



# Non-convex Technologies and Economic Efficiency Measures with Imprecise Data

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## Abstract

The economic efficiency (EE) measure in non-convex technologies requires the data of input/output vectors and prices to be known deterministically. But as regards the data of the production process in many real-world applications, rather than dealing with crisp real numbers and crisp intervals, one has to deal with "approximate" numbers or intervals of the type that can be described as "numbers that are close to a given real number." For the aforementioned reason, development of the economic efficiency models in such a way that they can deal with imprecise data, has become an issue of great interest. To this end, the notion of bounded data and fuzziness has been introduced. This paper develops a procedure to compute the economic efficiency measures with non-convex technologies in the presence of uncertain data. In this study, uncertain EE formulas are transformed into a family of crisp EE formulas and LP models, based on comparison intervals and  $\alpha$  - cuts. To obtain the bounds of the membership functions of efficiencies, we propose a family of parametric two-level programs. This pair of parametric programming problems gives the lower and upper bounds of  $\alpha$  - cuts corresponding to the membership function of EE. Then, we prove that the lower bound is computed by some closed form expressions, but to obtain the upper bound we solve a LP model. Since the efficiency measures are expressed by membership functions rather than by crisp values, more information is provided for the management. Moreover, two examples are provided for illustrating the proposed approaches.

*Keywords* : DEA; Economic Efficiency; Revenue Efficiency; Uncertain Data; Fuzzy Data.

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## 1 Introduction

In data envelopment analysis (DEA) methodology, the boundary of the production possibility set (PPS) plays a crucial role in efficiency and productivity measurement. The free disposal hull (FDH) (see [9, 21]) is a well-known approach and the closest inner approximation of the true, strongly disposable (possibly non-convex) technology. Though the non-convex technology does not dominate the convex ones, the non-convex FDH models have some advantages from the managerial viewpoint (see [3]). The managers may want to compare the unit under assessment with a really observed unit. Many authors have focused on the technical aspects of the production process for use in situations where output price and input cost vectors are not available (see [1, 5]). In most studies, at least some information on output prices is available from theory or practical knowledge of the industry under assessment. One source of price information can be the quality of the products, which is assessed by customers using fuzzy linguistic terms such as excellent, very good, good, average, poor, and very poor (see [11] for more details on linguistic terms). Technology and cost are the wheels that drive modern enterprises; some enterprises have advantages in terms of technology and others in cost (see [8]). Kuosmanen and Post [13, 14] were the first to examine cost efficiency with uncertain input prices. They derived upper and lower bounds for overall cost efficiency, assuming incomplete price data in the form of a convex polyhedral cone. Although they presented and proved a model for determining the lower bound of CE, they did not utilize the model in empirical application of their CE concepts and resorted to the Free Disposable Hull technology. In their method, for obtaining the bounds of cost efficiency, many linear programming problems should be solved. Thus, in large scale applications, the computational burden is increased. So, in the current paper we modify the models for obtaining the upper and lower bounds of revenue efficiency, which is interesting from theory and practical point of view. However, Camanho and Dyson [4], Jahanshahloo et al. [10], and Mostafaei and Saljooghi [18] developed cost efficiency models with convex technology in such a way that they can include uncertain price information. They provided a pair of mathematical programming problems to obtain the lower and upper bounds of cost efficiency. But, they did not deal with non-convex technologies whatsoever. Briec et al. [3] developed a series of nonparametric, non-convex technologies and cost functions with crisp data, and obtained some closed-form expressions for the non-convex cost functions.

In the current study, we extend the standard EE models to include uncertain output prices. First, we assume that the output prices are uncertain and only their lower and upper bounds can be estimated. In this case, the lower bound of revenue efficiency (RE) are computed by using an enumerative processes, similar to that in Briec et al. [3], based upon vector dominance reasoning. Also, a LP problems should be solved to obtain the upper bound. Next, we proceed to extend the non-convex economic efficiency models in order to include fuzzy data and linguistic terms. The basic idea is to transform the non-convex economic efficiency formulas into a family of parametric formulas based on  $\alpha$ -cuts, which is given by closed and bounded intervals. Since the efficiency measures are expressed via a crisp value additionally to its membership function, it provides more information to management. In the  $\alpha$ -cuts based approach, the fuzzy DEA model is solved by parametric programming by using  $\alpha$ -cuts. Solving the model at a given  $\alpha$ -cuts, produces a corresponding efficiency interval for the target DMU. A number of such intervals can be used to construct the corresponding fuzzy efficiency membership functions. More details

on the  $\alpha$  level based approach can be found in Meada et al. [17], Kao and Liu [12], and Lertworasirikul [15].

The rest of this paper unfolds as follows:

In section 2, some materials, required in the following sections are provided. In section 3, the non-convex RE models are developed to include uncertain data in the form of ranges. Also, a parametric closed-form expression is provided to obtain the lower bound of RE measures, but a linear programming problem should be solved to obtain the upper bound and a numerical example is given to illustrate how the proposed method is applied to find RE measures with various RTS assumptions of technology. Section 4 includes some definitions and concepts which are required in the next section. In section 5, the concept of fuzzy data is embedded in RE models; the main idea is to apply the  $\alpha$  – cuts and Zadeh’s extension principle [22, 23, 24] to transform the fuzzy RE formulas to a series of conventional crisp closed-form expressions. Also, a numerical example is provided to illustrate the proposed approach. Section 6 concludes the paper.

## 2 Preliminaries

Assume that we deal with  $n$  DMUs ( $DMU_j; j = 1, \dots, n$ ), which consume  $m$  homogenous inputs  $x_j^t = (x_{1j}, x_{2j}, \dots, x_{mj})^t$  to produce  $s$  homogenous outputs  $y_j^t = (y_{1j}, y_{2j}, \dots, y_{sj})^t$ . To accommodate eventual null components, when  $DMU_o$  is under evaluation, the following notation is introduced for any input  $x_o, o \in \{1, 2, \dots, n\}$ :  $I(x_o) = i \in \{1, \dots, m : x_{io} + x_i > 0\}$ . The non-convex technologies and RE measures of  $DMU_o$  with different returns to scale assumptions of technology can be computed by solving the following model:

$$RE_o^{NC,\Phi}(p_o, x_o) = \min\{(p_o^t y_o)/(p_o^t y) : y \in L^{NC,\Phi}(x_o)\} \tag{2.1}$$

