  $\rho$ -closed balls in  $X_\rho$ .

**Definition 2.3.** Suppose that  $X_\rho$  is a modular space and  $C$  its subset. We say that  $C$  is modular proximal, if for each  $x \in X_\rho$

$$C \cap B_\rho(x, \text{dist}_\rho(x, C)) \neq \emptyset$$

such that

$$\text{dist}_\rho(x, C) = \inf\{\rho(x - y) : y \in C\}$$

**Definition 2.4.** A subset  $E$  of modular space  $(X_\rho, \rho)$  is called externally modular hyperconvex (with respect to  $X_\rho$ ) if for each family of elements  $\{x_\alpha\}_{\alpha \in \Gamma}$  in  $X_\rho$  and each family of positive real number  $\{r_\alpha\}_{\alpha \in \Gamma}$  such that for every  $\alpha, \beta \in \Gamma$

$$\text{dist}_\rho(x_\alpha, E) \leq r_\alpha \quad , \quad \rho\left(\frac{1}{2}(x_\alpha - x_\beta)\right) \leq r_\alpha + r_\beta$$

the following holds

$$\bigcap_{\alpha \in \Gamma} B_\rho(x_\alpha, r_\alpha) \cap E \neq \emptyset.$$

$\mathcal{A}_\rho(X_\rho)$  is the notation of all the nonempty modular admissible subsets  $X_\rho$  and  $\varepsilon_\rho(X_\rho)$  is the notation of all the externally modular hyperconvex subsets  $X_\rho$  and  $\mathcal{H}_\rho(X_\rho)$  is the notation of all the modular hyperconvex subsets  $X_\rho$ .

**Definition 2.5.** Let  $A$  be a subset of a modular hyperconvex space  $X_\rho$ , set

$$\begin{aligned} r_x(A) &= \sup\{\rho(x - y) : y \in A\}, \quad x \in X_\rho; \\ r(A) &= \inf\{r_x(A) : x \in X_\rho\}; \\ R(A) &= \inf\{r_x(A) : x \in A\}; \\ \text{diam}(A) &= \sup\{\rho(x - y) : x, y \in A\}; \\ C(A) &= \{x \in X_\rho : r_x(A) = r(A)\}; \\ C_A(A) &= \{x \in A : r_x(A) = r(A)\}; \\ \text{cov}_\rho(A) &= \bigcap\{B : B \text{ is a } \rho\text{-ball and } B \supseteq A\}; \end{aligned}$$

$r(A)$  is called the reduce of  $A$  (relative to  $X_\rho$ ),  $\text{diam}(A)$  is called the diameter of  $A$ ,  $R(A)$  is called Chebyshev radius of  $A$ ,  $C(A)$  is called the Chebyshev center of  $A$ , and  $\text{cov}_\rho(A)$  is called the cover of  $A$ .

The reader can see the proof of the following Lemma in [8].

**Lemma 2.1.** Let  $A$  be a  $\rho$ -bounded subset of modular hyperconvex space  $X_\rho$ , then:

- 1)  $\text{cov}_\rho(A) = \bigcap\{B_\rho(x, r_x(A)) : x \in X_\rho\}$ .
- 2)  $r_x(\text{cov}_\rho(A)) = r_x(A)$ , for any  $x \in X_\rho$ .
- 3)  $r(\text{cov}_\rho(A)) = r(A)$ .
- 4)  $r(A) = 1/2(\text{diam}(A))$ .
- 5)  $\text{diam}(\text{cov}_\rho(A)) = \text{diam}(A)$ .
- 6) If  $A = \text{cov}_\rho(A)$ , then  $r(A) = R(A)$ . In particular we have  $R(A) = 1/2(\text{diam}(A))$ .

If  $A$  is a subset of modular space  $X_\rho$ , we denote the  $\varepsilon$ - closed neighborhood  $A$  with  $N_\varepsilon(A)$  in which

$$N_\varepsilon(A) = \{x \in X_\rho : \text{dist}_\rho(x, A) \leq \varepsilon\}.$$

**Definition 2.6.** Suppose that  $A$  is an arbitrary set. A map  $T$  from  $A$  to  $P(A)$  where  $P(A)$  is the power set of  $A$ , is called set-valued mapping.

