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Ranking Stochastic Efficient DMUs based on Reliability

G.R. Jahanshahloo [a], M. H. Behzadi [b], M. Mirbolouki[a]*

(a) *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.*

(b) *Department of Statistics, Science and Research Branch, Islamic Azad University, Tehran, Iran.*

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Abstract

Data Envelopment Analysis (DEA) is a nonparametric mathematical programming approach for evaluating efficiency of a set of decision making units (DMUs). Since the number of efficient DMUs is more than one, there is a necessity of having methods to discriminate between efficient DMUs. In classic DEA it is assumed that data have deterministic values. Through real world applications this assumption may not be satisfied. On the other hand, data may have stochastic essence. In this paper a method for ranking stochastic DMUs is suggested which is based on the reliability of efficiency of DMUs. Using numerical example, we demonstrate how to use the results.

Keywords : Data envelopment analysis; Quadratic programming; Ranking; Stochastic programming.

1 Introduction

Data Envelopment Analysis (DEA) concepts were originated by Charnes et al. [3] after that this idea was extended to an approach for evaluating the relative efficiency of DMUs [6,10]. The DMUs usually use a set of resources, referred to as input indices, and transform them into a set of outcomes, referred to as output indices. DEA models divide DMUs into two categories: efficient DMUs and inefficient DMUs. In many applications, we know that usually several DMUs are efficient. To discriminate between these efficient DMUs ranking methodologies had been initiated [9]. Sexton et al. [11] were pioneers in ranking field. They introduced a ranking method based on a cross-efficiency. Andersen and Petersen [1] evaluated a DMUs efficiency by excluding it from production possibility set and they started supper efficiency field. They tried to discriminate between these efficient DMUs,

*Corresponding author. Email address: m.mirbolouki@srbiau.ac.ir

by using different efficiency scores larger than 1.0. Cook et al. [4] developed prioritization models to rank the efficient units in DEA. Torgersen et al. [12] obtained a complete ranking of efficient DMUs by measuring their importance as a benchmark for inefficient DMUs.

In many applications through real world, data of problem are imprecise. One of methods in confronting with these kinds of data is considering them as stochastic or random variables. Cooper et al. [5] applied stochastic variables in DEA models. They also defined stochastic efficient DMUs. Huang and Li [8] introduced stochastic dominance conditions. behzadi et al. [2] and Hosseinzadeh Lotfi et al. [7] proposed methods for ranking stochastic efficient DMUs. In this paper we propose a ranking approach based on Huang an Li [8]. With this method an stochastic efficient DMU has a higher rank if it is dominated with lesser error.

The paper is organized as follows: First the preliminaries on stochastic models and stochastic efficiency are provided and then a model for ranking DMUs based on stochastic reliability is introduced. Using numerical example, we demonstrate how to use the result.

2 Preliminaries

Consider n DMUs with $\tilde{X}_j = (\tilde{x}_{1j}, ..., \tilde{x}_{mj})$ and $\tilde{Y}_j = (\tilde{y}_{1j}, ..., \tilde{y}_{sj})$ as random input and output vectors of DMU_j, $j = 1 \ldots, n$. Assume that $X_j = (x_{1j}, ..., x_{mj})$ and $Y_j = (y_{1j}, ..., y_{sj})$ stand for corresponding vectors of expected values of input and output for every DMU_j. All input and output components have been considered to be normally distributed. The chance constrained version of input oriented stochastic BCC model is as follows:

$$\begin{aligned}
\min \quad & \theta \\
s.t. \quad & p\{\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \geq \tilde{y}_{ro}\} \geq 1-\alpha, \quad r = 1, ..., s, \\
& p\{\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leq \theta \tilde{x}_{io}\} \geq 1-\alpha, \quad i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \qquad\qquad j = 1, ..., n.
\end{aligned} \tag{2.1}$$