where  $L^{NC,\Phi}(x_o) = \{y : (x_o, y) \in PPS^{NC,\Phi}\}$ ,

$$PPS^{NC,\Phi} = \{(x, y) : x \geq \sum_{j=1}^n \delta \lambda_j x_j, 0 \leq y \leq \sum_{j=1}^n \delta \lambda_j y_j, \sum_{j=1}^n \lambda_j = 1, \lambda_j \in \{0, 1\}, \delta \in \Phi\},$$

and  $\Phi$ , depends on the RTS assumption of the reference technology,

$$\Phi^{VRS} = \{\delta : \delta = 1\},$$

$$\Phi^{CRS} = \{\delta : \delta \geq 0\},$$

$$\Phi^{NIRS} = \{\delta : \delta \geq 1\}.$$

In the following sections, an important role is played by the set  $V_m$  of all  $\pm 1$ -vectors in  $R^m$ : i.e.,  $V_m = \{v \in R^m : |v| = e\}$ ; where  $e = (1, 1, \dots, 1)$  and “ $|\cdot|$ ” indicates the absolute value (see [11]). It is well known that the cardinality of  $V_m$  is  $2^m$ . For a given vector  $v$ ,

$$Q_v = \text{diag}(v_1, v_2, \dots, v_m) = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_m \end{pmatrix}.$$

In [18], an algorithm to generate all  $\pm 1$  - vectors forming set  $V_m$  systemically is provided. For each  $v \in V_m$ , we define vectors  $p_o^v = p_o^c + Q_v \gamma_o$ , where  $p_o^c = 1/2(p_o^L + p_o^R)$  and  $\gamma_o = 1/2(p_o^R - p_o^L)$ . Then, for any such  $v$  we have

$$(p_o^v)_i = (p_o^c)_i + v_i \gamma_{io} = \begin{cases} p_i^L & \text{if } v_i = -1 \\ p_i^R & \text{if } v_i = +1 \end{cases}$$

It is clear that the set  $\{p_o^v : v \in V_m\}$  forms all extreme points of the set  $P_o = \{p_o : p_o^L \leq p_o \leq p_o^R\}$ . While the lower bound of RE measures with convex technologies in DEA are computed by solving LP problems, the following Theorem shows that the lower bound of RE measures with various non-convex technologies can be obtained, using a type of implicit enumeration algorithm based upon vector comparison.

**Theorem 2.1.** *The non-convex RE measure  $RE_o^{NC,\Phi}(p_o, x_o)$  can be computed as follows:*

$$RE_o^{NC,\Phi}(p_o, x_o) = \begin{cases} \min_{j=1,\dots,n} \left\{ \frac{p_o^t y_o}{p_o^t y_j} : x_j \leq x_o \right\} & \text{for } \Phi = VRS \\ \min_{j=1,\dots,n} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = CRS \\ \min \left\{ j : \max_{i=1,\dots,m} (x_{ij}/x_{io}) \leq 1 \right\} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NIRS \\ \min_{j=1,\dots,n} \left\{ \max \left( \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right), 1 \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NDRS, \end{cases} \quad (2.2)$$

**Proof:** The proof can be adapted from Proposition 3 in [3] and (2.1).

In the next section, we extend the RE models with various returns to scale assumptions of non-convex technologies to include bounded data.

### 3 Non-convex revenue efficiency measures with uncertain data

As we can observe, the RE measure of DMUs can be attained when the data is exactly known.. But in many practical applications, the data cannot be estimated accurately enough to make good use of EE concepts, and only the lower and upper bounds of the data can be estimated. When the data of input, output, and price vectors are uncertain and can be expressed in the form of bounded data, the RE measure calculated from the uncertain data should be uncertain, as well. The bounds typically give a better approximation of true RE measures than a crisp value does. So, we propose the following model to obtain the lower bound of the nonconvex RE measures in the presence of bounded data:

$$RE_{o, LB}^{NC, \Phi} = \min_{\substack{x_j \in [x_j^L, x_j^R] \\ y_j \in [y_j^L, y_j^R] \\ p_o \in [p_o^L, p_o^R] \\ j = 1, \dots, n}} \left\{ \begin{array}{ll} \min_{j=1, \dots, n} \left\{ \frac{p_o^t y_o}{p_o^t y_j} : x_j \leq x_o \right\} & \text{for } \Phi = VRS \\ \min_{j=1, \dots, n} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = CRS \\ \min_{\left\{ j : \max_{i=1, \dots, m} (x_{ij}/x_{io}) \leq 1 \right\}} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NIRS \\ \min_{j=1, \dots, n} \left\{ \max \left( \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right), 1 \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NDRS, \end{array} \right. \tag{3.3}$$

The inner program gives the RE measures of the unit under assessment with different returns to scale assumptions of technology, based on the data provided by the outer program, while the objective of the outer program is to minimize the possible RE measures. In a similar way, we propose the following model to obtain the upper bound of RE; the only difference being the objective function of the outer program, which is of the maximizing type.

$$RE_{o, UB}^{NC, \Phi} = \max_{\substack{x_j \in [x_j^L, x_j^R] \\ y_j \in [y_j^L, y_j^R] \\ p_o \in [p_o^L, p_o^R] \\ j = 1, \dots, n}} \left\{ \begin{array}{ll} \min_{j=1, \dots, n} \left\{ \frac{p_o^t y_o}{p_o^t y_j} : x_j \leq x_o \right\} & \text{for } \Phi = VRS \\ \min_{j=1, \dots, n} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = CRS \\ \min_{\left\{ j : \max_{i=1, \dots, m} (x_{ij}/x_{io}) \leq 1 \right\}} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NIRS \\ \min_{j=1, \dots, n} \left\{ \max \left( \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right), 1 \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NDRS, \end{array} \right. \tag{3.4}$$

In the two-level mathematical programming problem (3.4), the inner program calculates the RE measure of the DMU under assessment with the data provided by the outer program, while the outer program chooses the data that maximize the possible RE measures. The following theorem proves that the lower bound RE measure, computed from the pessimistic point of view is characterized by some closed-form expressions. As we can observe, the optimal output prices is obtained in an extreme point of the output prices. The optimal output prices for the upper bound may not occur at extreme points of output prices, but it may occur at an interior point of output ranges. So, a linear programming problem is provided for computing the upper bound. In the next theorem the following notation will be useful,

$$\lambda_j^{oR} = \max_{i \in I(x_o^R)} \left( \frac{x_{ij}^L}{x_{io}^R} \right), \lambda_j^{oL} = \max_{i \in I(x_o^L)} \left( \frac{x_{ij}^R}{x_{io}^L} \right)$$