**Definition 2.7.** Let  $T^* : X_\rho \rightarrow B_\rho(X_\rho)$  is a set-valued mapping. A selection is a map such as  $T : X_\rho \rightarrow X_\rho$  such that  $T(x) \in T^*(x)$  for each  $x \in X_\rho$ .

**Definition 2.8.** Suppose that  $(X_1, \rho_1)$  and  $(X_2, \rho_2)$  are modular spaces. we say the mapping  $T : X_1 \rightarrow X_2$  is  $\lambda$ - lipschitzian when there exists  $\lambda \geq 0$  such that for each  $x, y \in X_1$  the following satisfies  $\rho_2(T(x) - T(y)) \leq \lambda\rho_1(x - y)$ .

The smallest  $\lambda$  that satisfies in the above relation is called lipschitzian constant and we note it with  $\text{Lip}(T)$ . If  $\lambda = 1$  then we say the above map is nonexpansive.

**Lemma 2.2.** (lemma 2.2 [8]) If  $A$  is a external modular hyperconvex subset or a admissible modular subset of a modular hyperconvex space  $H_\rho$ , then  $A$  is the modular proximal in  $H_\rho$ .

**Lemma 2.3.** (lemma 3.2 [8]) Let  $X_\rho$  be a modular hyperconvex space and  $D = \bigcap_{\alpha \in \Gamma} B_\rho(x_\alpha, r_\alpha)$ . In this case for each  $\varepsilon > 0$  we have

$$N_\varepsilon(D) = \bigcap_{\alpha \in \Gamma} B_\rho(x_\alpha, r_\alpha + \varepsilon).$$

**Definition 2.9.** If  $A$  and  $B$  are bounded and closed subsets in modular space  $X_\rho$  we define Hausdorff distance  $\rho_H$  as follows:

$$\rho_H(A, B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ , } B \subset N_\varepsilon(A)\}.$$

**Theorem 2.1.** (Theorem 2.2 [8]) If  $X_\rho$  is a modular hyperconvex space then we have

$$\mathcal{A}_\rho(X_\rho) \subset \varepsilon_\rho(X_\rho) \subset \mathcal{H}_\rho(X_\rho).$$

### 3 Main results

In this section, we develop some results getting in [4, 7], for modular hyperconvex space.

**Theorem 3.1.** Let  $H_\rho$  be a modular hyperconvex space ,  $S$  is an arbitrary set and  $T^* : S \rightarrow \varepsilon_\rho(H_\rho)$ . Then there exists the map  $T : S \rightarrow H_\rho$  such that for each  $x \in S$ ,  $T(x) \in T^*(x)$  and for each  $x, y \in S$  we have

$$\rho(T(x) - T(y)) \leq \rho_H(T^*(x), T^*(y)).$$

**Proof:** Let  $F$  denote the collection of all pairs  $(D, T)$ , where  $D \subseteq S$  and for all  $d \in D$  and for all  $x, y$  in  $D$  we have

$$T : D \rightarrow H$$

$$T(d) \in T^*(d)$$

$$\rho(T(x) - T(y)) \leq \rho_H(T^*(x), T^*(y))$$

We note that for each  $x_0 \in S$ ,  $T(x_0) \in T^*(x_0)$ , then  $(\{x_0\}, T) \in F$ . Thus we have  $F \neq \emptyset$ . Now, we define the order relation  $\leq$  on  $F$  as follows

$$(D_1, T_1) \leq (D_2, T_2) \Leftrightarrow D_1 \subset D_2 \text{ , } T_2|_{D_1} = T_1.$$

Suppose  $\{(D_\alpha, T_\alpha)\}$  is the increasing chain in  $(F, \leq)$ . So it follows that  $(\bigcup_{\alpha \in \Gamma} D_\alpha, T) \in F$  where  $T|_{D_\alpha} = T_\alpha$ . It is clear that this member is an upper bound for above chain. with defining order by Zorn's lemma, the maximal element such as  $(D, T)$  in  $(F, \leq)$  exists. Now, we show that  $D = S$ . Assume  $D \neq S$ , therefore there exists the element  $x_0 \in S \setminus D$ . Let  $\tilde{D} = D \cup \{x_0\}$ .