Model (2.1) can be converted into the following two-stage model with equality constraints:

$$\begin{aligned}
\min \quad & \theta - \varepsilon(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-) \\
s.t. \quad & p\{\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - s_r^+ \geq \tilde{y}_{ro}\} = 1-\alpha, \quad r = 1, ..., s, \\
& p\{\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} + s_i^- \leq \theta \tilde{x}_{io}\} = 1-\alpha, \quad i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& s_i^- \geq 0, \quad s_r^+ \geq 0, \quad i = 1, ..., m, \quad r = 1, ..., s. \\
& \lambda_j \geq 0, \quad j = 1, ..., n.
\end{aligned} \tag{2.2}$$

where in the above models, p means "probability" and α is a predetermined number between 0 and 1. On basis of normal distribution characteristics, the deterministic model for (2.2) can be attained as follows:

$$\begin{aligned}
\min \quad & \theta - \varepsilon(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ + \Phi^{-1}(\alpha) u_r^o = y_{ro}, \quad r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- - \Phi^{-1}(\alpha) v_i^I = \theta x_{io}, \quad i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& s_i^- \geq 0, \quad s_r^+ \geq 0, \quad i = 1, ..., m, \quad r = 1, ..., s, \\
& \lambda_j \geq 0, \quad j = 1, ..., n.
\end{aligned} \tag{2.3}$$

where

$$(u_r^o)^2 = \sum_{\substack{j=1 \\ j \neq o}}^{n} \sum_{\substack{k=1 \\ k \neq o}}^{n} \lambda_j \lambda_k cov(\tilde{y}_{rj}, \tilde{y}_{rk}) + 2(\lambda_o - 1) \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j cov(\tilde{y}_{rj}, \tilde{y}_{ro}) + (\lambda_o - 1)^2 var(\tilde{y}_{ro}),$$

and

$$(v_i^I)^2 = \sum_{\substack{j=1 \\ j \neq o}}^{n} \sum_{\substack{k=1 \\ k \neq o}}^{n} \lambda_j \lambda_k cov(\tilde{x}_{ij}, \tilde{x}_{ik}) + 2(\lambda_o - \theta) \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j cov(\tilde{x}_{ij}, \tilde{x}_{io}) + (\lambda_o - \theta)^2 var(\tilde{x}_{io}).$$

Here, Φ is the cumulative distribution function of the standard normal distribution and $\Phi^{-1}(\alpha)$, is its inverse in level of α. The above model is a quadratic nonlinear programming model.

Definition 2.1. *DMU_o is stochastic efficient if and only if in the optimal solution of model (2.3), the following conditions are both satisfied:*
(i) $\theta^* = 1$
(ii) Slack values are all zeros.

3 Reliability ranking method

Let $\tilde{X} = (\tilde{x}_1, ..., \tilde{x}_n)$ and $\tilde{Y} = (\tilde{y}_1, ..., \tilde{y}_n)$ be the matrixes of input vectors and output vectors respectively, and $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$ be their mean matrixes. Huang and Li [5] defined β-stochastic efficient DMU as follows,

Definition 3.1. *DMU_o is β-stochastic efficient if it is efficient with reliability of at least* $(1 - \beta)$.

The above definition indicates that an stochastic efficient DMU may be dominated with a probability of at most β. On the other hand DMU_o is β-stochastic efficient if the following expression is hold and strictly for at least one.

$$P\left(\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io}, \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \geq \tilde{y}_{ro}\right) \leq \beta.$$