**Theorem 3.1.** We have

$$RE_{o, LB}^{NC, \Phi} = \min_{v \in V_m} \begin{cases} \min \left( \min_{j \neq o, j=1, \dots, n} \left\{ \frac{p_o^{v^t} y_o^L}{p_o^{v^t} y_j^R} : x_j^L \leq x_o^R \right\}, 1 \right) & \text{for } \Phi = VRS \\ \min \left( \min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{o^R} \frac{p_o^{v^t} y_o^L}{p_o^{v^t} y_j^R} \right\}, 1 \right) & \text{for } \Phi = CRS \\ \min \left( \min_{\{j \neq o : \lambda_j^{o^R} \leq 1\}} \left\{ \lambda_j^{o^R} \cdot \frac{p_o^{v^t} y_o^L}{p_o^{v^t} y_j^R} \right\}, 1 \right) & \text{for } \Phi = NIRS \\ \min \left( \min_{j \neq o, j=1, \dots, n} \left\{ \max \left( \lambda_j^{o^R}, 1 \right) \cdot \frac{p_o^{v^t} y_o^L}{p_o^{v^t} y_j^R} \right\}, 1 \right) & \text{for } \Phi = NDRS, \end{cases} \tag{3.5}$$

and

$$RE_{o, UB}^{NC, \Phi} = \begin{cases} \max \hat{p}_o^t y_o^R : \hat{p}_o^t y_o^R \leq 1, \hat{p}_o^t y_j^L \leq 1; j \neq o, x_j^R \leq x_o^L, tp_o^L \leq \hat{p}_o \leq tp_o^R, \Phi = VRS \\ \max \hat{p}_o^t y_o^R : \hat{p}_o^t y_o^R \leq 1, \frac{\hat{p}_o^t y_j^L}{\lambda_j^{o^L}} \leq 1; j \neq o, tp_o^L \leq \hat{p}_o \leq tp_o^R, \Phi = CRS \\ \max \hat{p}_o^t y_o^R : \hat{p}_o^t y_o^R \leq 1, \frac{\hat{p}_o^t y_j^L}{\lambda_j^{o^L}} \leq 1; j \neq o, \lambda_j^{o^L} \leq 1, tp_o^L \leq \hat{p}_o \leq tp_o^R, \Phi = NIRS \\ \max \hat{p}_o^t y_o^R : \hat{p}_o^t y_o^R \leq 1, \frac{\hat{p}_o^t y_j^L}{\lambda_j^{o^L}} \leq 1; j \neq o, \lambda_j^{o^L} \geq 1, tp_o^L \leq \hat{p}_o \leq tp_o^R, \Phi = NDRS \end{cases} \tag{3.6}$$

**Proof:** We only prove the theorem in the case of  $\Phi = CRS$ ; for the other non-convex assumptions of technology, the proof is similar and hence removed.

a) The proof for the lower bound

First, we assume that output prices are fixed and known deterministically, and only input/output data is imprecise and can be expressed in the form of bounded data. If  $x_{ij}^L \leq x_{ij} \leq x_{ij}^R, y_{rj}^L \leq y_{rj} \leq y_{rj}^R; i = 1, \dots, m, r = 1, \dots, s, j = 1, \dots, n$ , then we have

$$\max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \geq \max_{i \in I(x_o^R)} \left( \frac{x_{ij}^L}{x_{io}^R} \right) \cdot \frac{p_o^t y_o^L}{p_o^t y_j^R}; j \neq o.$$

This implies that

$$\begin{aligned} \min_{j=1, \dots, n} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} &\geq \min \left( \min_{j \neq o, j=1, \dots, n} \left\{ \max_{i \in I(x_o^R)} \left( \frac{x_{ij}^L}{x_{io}^R} \right) \cdot \frac{p_o^t y_o^L}{p_o^t y_j^R} \right\}, 1 \right) \\ &= \min \left( \min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{o^R} \cdot \frac{p_o^t y_o^L}{p_o^t y_j^R} \right\}, 1 \right). \end{aligned}$$

Now, let  $p_o \in P_o = \{p_o : p_o^L \leq p_o \leq p_o^R\}$ . To complete the proof, it is sufficient to show that the optimal output prices occur at an extreme point of set  $P_o$ . Toward this

end, let  $p_o^* \in P_o$  be an optimal output price; hence, considering the representation theorem (see Theorem 2.1 on p. 69 of [2]), we have

$$p_o^* = \sum_{v \in V_m} \lambda_v p_o^v, \quad \sum_{v \in V_m} \lambda_v = 1, \quad \lambda_v \geq 0$$

By contradiction, let for each  $v \in V_m$ ,

$$\lambda_k^{oR} \cdot \frac{p_o^{*t} y_o^L}{p_o^{*t} y_k^R} = \min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{oR} \cdot \frac{p_o^{*t} y_o^L}{p_o^{*t} y_j^R} \right\} < \min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{oR} \cdot \frac{p_o^{v^t} y_o^L}{p_o^{v^t} y_j^R} \right\} \leq \lambda_k^{oR} \cdot \frac{p_o^{v^t} y_o^L}{p_o^{v^t} y_k^R}$$

We have  $(p_o^{v^t} y_k^R)(p_o^{*t} y_o^L) < (p_o^{v^t} y_o^L)(p_o^{*t} y_k^R)$ . By multiplying the both sides of the above inequality by  $\lambda_v; v \in V_m$  and summation on  $v \in V_m$ , we have

$$\sum_{v \in V_m} \lambda_v p_o^{v^t} y_k^R (p_o^{*t} y_o^L) < \sum_{v \in V_m} \lambda_v p_o^{v^t} y_o^L (p_o^{*t} y_k^R)$$

This implies that

$$(p_o^{*t} y_k^R)(p_o^{*t} y_o^L) < (p_o^{*t} y_o^L)(p_o^{*t} y_k^R),$$

but this inequality is a contradiction. Therefore, there exists a  $v \in V_m$  such that

$$\min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{oR} \cdot \frac{p_o^{*t} y_o^L}{p_o^{*t} y_j^R} \right\} = \min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{oR} \cdot \frac{p_o^{v^t} y_o^L}{p_o^{v^t} y_j^R} \right\}$$

and this in turn implies that

$$RE_{o, LB}^{NC, CRS} = \min_{v \in V_m} \min \left( \min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{oR} \cdot \frac{p_o^{v^t} y_o^L}{p_o^{v^t} y_j^R} \right\}, 1 \right)$$

This completes the proof for the lower bound.

b) The proof for the upper bound

If

$$x_{ij}^L \leq x_{ij} \leq x_{ij}^R, \quad y_{rj}^L \leq y_{rj} \leq y_{rj}^R; \quad i = 1, \dots, m, \quad r = 1, \dots, s, \quad j = 1, \dots, n,$$

then we have

$$\max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \leq \max_{i \in I(x_o^L)} \left( \frac{x_{ij}^R}{x_{io}^L} \right) \cdot \frac{p_o^t y_o^R}{p_o^t y_j^L}; \quad j \neq o.$$