Now, consider the following set :

$$J = \left( \bigcap_{x \in D} B_\rho(T(x), \rho_H(T^*(x), T^*(x_0))) \right) \cap T^*(x_0).$$

Since  $T^*(x_0) \in \varepsilon_\rho(H_\rho)$  we have  $J \neq \emptyset$  if and only if for each  $x \in D$  :

$$\text{dist}_\rho(T(x), T^*(x_0)) \leq \rho_H(T^*(x), T^*(x_0)).$$

by lemma 2.2 we have  $T^*(x_0) \in \varepsilon_\rho(H_\rho)$  as a modular proximal subset from  $H_\rho$ . The above is true if and only if for each  $x \in D$

$$B_\rho(T(x), \rho_H(T^*(x), T^*(x_0))) \cap T^*(x_0) \neq \emptyset.$$

By the definition of Hausdorff distance

$$T^*(x) \subset N_{\rho_H(T^*(x), T^*(x_0)) + \varepsilon}(T^*(x_0)).$$

However by assumption  $T(x) \in T^*(x)$  so it must be the case that for each  $\varepsilon > 0$

$$B_\rho(T(x), \rho_H(T^*(x), T^*(x_0)) + \varepsilon) \cap T^*(x_0) \neq \emptyset.$$

Since  $T^*(x_0)$  is a modular proximial in  $H_\rho$ , this implies that

$$B_\rho(T(x), \rho_H(T^*(x), T^*(x_0))) \cap T^*(x_0) \neq \emptyset.$$

Thus  $J \neq \emptyset$ . Now choose  $y_0 \in J$  and define

$$\tilde{T}(x) = \begin{cases} y_0 & \text{if } x = x_0 \\ T(x) & \text{if } x \in D \end{cases}$$

On the other hand  $\rho(\tilde{T}(x_0) - \tilde{T}(x)) = \rho(y_0 - T(x)) \leq \rho_H(T^*(x), T^*(x_0))$  ( $\forall x \in D$ ). Thus  $(D \cup \{x_0\}, \tilde{T}) \in F$  and it has contradiction with maximality of  $(D, T)$ . Therefore  $D = S$ .

**Corollary 3.1.** *Suppose that  $H_\rho$  is a modular hyperconvex space and  $(M, \rho_1)$  is a modular space and the set-valued mapping  $T^* : M \rightarrow \varepsilon_\rho(H_\rho)$  is nonexpansive i.e, for each  $x, y \in M$ ,  $\rho_H(T^*(x), T^*(y)) \leq \rho_1(x - y)$ . Then the nonexpansive map  $T : M \rightarrow H_\rho$  exists such that for each  $x \in M$ ,  $T(x) \in T^*(x)$ .*

**Proof:** By theorem 3.1 and nonexpansive the  $T^*$ , there exists the selection  $T : M \rightarrow H_\rho$  that for each  $x \in M$ ,  $T(x) \in T^*(x)$  and

$$\rho(T(x) - T(y)) \leq \rho_H(T^*(x), T^*(y)) \leq \rho_1(x - y), \quad (\forall x, y \in M).$$

Therefore  $\rho(T(x) - T(y)) \leq \rho_1(x - y)$ , thus  $T$  is nonexpansive.

**Theorem 3.2.** *Let  $M_\rho$  be a bounded modular space and  $(H_\beta)_{\beta \in \Gamma}$  be a decreasing family of nonempty modular hyperconvex subsets of  $M_\rho$ , where  $\Gamma$  is totally ordered. Then  $\bigcap_{\beta \in \Gamma} H_\beta$  is nonempty and modular hyperconvex.*

**Proof:** Define  $F$  as the follows:

$$F = \{A = \Pi_{\beta \in \Gamma} A_\beta, A_\beta \in \mathcal{A}_\rho(H_\beta) \text{ and } (A_\beta) \text{ is decreasing and nonempty}\}.$$

Since  $M_\rho$  is bounded so  $H_\beta$  is bounded. Thus  $H_\beta \in \mathcal{A}_\rho(H_\beta)$ . So  $H_\beta$  is not empty and decreasing then  $\Pi_{\beta \in \Gamma} H_\beta \in F$  and  $F \neq \emptyset$ .