By the specifications of sets and probability concepts it is resulted that,

$$\left\{\sum_{j=1}^{n}\lambda_j\tilde{x}_j \leq \tilde{x}_o, \sum_{j=1}^{n}\lambda_j\tilde{y}_j \geq \tilde{y}_o\right\} \subset \left\{1^T(\sum_{j=1}^{n}\lambda_j\tilde{x}_j - \tilde{x}_o) + 1^T(\tilde{y}_o - \sum_{j=1}^{n}\lambda_j\tilde{y}_j) < 0\right\}.$$

where $1^T = (1, ..., 1)$. From the above expression we have,

$$P\left(\sum_{j=1}^{n}\lambda_j\tilde{x}_j \leq \tilde{x}_o, \sum_{j=1}^{n}\lambda_j\tilde{y}_j \geq \tilde{y}_o\right) \leq P\left(1^T(\sum_{j=1}^{n}\lambda_j\tilde{x}_j - \tilde{x}_o) + 1^T(\tilde{y}_o - \sum_{j=1}^{n}\lambda_j\tilde{y}_j) < 0\right).$$

Therefore the following expression is the necessary condition for β-stochastic efficiency.

$$P\left(1^T(\sum_{j=1}^{n}\lambda_j\tilde{x}_j - \tilde{x}_o) + 1^T(\tilde{y}_o - \sum_{j=1}^{n}\lambda_j\tilde{y}_j) < 0\right) \leq \beta. \tag{3.4}$$

Let $\tilde{h} = 1^T(\tilde{X}\lambda - \tilde{x}_o) + 1^T(\tilde{y}_o - \tilde{Y}\lambda)$. Therefore expression (3.4) is equal to $P\left(\tilde{h} \leq 0\right) \leq \beta$. It can be converted to an equal form by adding nonnegative variables as $P\left(\tilde{h} \leq 0\right) = \beta - \varepsilon$. Applying slack variables results that,

$$P\left(\tilde{h} \leq s'\right) = \beta. \tag{3.5}$$

$\tilde{h}$ has normal distribution with parameters h and σ_h^2 where,

$$\begin{aligned} h &= E(\tilde{h}) = 1^T(X\lambda - x_o) + 1^T(y_o - Y\lambda), \\ \sigma_h^2 &= Var(\tilde{h}) = 1^T\Sigma 1 + 1^T\Sigma' 1 + 2(1^T\Sigma'' 1). \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\Sigma = [\xi_{ik}]_{m\times m}, \\ &\xi_{ik} = Cov(\sum_{j=1}^{n}\lambda_j x_{ij} - x_{io}, \sum_{j=1}^{n}\lambda_j x_{kj} - x_{ko}), \\ &\Sigma' = [\xi'_{rk}]_{s\times s}, \\ &\xi'_{rk} = Cov(y_{ro} - \sum_{j=1}^{n}\lambda_j y_{rj}, y_{ko} - \sum_{j=1}^{n}\lambda_j y_{kj}), \\ &\Sigma'' = [\xi''_{ir}]_{m\times s}, \\ &\xi''_{ir} = Cov(\sum_{j=1}^{n}\lambda_j x_{ij} - x_{io}, y_{ro} - \sum_{j=1}^{n}\lambda_j y_{rj}). \end{aligned}$$

From expressions (3.5) and (3.6) we have,

$$\begin{aligned} &P\left(\tfrac{\tilde{h}-h}{\sigma_h} \leq \tfrac{s'-h}{\sigma_h}\right) = \beta, \\ &\Rightarrow P\left(\tilde{Z} \leq \tfrac{s'-h}{\sigma_h}\right) = \beta, \\ &\Rightarrow \tfrac{s'-h}{\sigma_h} = \Phi^{-1}(\beta). \end{aligned}$$

Thus deterministic equivalent of (3.4) is $h - s' + \sigma_h \Phi^{-1}(\beta) = 0$.
Suppose DMU_o is an stochastic efficient DMU. Therefore from definition 2.1 and optimal solution of model (2.3),

$$\sum_{j=1}^{n} \lambda_j^* x_{ij} - \Phi^{-1}(\alpha) v_i^* = x_{io}, i = 1, ..., m,$$
$$\sum_{j=1}^{n} \lambda_j^* y_{rj} + \Phi^{-1}(\alpha) u_r^* = y_{ro}, r = 1, ..., s.$$