This implies that

$$\min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^o \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} \leq \min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{oL} \cdot \frac{p_o^t y_o^R}{p_o^t y_j^L} \right\}.$$

So,

$$\begin{aligned} RE_{o, UB}^{NC, CRS} &= \max_{p_o^L \leq p_o \leq p_o^R} \min_{j \neq o, j=1, \dots, n} \left\{ \lambda_j^{oL} \cdot \frac{p_o^t y_o^R}{p_o^t y_j^L} \right\} \\ &= \max_{p_o^L \leq p_o \leq p_o^R} \left( \frac{p_o^t y_o^R}{\max_{j \neq o, j=1, \dots, n} \left\{ \max_{j \neq o, j=1, \dots, n} \frac{p_o^t y_j^L}{\lambda_j^{oL}}, p_o^t y_o^R \right\}} \right) \end{aligned}$$

We set,

$$\max \left\{ \max_{j \neq o, j=1, \dots, n} \frac{p_o^t y_j^L}{\lambda_j^{oL}}, p_o^t y_o^R \right\} = \frac{1}{t},$$

We have,

$$\frac{tp_o^t y_j^L}{\lambda_j^{oL}} \leq 1, \quad j \neq o, \quad j = 1, \dots, n, \quad tp_o^t y_o^R \leq 1, \quad tp_o^L \leq tp_o \leq tp_o^R.$$

Let  $\hat{p}_o^t = tp_o$ , it implies that,

$$\frac{\hat{p}_o^t y_j^L}{\lambda_j^{oL}} \leq 1, \quad j \neq o, \quad j = 1, \dots, n, \quad \hat{p}_o^t y_o^R \leq 1, \quad tp_o^L \leq \hat{p}_o \leq tp_o^R.$$

Therefore,

$$RE_{o,UB}^{NC,CRS} = \max \hat{p}_o^t y_o^R : \hat{p}_o^t y_o^R \leq 1, \quad \frac{\hat{p}_o^t y_j^L}{\lambda_j^{oL}} \leq 1; \quad j \neq o, \quad tp_o^L \leq \hat{p}_o \leq tp_o^R \quad \text{for } \Phi = CRS$$

This completes the proof.

**Remark 3.1.** Although, the optimal output prices for the lower bound is occurred in an extreme point of the set  $p_o \in P_o = \{p_o : p_o^L \leq p_o \leq p_o^R\}$ , but optimal output prices for the upper bound may occur in an interior point of the set  $P_o$ . Consequently, the lower bound is obtained without solving any linear programming problems, but by a simple implicit formula, and also a linear programming problem should be solved to obtain the upper bound. Since in the proof of Theorem (3.1) we only made use of convexity of output prices. The method proposed in this paper is applicable not only when the input prices are in the form of ranges but also when they are in the form of a convex set. Specially, when the output prices are uncertain in the form of a polyhedral convex cone, as was used in Kuosmanen and Post [9, 10],

$$W = \{w \in \mathfrak{R}_+^s \mid Aw \geq 0\}$$

Where  $A$  is an  $l \times n$  matrix. For obtaining the lower bound of revenue efficiency, it is sufficient to consider the extreme points of the following normalized output prices,

$$\hat{W} = \{w \in \mathfrak{R}_+^s \mid Aw \geq 0, e^t w = 1\}$$

Where  $e^t = (1, 1, \dots, 1)$  is an  $s$ -vector with all components equal to one.

**Example 3.1.** For illustrating the given formulas, we consider an example with two inputs  $x_1, x_2$ , and two outputs  $y_1, y_2$ . We assume that the data is uncertain, and only their lower and upper bounds can be estimated. The input, output, and output price data are listed in Table 1. Note that when the data is precise, the lower bound is equal to the upper bound. We obtained the bounds of RE measures with various returns to scale assumptions of technology using models (3.5) and (3.6). Consider DMU C for instance; the lower and upper bounds of the second input are 10.5 and 11, respectively. Also, the output prices can

be expressed as follows:  $P_o = \{(p_1, p_2) \mid 9 \leq p_1 \leq 9.5, 3 \leq p_2 \leq 4\}$ . The extreme points of output prices required to calculate the bounds of RE measures are as follows:

$$\left\{ \begin{pmatrix} 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ 4 \end{pmatrix}, \begin{pmatrix} 9.5 \\ 3 \end{pmatrix}, \begin{pmatrix} 9.5 \\ 4 \end{pmatrix} \right\}$$

The RE measures of the eight DMUs calculated by Models (3.5) and (3.6) are given in Table 2. The second and third columns of Table 2 show the lower and upper bounds of RE with variable returns to scale (VRS) assumption of technology, the bounds of RE with constant returns to scale assumption of technology are given in columns 4 and 5, the bounds of RE with non-increasing returns to scale (NIRS) assumption of technology are provided in columns 6 and 7, and the eighth and ninth columns show the bounds of RE with non-decreasing returns to scale (NDRS) assumption of technology. For instance, the lower bound of RE of DMU G with NIRS assumption of technology is 0.433 and the upper bound is 0.8151. The extreme points of output prices corresponding to the lower and upper bounds are  $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 6.8 \\ 3.75 \end{pmatrix}$ , respectively. As we can observe, DMUs B and G are inefficient with all RTS assumptions of technology, but the others are efficient when calculated from an optimistic point of view. DMUs C and D are efficient with VRS and NDRS assumptions of technology from both optimistic and pessimistic viewpoints, but they are inefficient from pessimistic viewpoints with other RTS assumptions.