Since  $H_\beta$  is modular hyperconvex then  $\mathcal{A}_\rho(H_\beta)$  is compact for every  $\beta \in \Gamma$ . Thus  $F$  satisfies the assumptions of zorn's lemma when ordered by set inclusion. Hence for every  $D \in F$  there exists a minimal element  $A \in F$  such that,  $A \subset D$ .

We claim that if  $A = \Pi_{\beta \in \Gamma} A_\beta$  is minimal then there exists  $\beta_0 \in \Gamma$  such that  $\delta(A_\beta) = 0$  for every  $\beta \geq \beta_0$ , where  $\delta(A) = \text{diam}(A)$ . Let  $\beta \in \Gamma$  be fixed. For every  $D \subset M_\rho$  define

$$\text{cov}_{\rho\beta}(D) = \bigcap_{x \in H_\beta} \beta_\rho(x, r_x(D)).$$

Consider  $A' = \prod_{\alpha \in \Gamma} A'_\alpha$  where  $A'_\alpha = \text{cov}_{\rho\beta}(A_\beta) \cap A_\alpha$  if  $\alpha \leq \beta$  and  $A'_\alpha = A_\alpha$  if  $\alpha \geq \beta$ .

Since  $A \in F$  then the family  $(A'_{\alpha \geq \beta})$  is decreasing. Let  $\alpha \leq \gamma \leq \beta$ . Since  $A_\gamma \subset A_\alpha$  and  $A_\beta = \text{cov}_{\rho\beta}(A_\beta) \cap A_\beta$  so  $A'_\gamma \subset A'_\alpha$ . Hence the family  $(A'_\alpha)$  is decreasing. On the other hand if  $\alpha \leq \beta$  then  $\text{cov}_{\rho\beta}(A_\beta) \cap A_\alpha \in \mathcal{A}_\rho(H_\alpha)$ . Since  $H_\beta \subset H_\alpha$  so  $A'_\alpha \in \mathcal{A}_\rho(H_\alpha)$ . Thus  $A' \in F$ . Since  $A$  is minimal this implies that  $A = A'$  which implies

$$A_\alpha = \text{cov}_{\rho\beta}(A_\beta) \cap A_\alpha \quad \forall \alpha \leq \beta.$$

Let  $x \in H_\beta$  and  $\alpha \leq \beta$ . Since  $A_\beta \subset A_\alpha$ , then  $r_x(A_\beta) \leq r_x(A_\alpha)$ . Now  $\text{cov}_{\rho\beta}(A_\beta) = \bigcap_{x \in H_\beta} \beta_\rho(x, r_x(A_\beta))$ , then we have  $\text{cov}_{\rho\beta}(A_\beta) \subset \beta_\rho(x, r_x(A_\beta))$ , which implies

$$r_x(\text{cov}_{\rho\beta}(A_\beta)) \leq r_x(A_\beta).$$

Additionally  $A_\alpha \subset \text{cov}_{\rho\beta}(A_\beta)$  so

$$r_x(A_\beta) \leq r_x(A_\alpha) \leq r_x(\text{cov}_{\rho\beta}(A_\beta)) \leq r_x(A_\beta).$$

Therefore we have  $r_x(A_\beta) = r_x(A_\alpha)$  for every  $x \in H_\beta$ . Using the definition of  $r$ , we get

$$r(A_\alpha) \leq r(A_\beta).$$

Let  $a \in A_\alpha$  and  $s = r_a(A_\alpha)$ , then  $a \in \text{cov}_{\rho\beta}(A_\beta)$ . Since  $A_\alpha \subset \text{cov}_{\rho\beta}(A_\beta)$  so

$$a \in \bigcap_{x \in A_\beta} \beta_\rho(x, s) \cap \text{cov}_{\rho\beta}(A_\beta).$$