Let Λ be the optimality solutions space of model (2.3). In our ranking method we seek the minimum level of error in which a DMU is dominated probabilistically on set Λ. i.e. we seek the optimal solution of the following model:

$$\begin{aligned}
\beta^* = \min \quad & \beta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} - \Phi^{-1}(\alpha) v_i = x_{io}, \qquad i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \Phi^{-1}(\alpha) u_r = y_{ro}, \quad r = 1, ..., s, \\
& h - s' + \Phi^{-1}(\beta) w = 0, \\
& v_i^2 = \sum_{\substack{j=1 \\ j \neq o}}^{n} \sum_{\substack{k=1 \\ k \neq o}}^{n} \lambda_j \lambda_k Cov(\tilde{x}_{ij}, \tilde{x}_{ik}) + (\lambda_o - 1)^2 Var(\tilde{x}_{io}) \\
& + 2(\lambda_o - 1) \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j Cov(\tilde{x}_{ij}, \tilde{x}_{io}), i = 1, ..., m, \\
& u_r^2 = \sum_{\substack{j=1 \\ j \neq o}}^{n} \sum_{\substack{k=1 \\ k \neq o}}^{n} \lambda_j \lambda_k Cov(\tilde{y}_{rj}, \tilde{y}_{rk}) + (\lambda_o - 1)^2 Var(\tilde{y}_{ro}) \\
& + 2(\lambda_o - 1) \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j Cov(\tilde{y}_{rj}, \tilde{y}_{ro}), r = 1, ..., s, \\
& w^2 = 1^T \Sigma 1 + 1^T \Sigma' 1 + 2\left(1^T \Sigma'' 1\right), \\
& w \geq 0, \lambda_j \geq 0, j = 1, ..., n, \\
& v_i \geq 0, u_r \geq 0, i = 1, ..., m, r = 1, ..., s.
\end{aligned} \tag{3.7}$$

Model (3.7) is a nonlinear programming which is not clear because of the existence of ill defined term Φ^{-1}. Minimizing β equals minimizing $\Phi^{-1}(\beta)$. Then model (3.7) can be

converted to the following model:

$$
\begin{aligned}
\gamma^* = \min \quad & \gamma \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} - \Phi^{-1}(\alpha) v_i = x_{io}, \qquad i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \Phi^{-1}(\alpha) u_r = y_{ro}, \quad r = 1, ..., s, \\
& h - s' + \gamma w = 0, \\
& v_i^2 = \sum_{\substack{j=1 \\ j \neq o}}^{n} \sum_{\substack{k=1 \\ k \neq o}}^{n} \lambda_j \lambda_k Cov(\tilde{x}_{ij}, \tilde{x}_{ik}) + (\lambda_o - 1)^2 Var(\tilde{x}_{io}) \\
& + 2(\lambda_o - 1) \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j Cov(\tilde{x}_{ij}, \tilde{x}_{io}), i = 1, ..., m, \\
& u_r^2 = \sum_{\substack{j=1 \\ j \neq o}}^{n} \sum_{\substack{k=1 \\ k \neq o}}^{n} \lambda_j \lambda_k Cov(\tilde{y}_{rj}, \tilde{y}_{rk}) + (\lambda_o - 1)^2 Var(\tilde{y}_{ro}) \\
& + 2(\lambda_o - 1) \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j Cov(\tilde{y}_{rj}, \tilde{y}_{ro}), r = 1, ..., s, \\
& w^2 = 1^T \Sigma 1 + 1^T \Sigma' 1 + 2\left(1^T \Sigma'' 1\right), \\
& -3.8 \leq \gamma \leq 3.8, \\
& w \geq 0, \lambda_j \geq 0, j = 1, ..., n, \\
& v_i \geq 0, u_r \geq 0, i = 1, ..., m, r = 1, ..., s.
\end{aligned}
\tag{3.8}
$$

Model (3.8) is a nonlinear quadratic programming model. The constraint $-3.8 \leq \gamma \leq +3.8$ is added to prevent unboundedness of this model. The optimal objective function of model (3.8) is our reliability ranking indicator. Less γ^* indicates better rank.