Table 1  
The input, output, and output price data

DMUs	X1L	X1R	X2L	X2R	Y1L	Y1R	Y2L	Y2R	P1L	P1R	P2L	P2R
DMU A	3	3.5	10	11	8	9	4	5	7	8	4	4
DMU B	4	4	11	13	6	7	3	3	6	7	3.5	5
DMU C	3	5	10.5	11	9	9.5	7	7	9	9.5	3	4
DMU D	2.5	2.5	7.875	8.25	6	6	3	3.25	8	8.5	3	5
DMU E	3.75	4	12.5	14	7.5	8	6.75	7	5.75	6	4	4
DMU F	3	5.5	13	15	8.5	9	3.5	3.5	9.5	9.5	2.25	2.5
DMU G	4	5	12	13	5	6.5	4	4.75	6.75	8	3	3.75
DMU H	2	3	10	11	7.5	8	5	6	8	9	4	5

Table 2  
The bounds of RE with various RTS assumptions of technology

DMUs	VRSL	VRSU	CRSL	CRSU	NIRSL	NIRSU	NDRSL	NDRSU
DMU A	0.7619	1	0.7273	1	0.7273	1	0.7619	1
DMU B	0.5543	0.85	0.44773	0.85	0.4477	0.85	0.5543	0.84
DMU C	1	1	0.68533	1	0.6853	1	1	1
DMU D	1	1	0.7224	0.9245	1	1	0.7224	0.9245
DMU E	0.8471	1	0.6353	0.7037	0.6353	0.7037	0.8471	1
DMU F	0.8306	1	0.5814	1	0.5814	1	0.8306	1
DMU G	0.5361	0.8892	0.433	0.8151	0.433	0.8151	0.5361	0.8905
DMU H	0.7658	1	0.7658	1	0.7658	1	0.7658	1

### 4 Extension to fuzzy data

In the preceding sections, we assumed that the data is uncertain in the form of bounded data. In an empirical study, we may encounter cases in which the quality of product is

assessed by customers using, fuzzy linguistic terms such as excellent, very good, good, average, poor, and very poor (see [11] for more details on linguistic terms). Though we assume that the whole data is uncertain, the main focus of this section is on the development of the EE models to include fuzzy price information, which is more practical, and to take into account managerial viewpoints, which other authors have failed to deal with conveniently. One source of price information could be the prior knowledge of the quality of different inputs and outputs. For example, primary outputs are typically more expensive than secondary outputs. In this case, the price information is best described by fuzzy data rather than by other types of uncertain data. Even if the data is expressed in the form of ordinal data or some scale rates, it is better to transform them into fuzzy data with convenient membership functions. Consequently, there exists the opportunity and need to embed fuzzy data into EE models, so as to better describe EE concepts.

### 4.1 Membership function of linguistic terms

Let the meanings of the linguistic terms excellent, very good, good, average, poor, and very poor be defined by the following membership functions:

$$\mu_{Verypoor}(x) = \begin{cases} 1 - x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise,} \end{cases}, \quad \mu_{Poor}(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 2 - x, & \text{if } 1 \leq x \leq 2 \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu_{Average}(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 2 - x, & \text{if } 1 \leq x \leq 2 \\ 0, & \text{otherwise,} \end{cases}, \quad \mu_{Good}(x) = \begin{cases} x - 2, & \text{if } 2 \leq x \leq 3 \\ 3 - x, & \text{if } 3 \leq x \leq 4 \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu_{Verygood}(x) = \begin{cases} x - 3, & \text{if } 3 \leq x \leq 4 \\ 5 - x, & \text{if } 4 \leq x \leq 5 \\ 0, & \text{otherwise,} \end{cases}, \quad \mu_{Excellent}(x) = \begin{cases} 5 - x, & \text{if } 4 \leq x \leq 5 \\ 0, & \text{otherwise,} \end{cases}$$

where the variable  $x$  is the quality or importance of a given product, with the interval  $[0, 5]$  as the universe of discourse. Figure 1 shows the membership functions for the primary linguistic terms very poor, poor, average, good, very good, and excellent;

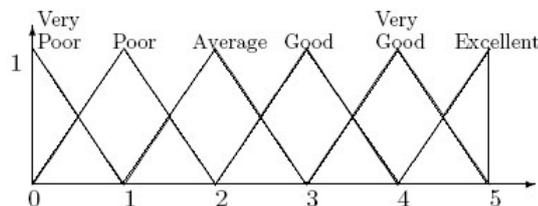


Fig 1. Fuzzy sets for output quality assessment

### 4.2 Basic definitions

In this subsection, we summarize basic notions and definitions from fuzzy set theory that are useful in the next section.

**Definition 4.1.** *If  $X$  is the universe of discourse, then a fuzzy set  $\tilde{A}$  in  $X$  is a set of ordered pairs as follows:*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

$\mu_{\tilde{A}}(x)$  is called the membership function (MF) or membership degree (also degree of compatibility or degree of truth) of  $x$  in the fuzzy set  $\tilde{A}$ . The range of the membership function is  $[0, 1]$ . Notice that the membership function of a crisp fuzzy subset of  $X$  is equal to the characteristic function of the corresponding set. If  $\sup\{\mu_{\tilde{A}}(x) \mid x \in X\} = 1$ , then  $\tilde{A}$  is said to be normalized. The upper-level set of the membership function  $\mu_{\tilde{A}}$  of  $\tilde{A}$  at  $\alpha \in [0, 1]$  is denoted by  $(\tilde{A})_{\alpha}$  and called the  $\alpha$  – cut or  $\alpha$  – level set of  $A$ ; that is

$$(\tilde{A})_{\alpha} = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}.$$

(i) A fuzzy set  $\tilde{A}$  is called a fuzzy interval if for each  $\alpha \in [0, 1]$ ,  $(\tilde{A})_{\alpha}$  is a nonempty and convex subset of  $\mathfrak{R}$ .

(ii) A fuzzy interval  $\tilde{A}$  is called a fuzzy number if its core is a singleton, where

$$\text{Core}(\tilde{A}) = \{x \in X \mid \mu_{\tilde{A}}(x) = 1\}.$$

(iii) Let  $X = \mathfrak{R}$ . A fuzzy number  $\tilde{A}$  is said to be nonnegative if for all  $x \in (-\infty, 0)$ , we have  $\mu_{\tilde{A}}(x) = 0$ .

## 5 Economic efficiency with fuzzy data

In this section, we extend the classical non-convex Farrells theoretical framework to include linguistic terms. Toward this end, we transform the linguistic terms or ordinal data to fuzzy data with corresponding membership functions, and then transform the non-convex RE formulas to a family of crisp closed-form expressions based upon  $\alpha$  – cuts concepts. Since in the RE models, the objective function is a special fractional program with the same coefficients in the denominator and numerator, it is too complicated to solve. To remedy this problem, we make use of extreme points of  $\alpha$  – level of output prices as provided in the previous sections.