By hyperconvexity of  $H_\beta$ ,

$$S_\beta = H_\beta \cap \bigcap_{x \in A_\beta} \beta_\rho(x, s) \cap \text{cov}_{\rho\beta}(A_\beta) \neq \emptyset.$$

Let  $z \in S_\beta$ , then  $z \in \bigcap_{x \in A_\beta} \beta_\rho(x, s)$ . Since

$$A_\beta = H_\beta \cap \text{cov}_{\rho\beta}(A_\beta).$$

It follows that  $r_z(A_\beta) \leq s$ , which implies

$$r(A_\beta) \leq s = r_a(A_\alpha)$$

for every  $a \in A_\alpha$ . Hence

$$r(A_\beta) = r(A_\alpha) \quad \forall \alpha, \beta \in \Gamma.$$

Assume that  $\delta(A_\beta) > 0$  for every  $\beta \in \Gamma$ . Set  $A''_\beta = C(A_\beta)$  for every  $\beta \in \Gamma$ . The family  $(A''_\beta)$  is decreasing. Let  $\alpha \leq \beta$  and  $x \in A''_\beta$ , then  $r_x(A_\beta) = r(A_\beta)$ . Since we proved that  $r_z(A_\beta) = r_z(A_\alpha)$  for every  $z \in H_\beta$  then  $r_x(A_\alpha) = r_x(A_\beta) = r(A_\beta) = r(A_\alpha)$ , which implies that  $x \in A''_\alpha$ . Therefore

$$A'' = \prod_{\beta \in \Gamma} A''_\beta \in F.$$

Since  $A'' \subset A$  and  $A$  is minimal, we get that  $A'' = A$ . Therefore  $A_\beta = C(A_\beta)$  for every  $\beta \in \Gamma$ . This is in contradiction to hyperconvexity of  $H_\beta$  for each  $\beta \in \Gamma$ . Thus there exists  $\beta_0 \in \Gamma$  such that  $\delta(A_\beta) = 0$  for every  $\beta \geq \beta_0$ . So  $A_\beta = \{a\}$  for every  $\beta \geq \beta_0$ , which implies that  $a \in \bigcap_{\beta \in \Gamma} H_\beta \neq \emptyset$ .

In order to complete the proof, we need to show that  $S = \bigcap_{\beta \in \Gamma} H_\beta$  is modular hyperconvex. Let  $(\beta_{\rho_i})_{i \in I}$  be a family of balls centered in  $S$  such that  $\bigcap_{i \in I} \beta_{\rho_i} \neq \emptyset$ . Define  $D_\beta = \bigcap_{i \in I} \beta_{\rho_i} \cap H_\beta$  for all  $\beta \in \Gamma$ . Since  $H_\beta$  is modular hyperconvex and the family  $(\beta_{\rho_i})_{i \in I}$  centered in  $H_\beta$  then  $D_\beta$  is not empty and  $D_\beta \in \mathcal{A}_\rho(H_\beta)$ . Therefore  $D_\beta$  is modular hyperconvex. the above proof shows that  $\bigcap_{\beta \in \Gamma} D_\beta \neq \emptyset$ .

**Lemma 3.1.** *Let  $H_\rho$  be a modular hyperconvex space and  $E \subset H_\rho$  be externally modular hyperconvex related to  $H_\rho$ . Suppose  $A$  is a modular admissible subset of  $H_\rho$ . Then  $E \cap A$  is externally modular hyperconvex related to  $H_\rho$ .*

**Proof:** Suppose  $\{x_\alpha\}$  and  $\{r_\alpha\}$  satisfy  $\rho(\frac{1}{2}(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$  and  $dist_\rho(x_\alpha, E \cap A) \leq r_\alpha$ . Since  $A$  is admissible,  $A = \bigcap_{i \in I} \beta_\rho(x_i, r_i)$  and since  $\beta_\rho(x_\alpha, r_\alpha) \cap A \neq \emptyset$ . It follows that  $\rho(\frac{1}{2}(x_\alpha - x_i)) \leq r_\alpha + r_i$  for each  $i \in I$ . Since  $A \subset \beta_\rho(x_i, r_i)$  it follows that

$$dist_\rho(x_i, E \cap A) \leq r_i \quad , \quad \rho(\frac{1}{2}(x_i - x_j)) \leq r_i + r_j \quad \forall i, j \in I.$$

Therefore by external modular hyperconvexity of  $E$

$$(\bigcap_i \beta_\rho(x_i, r_i)) \cap (\bigcap_\alpha \beta_\rho(x_\alpha, r_\alpha)) \cap E = \bigcap_\alpha \beta_\rho(x_\alpha, r_\alpha) \cap (A \cap E) \neq \emptyset.$$

Thus  $E \cap A$  is externally modular hyperconvex related to  $H_\rho$ .