4 An application

In this section, we consider 10 branches of an Iranian bank with two stochastic inputs and two stochastic outputs and run the mentioned model in order to fully rank the stochastic efficient units. In this model, "payable benefit" and "delayed requisitions" are inputs and "amount of deposits " and "received benefit" are outputs. These data based on considering ten successive months have normal distribution and their scaled parameters are presented in Table 1. We want to assess the total performance of these units. In this example these DMUs have been assessed by using model (2.3). Then stochastic efficient DMUs have been ranked by their inefficiency scores by applying model (3.8). We consider $\alpha = 0.05$ or on the other hand at least 95% confidence to results which are stated in Table 2.

Table 1
Predicted inputs and outputs.

inputs				outputs			
x_{ij}	$N(\mu,\sigma)$	x_{ij}	$N(\mu,\sigma)$	y_{rj}	$N(\mu,\sigma)$	y_{rj}	$N(\mu,\sigma)$
X1,1	N(18.79,9.41)	X2,1	N(7.28,0.76)	Y1,1	N(49.6,6.93)	Y2,1	N(4.7,0.64)
X1,2	N(44.3,25.3)	X2,2	N(1.11,0.15)	Y1,2	N(73.13,3.62)	Y2,2	N(1.85,0.15)
X1,3	N(19.73,16.63)	X2,3	N(19.2,0.69)	Y1,3	N(108.04,15.02)	Y2,3	N(6.06,0.12)
X1,4	N(17.43,11.06)	X2,4	N(59.47,0.92)	Y1,4	N(44.97,3.71)	Y2,4	N(4.9,1.29)
X1,5	N(10.38,4.59)	X2,5	N(12.23,7.74)	Y1,5	N(31.63,6.24)	Y2,5	N(2.78,0.66)
X1,6	N(16.67,10.42)	X2,6	N(568.63,37.42)	Y1,6	N(71.98,8.37)	Y2,6	N(13.19,3.03)
X1,7	N(25.46,13.67)	X2,7	N(552.85,20.78)	Y1,7	N(78.05,13.99)	Y2,7	N(7.79,1.89)
X1,8	N(123.06,65.3)	X2,8	N(14.78,0.25)	Y1,8	N(219.69,19.38)	Y2,8	N(35.3,3.92)
X1,9	N(36.16,19.59)	X2,9	N(361.88,34.11)	Y1,9	N(86.25,6.95)	Y2,9	N(17.64,1.92)
X1,10	N(46.41,23.06)	X2,10	N(12.81,0.62)	Y1,10	N(194.58,42.15)	Y2,10	N(25.9,3.52)

Table 2
Results

efficient DMU	$\widehat{\psi}$	RANK
DMU3	-1.02	3
DMU5	-0.98	4
DMU6	-2.08	1
DMU10	-1.19	2

5 Conclusion

There are several models in DEA field which have been formulated for evaluating efficiency and ranking DMUs in various fields with different data such as: deterministic, interval, fuzzy, e.t.c. In real world application managers may encounter the data which are not deterministic. Nowadays the extent of probability has a significant importance. Therefore DEA models have been extended to stochastic data by researchers. Thus the necessity of having models that are able to rank DMUs has been under consideration. They have been assessed and in such way they have defined the stochastic efficient DMUs. In this paper on basis of the efficiency reliability of efficient DMUs, a model for ranking stochastic efficient DMUs has been presented. This model is a quadratic programming model and the objective function is a function of α, which is the level of error that should be determined by the managers. In this paper we have applied normal distributions. Different distributions as well as normal distribution can be considered from this point of view.

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