Suppose that each  $DMU_j$  consumes  $m$  fuzzy inputs  $\tilde{x}_j^t = (\tilde{x}_{1j}, \tilde{x}_{2j}, \dots, \tilde{x}_{mj})$  to produce  $s$  fuzzy outputs  $\tilde{y}_j^t = (\tilde{y}_{1j}, \tilde{y}_{2j}, \dots, \tilde{y}_{sj})$ . Also, we assume that output prices are uncertain in the form of linguistic terms, and we have expressed them by fuzzy data with the membership function provided in the previous section as:  $\tilde{p}_o^t = (\tilde{p}_{1o}, \tilde{p}_{2o}, \dots, \tilde{p}_{so})$ . To measure the fuzzy efficiency of  $DMU_o$  with non-convex assumption of technology relative to other DMUs, the following fuzzy DEA model is provided:

$$RE_o^{NC, \Phi}(\tilde{p}_o, \tilde{x}_o) = \min \left\{ \frac{\tilde{p}_o^t \tilde{y}_o}{\tilde{p}_o^t \tilde{y}} : \tilde{y} \in L^{NC, \Phi}(\tilde{x}_o) \right\} \tag{5.7}$$

The following notations are useful in the remaining sections. Given  $\alpha \in [0, 1]$ , for all  $i, s, j$ , we denote

$$(\tilde{x}_{ij})_{\alpha}^L = \inf \{t \in \mathfrak{R} \mid t \in (\tilde{x}_{ij})_{\alpha}\} = \inf(\tilde{x}_{ij})_{\alpha}, \quad (\tilde{x}_{ij})_{\alpha}^R = \sup \{t \in \mathfrak{R} \mid t \in (\tilde{x}_{ij})_{\alpha}\} = \sup(\tilde{x}_{ij})_{\alpha}$$

$$(\tilde{y}_{rj})_{\alpha}^L = \inf \{t \in \mathfrak{R} \mid t \in (\tilde{y}_{rj})_{\alpha}\} = \inf(\tilde{y}_{rj})_{\alpha}, \quad (\tilde{y}_{rj})_{\alpha}^R = \sup \{t \in \mathfrak{R} \mid t \in (\tilde{y}_{rj})_{\alpha}\} = \sup(\tilde{y}_{rj})_{\alpha}$$

$$(\tilde{p}_{rj})_{\alpha}^L = \inf \{t \in \mathfrak{R} \mid t \in (\tilde{p}_{rj})_{\alpha}\} = \inf(\tilde{p}_{rj})_{\alpha}, \quad (\tilde{p}_{rj})_{\alpha}^R = \sup \{t \in \mathfrak{R} \mid t \in (\tilde{p}_{rj})_{\alpha}\} = \sup(\tilde{p}_{rj})_{\alpha}$$

Let  $(RE_o)_{\alpha}^L$  and  $(RE_o)_{\alpha}^R$  denote the lower and upper bounds of the  $\alpha$ -cut of the membership function of the RE measure for the DMU under evaluation,  $DMU_o$ . The approach for constructing the membership function  $\mu_{\widetilde{RE}}$  proposed in this article is to derive the  $\alpha$ -cut of  $\mu_{\widetilde{RE}}$ .

We propose the following model to obtain the lower bound of the  $\alpha$ -cut of the membership function of the RE measure for  $DMU_o$ .

$$(RE_o^{NC,\Phi})_{\alpha}^L = \min_{\substack{x_j \in [(\tilde{x}_j)_{\alpha}^L, (\tilde{x}_j)_{\alpha}^R] \\ y_j \in [(\tilde{y}_j)_{\alpha}^L, (\tilde{y}_j)_{\alpha}^R] \\ p_o \in [(\tilde{p}_o)_{\alpha}^L, (\tilde{p}_o)_{\alpha}^R] \\ j = 1, \dots, n}} \begin{cases} \min_{j=1, \dots, n} \left\{ \frac{p_o^t y_o}{p_o^t y_j} : x_j \leq x_o \right\} & \text{for } \Phi = VRS \\ \min_{j=1, \dots, n} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = CRS \\ \min_{\left\{ j : \max_{i=1, \dots, m} (x_{ij}/x_{io}) \leq 1 \right\}} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NIRS \\ \min_{j=1, \dots, n} \left\{ \max \left( \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right), 1 \right) \cdot \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NDRS, \end{cases} \tag{5.8}$$

From Theorem (3.3),  $(RE_o^{NC,\Phi})_{\alpha}^L$  can be computed easily as follows:

$$(RE_o^{NC,\Phi})_{\alpha}^L = \min_{v \in V_m} \begin{cases} \min \left( \min_{j \neq o, j=1, \dots, n} \left\{ \frac{p_o^{v^t} (\tilde{y}_o)_{\alpha}^L}{p_o^{v^t} (\tilde{y}_j)_{\alpha}^R} : (\tilde{x}_j)_{\alpha}^L \leq (\tilde{x}_o)_{\alpha}^R \right\}, 1 \right) & \text{for } \Phi = VRS \\ \min \left( \min_{j \neq o, j=1, \dots, n} \left\{ (\tilde{\lambda}_j)_{\alpha}^{oR} \cdot \frac{(\tilde{p}_o)_{\alpha}^{v^t} (\tilde{y}_o)_{\alpha}^L}{(\tilde{p}_o)_{\alpha}^{v^t} (\tilde{y}_j)_{\alpha}^R} \right\}, 1 \right) & \text{for } \Phi = CRS \\ \min \left( \min_{\left\{ j \neq o : (\tilde{\lambda}_j)_{\alpha}^{oR} \leq 1 \right\}} \left\{ \max_{i \in I(y_o)_{\alpha}^L} (\tilde{\lambda}_j)_{\alpha}^{oR} \cdot \frac{(\tilde{p}_o)_{\alpha}^{v^t} (\tilde{y}_o)_{\alpha}^L}{(\tilde{p}_o)_{\alpha}^{v^t} (\tilde{y}_j)_{\alpha}^R} \right\}, 1 \right) & \text{for } \Phi = NIRS \\ \min \left( \min_{j=1, \dots, n} \left\{ \max((\tilde{\lambda}_j)_{\alpha}^{oR}, 1) \cdot \frac{(\tilde{p}_o)_{\alpha}^{v^t} (\tilde{y}_o)_{\alpha}^L}{(\tilde{p}_o)_{\alpha}^{v^t} (\tilde{y}_j)_{\alpha}^R} \right\}, 1 \right) & \text{for } \Phi = NDRS \end{cases} \tag{5.9}$$

Where

$$(\tilde{\lambda}_j)_{\alpha}^R = \max_{i \in I(x_o)_{\alpha}^R} \left( \frac{x_{ij}^L}{x_{io}^R} \right).$$

We propose the following model for obtaining the upper bound of the  $\alpha$ -cut of the

membership function of the RE measure for  $DMU_o$ .