**Theorem 3.3.** *Let  $\{H_i\}$  be a decreasing chain of nonempty modular externally hyperconvex subsets of a bounded modular hyperconvex space  $H_\rho$ . Then  $\bigcap_i H_i$  is nonempty and externally modular hyperconvex in  $H_\rho$ .*

**Proof:** By Theorem (3.2) and Theorem (2.1), we have  $D = \bigcap_i H_i \neq \emptyset$ . To prove  $D$  is externally modular hyperconvex, let  $\{x_\alpha\} \subset H$  and  $\{r_\alpha\} \subset R$  satisfy

$$\rho(\frac{1}{2}(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta \quad , \quad dist_\rho(x_\alpha, D) \leq r_\alpha.$$

Since  $H_\rho$  is modular hyperconvex we know that  $A = \bigcap_\alpha \beta_\rho(x_\alpha, r_\alpha) \neq \emptyset$ . also  $dist_\rho(x_\alpha, D) \leq r_\alpha$  and  $dist_\rho(x_\alpha, H_i) \leq r_\alpha$  for each  $i$ , so by externally modular hyperconvexity of  $H_i$  implies that for each  $i$  we have  $A \cap H_i \neq \emptyset$ . By Theorem 2.1 and lemma 3.1,  $\{A \cap H_i\}$  is a decreasing chain of nonempty modular hyperconvex subsets of  $H_\rho$ . Now by Theorem 3.2, we have

$$\bigcap_i (A \cap H_i) = A \cap D \neq \emptyset.$$

Thus  $\bigcap_i H_i$  is nonempty and externally modular hyperconvex in  $H_\rho$ .

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**References**

[1] N. Aronszajn, P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric space, Pacific J. Math,6 (1956), 405-439.  
 [2] J. B. Baillon, Nonexpansive mapping and hyperconvex space, Contemp. Math,72 (1986), 11-22.

- [3] J. G. Dix and G. L. Karakostas, A fixed-point theorem for S-type operators on Banach spaces and its applications to boundary-value problems, *Nonlinear Analysis* 71 (2009).
- [4] R. Espinola and M.A.Khamsi, Introduction to hyperconvex spaces *Handbook of Metric Fixed Point Theory*, Kluwer, Dordrecht, 2001, 391-435.
- [5] M. A. Japon, Some geometric properties in modular spaces and application to fixed point theorem, *J Math. Appl.* 295(2004), 576-594.
- [6] M. A. Khamsi , ON asymptotically nonexpansive mappings in hyperconvex metric spaces, *Volume. 132, number 2, page 365-373 (2003)*.
- [7] M.A. Khamsi, W.A. Kirk, And Carlos Martinez Yanez (communicated by David R. Larson) , Fixed point and selection theorems in hyperconvex spaces, *volume 128, number 11, pages 3275-3283 (2000)*
- [8] H.R. Rahimi, M. Firooznasab, Some Results On Modular Hyperconvex Spaces. *Int. J. Industrial Mathematics Vol. 2, No.1, (2010) 43-46*.
- [9] Robert F. Brown, *A Topological Introduction to Nonlinear Analysis*, Birkhuser, Boston, 1993.
- [10] Serge Lang, *Undergraduate Analysis*, Springer Science Business Media, New York, 1997.
- [11] R. Sine, Hyperconvexity and approximate fixed points, *Nonlinear, Anal.* 13 (1989), 893-869
- [12] D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, 1974.
- [13] P. Soardi, Existence of fixed points for nonexpansive mapping in certain Banach lattices, *Proc. Amer. Math. Soc* 13 (1972) 25-29