$$(RE_o^{NC,\Phi})_\alpha^R = \max_{\substack{x_j \in [(\tilde{x}_j)_\alpha^L, (\tilde{x}_j)_\alpha^R] \\ y_j \in [(\tilde{y}_j)_\alpha^L, (\tilde{y}_j)_\alpha^R] \\ p_o \in [(\tilde{p}_o)_\alpha^L, (\tilde{p}_o)_\alpha^R] \\ j = 1, \dots, n}} \left\{ \begin{array}{ll} \min_{j=1, \dots, n} \left\{ \frac{p_o^t y_o}{p_o^t y_j} : x_j \leq x_o \right\} & \text{for } \Phi = VRS \\ \min_{j=1, \dots, n} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = CRS \\ \min \left\{ \max_{i=1, \dots, m} (x_{ij}/x_{io}) \leq 1 \right\} \left\{ \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right) \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NIRS \\ \min_{j=1, \dots, n} \left\{ \max \left( \max_{i \in I(x_o)} \left( \frac{x_{ij}}{x_{io}} \right), 1 \right) \frac{p_o^t y_o}{p_o^t y_j} \right\} & \text{for } \Phi = NDRS, \end{array} \right. \tag{5.10}$$

From Theorem (3.3),  $(RE_o^{NC,\Phi})_\alpha^R$  can be computed by the following model,

$$(RE_o^{NC,\Phi})_\alpha^R = \left\{ \begin{array}{ll} \max \hat{p}_o^t (\tilde{y}_o)_\alpha^R : \hat{p}_o^t (\tilde{y}_o)_\alpha^R \leq 1, \quad \hat{p}_o^t (\tilde{y}_j)_\alpha^L \leq 1; \quad j \neq o, \quad (\tilde{x}_j)_\alpha^R \leq (\tilde{x}_o)_\alpha^L, \quad t(\tilde{p}_o)_\alpha^L \leq \hat{p}_o \leq t(\tilde{p}_o)_\alpha^R & \Phi = VRS \\ \max \hat{p}_o^t (\tilde{y}_o)_\alpha^R : \hat{p}_o^t (\tilde{y}_o)_\alpha^R \leq 1, \quad \frac{\hat{p}_o^t (\tilde{y}_j)_\alpha^L}{(\tilde{\lambda}_j)_\alpha^L} \leq 1; \quad j \neq o, \quad t(\tilde{p}_o)_\alpha^L \leq \hat{p}_o \leq t(\tilde{p}_o)_\alpha^R & \Phi = CRS \\ \max \hat{p}_o^t (\tilde{y}_o)_\alpha^R : \hat{p}_o^t (\tilde{y}_o)_\alpha^R \leq 1, \quad \frac{\hat{p}_o^t (\tilde{y}_j)_\alpha^L}{(\tilde{\lambda}_j)_\alpha^L} \leq 1; \quad j \neq o, \quad (\tilde{\lambda}_j)_\alpha^L \leq 1, \quad t(\tilde{p}_o)_\alpha^L \leq \hat{p}_o \leq t(\tilde{p}_o)_\alpha^R & \Phi = NIRS \\ \max \hat{p}_o^t (\tilde{y}_o)_\alpha^R : \hat{p}_o^t (\tilde{y}_o)_\alpha^R \leq 1, \quad \frac{\hat{p}_o^t (\tilde{y}_j)_\alpha^L}{(\tilde{\lambda}_j)_\alpha^L} \leq 1; \quad j \neq o, \quad (\tilde{\lambda}_j)_\alpha^L \geq 1, \quad t(\tilde{p}_o)_\alpha^L \leq \hat{p}_o \leq t(\tilde{p}_o)_\alpha^R & \Phi = NDRS \end{array} \right. \tag{5.11}$$

where  $(\tilde{\lambda}_j)_\alpha^L = \max_{i \in I(x_o^L)} \left( \frac{(\tilde{x}_{ij})_\alpha^R}{(\tilde{x}_{io})_\alpha^L} \right)$ .

The family of intervals  $[(RE_o^{NC,\Phi})_\alpha^R, (RE_o^{NC,\Phi})_\alpha^R]; \alpha \in [0, 1]$  reveals the shape of  $\mu_{RE}$ , although the exact function form is not known explicitly. After the RE scores of all DMUs are obtained, a subsequent task is to rank the DMUs to determine the better ones. One can refer to [6, 7, 12, 16, 17, 19, 20], for example, for ranking methods of DMUs.

**Example 5.1.** In order to illustrate how the proposed method is employed to find RE measures, consider eight DMUs with input, output, and output price data as given in Table 3. We assume that the output prices are expressed in linguistic terms. Also, for simplicity, we assume that the input and output data are symmetric triangular fuzzy numbers.

Since an analytical solution is not attainable in this example, we obtain the bounds of RE with different values of  $\alpha$ : 0, 0.2, 0.4, 0.6, 0.8, 1. Figure 2 shows the rough shape of the membership function of DMU F with various RTS assumptions of technology constructed by these  $\alpha$  values. If we obtain the  $\alpha$  – cuts of the membership functions of RE with more values of  $\alpha$ , the shape of the membership functions will be closer to the real shape.

Though the data consists of symmetric triangular fuzzy numbers, Figure 2 shows that the RE measure is not triangular. Also, the efficiency measure of DMU H is a crisp value, despite the fact that the whole data includes fuzzy numbers.

Table 3

The fuzzy inputs, outputs, and output prices.

DMUs	x1	x2	y1	y2	p1	p2	p1f	p2f
DMU A	(4, 0.5)	(2.1,0.2)	(2.6,0.3)	(4.1,0.3)	Good	Average	(3,1)	(2,1)
DMU B	(2.9,0.1)	(1.5,0.1)	(2.2,0)	(3.5,0.2)	Very Good	Good	(4,1)	(3,1)
DMU C	(4.9,0.5)	(2.6,0.4)	(3.2,0.5)	(5.1,0.8)	Poor	Average	(1,1)	(2,1)
DMU D	(4.1, 0.7)	(2.3,0.1)	(2.9,0.4)	(5.7,0.2)	Excellent	Average	(5,1)	(2,1)
DMU E	(6.5,0.6)	(4.1,0.5)	(5.1,0.7)	(7.4,0.9)	Good	Good	(3,1)	(3,1)
DMU F	(3, 0.5)	(2,0.4)	(2.5,0.2)	(5.5,0.4)	Very Poor	Good	(4,1)	(1,1)
DMU G	(4,0.3)	(3,0.1)	(5,0.3)	(6,0.5)	Excellent	Good	(1,1)	(3,1)
DMU H	(2,0.1)	(3,0.4)	(2,0.2)	(5,0.8)	Poor	Average	(1,1)	(2,1)

Table 4

The  $\alpha$  - cuts of the RE measures at six  $\alpha$  values

DMUs	$\alpha = 0$							
	VRSL	VRSU	CRSL	CRSU	NIRSL	NIRSU	NDRSL	NDRSU
DMU A	0.628019	1	0.481837	1	0.481837	1	0.591744	1
DMU B	0.385965	1	0.624606	1	0.624606	1	0.385965	0.921752
DMU C	0.567251	1	0.388701	1	0.388701	1	0.567251	1
DMU D	0.737705	1	0.618557	1	0.618557	1	0.659091	1
DMU E	0.665254	1	0.321405	1	0.321405	1	0.665254	1
DMU F	0.69697	1	0.524371	1	0.677056	1	0.524371	1
DMU G	1	1	0.662094	1	0.662094	1	0.978904	1
DMU H	1	1	0.821745	1	1	1	0.803676	1
DMUs	$\alpha = 0.2$							
	VRSL	VRSU	CRSL	CRSU	NIRSL	NIRSU	NDRSL	NDRSU
DMU A	0.681083	1	0.537015	1	0.537015	1	0.63684	1
DMU B	1	1	0.685022	1	1	1	0.422842	0.857509
DMU C	0.607293	1	0.299968	1	0.299968	1	0.607293	1
DMU D	0.785439	1	0.6771	1	0.6771	1	0.707888	1
DMU E	0.722283	1	0.370136	1	0.370136	1	0.722283	1
DMU F	1	1	0.584169	1	0.926663	1	0.584169	1
DMU G	1	1	0.760283	1	0.760283	1	1	1
DMU H	1	1	0.907238	1	1	1	0.902578	1
DMUs	$\alpha = 0.4$							
	VRSL	VRSU	CRSL	CRSU	NIRSL	NIRSU	NDRSL	NDRSU
DMU A	0.752587	1	0.596646	0.99689	0.596646	0.99689	0.68463	1
DMU B	1	1	0.747874	1	1	1	0.462775	0.785746
DMU C	0.803759	1	0.345089	1	0.345089	1	0.672149	1
DMU D	0.997438	1	0.743852	1	0.743852	1	0.762527	1
DMU E	0.806039	1	0.441687	1	0.441687	1	0.806039	1
DMU F	1	1	0.64559	1	0.984534	1	0.64559	1
DMU G	1	1	0.83499	1	0.83499	1	1	1
DMU H	1	1	0.998084	1	1	1	0.998084	1

		$\alpha = 0.6$						
DMUs	VRSL	VRSU	CRSL	CRSU	NIRSL	NIRSU	NDRSL	NDRSU
DMU A	0.788488	1	0.665513	0.942628	0.665513	0.942628	0.740872	0.962533
DMU B	1	1	0.818591	1	1	1	0.506065	0.719946
DMU C	0.84492	1	0.584721	0.991297	0.584721	0.93364	0.747268	1
DMU D	1	1	0.806097	1	0.806097	1	0.815003	1
DMU E	0.906175	1	0.529874	1	0.529874	1	0.906175	1
DMU F	1	1	0.730541	1	1	1	0.730541	1
DMU G	1	1	0.888173	1	0.888173	1	1	1
DMU H	1	1	1	1	1	1	1	1

		$\alpha = 0.8$						
DMUs	VRSL	VRSU	CRSL	CRSU	NIRSL	NIRSU	NDRSL	NDRSU
DMU A	0.798908	1	0.716777	0.888022	0.716777	0.888022	0.797052	0.888022
DMU B	1	1	0.892311	1	1	1	0.553041	0.659554
DMU C	0.889155	0.972999	0.66835	0.872162	0.66835	0.84441	0.822413	1
DMU D	1	1	0.834421	0.972231	0.856838	1	0.834421	0.972231
DMU E	1	1	0.634905	0.925568	0.634905	0.925568	1	1
DMU F	1	1	0.804176	0.994664	1	1	0.804176	0.994664
DMU G	1	1	0.97769	1	0.97769	1	1	1
DMU H	1	1	1	1	1	1	1	1

		$\alpha = 1.0$						
DMUs	VRSL	VRSU	CRSL	CRSU	NIRSL	NIRSU	NDRSL	NDRSU
DMU A	0.8375	0.8375	0.797619	0.797619	0.797619	0.797619	0.8375	0.8375
DMU B	1	1	0.971069	0.971069	1	1	0.604069	0.604069
DMU C	0.937063	0.937063	0.763533	0.763533	0.763533	0.763533	0.909502	0.909502
DMU D	1	1	0.913043	0.913043	0.958372	0.958372	0.913043	0.913043
DMU E	1	1	0.762195	0.762195	0.762195	0.762195	1	1
DMU F	1	1	0.894231	0.894231	1	1	0.894231	0.894231
DMU G	1	1	1	1	1	1	1	1
DMU H	1	1	1	1	1	1	1	1

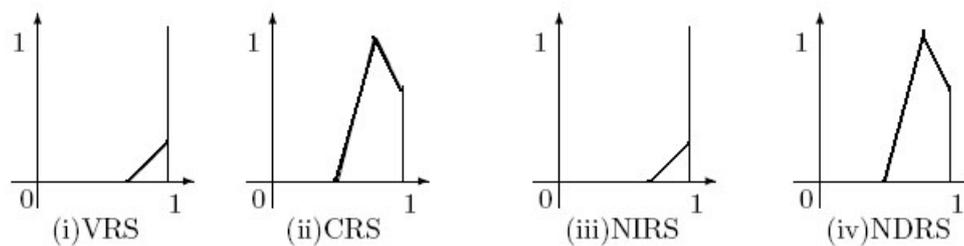


Fig. 2. Membership function of  $\mu_{RE}$  with (i)VRS, (ii)CRS,(iii)NIRS, and (iv)NDRS assumptions of RTS

## 6 Conclusion

In this paper, non-convex RE models with various RTS assumptions of technology were developed to include the bounded inputs, outputs, and output prices. A pair of simple closed-form expressions was provided for obtaining the bounds of RE. Thus, we are able to obtain the bounds of RE without solving any LP. In an empirical study, we may encounter cases in which the output price information has been collected by polling, where linguistic terms such as excellent, good, medium, and bad are used to reflect a kind of general

situation. Therefore, we extended the RE models to include fuzzy data. In order to obtain the membership functions of RE measures, we made use of the well-known  $\alpha - cut$  approach, which is simple to apply. Since in EE models the objective function is in a special fractional LP form, in which the numerator and denominator have the same coefficient, solving the model is complicated. In this study, we proved that the optimal output price vector occurs at an extreme point of output prices. Though we only deal with RE measures, the proposed approach can be directly adapted to cost efficiency and profit efficiency measures.

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