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Research Article

Differential transformation method for solving fuzzy differential inclusions by fuzzy partitions

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Abstract

There are some situations in the real word which can not be modeled by the proposed method until now, to get rid of this difficulty, the concept of Fuzzy Differential Inclusion (FDI) is introduced and then an extension of differential transformation method for solving it is given by defining a fuzzy partition. Proposed algorithms are illustrated by numerical example.

Keywords : Fuzzy number; Triangular fuzzy number; fuzzy partition; Fuzzy differential equation; Fuzzy differential inclusion.

1 Introduction

The fuzzy differential equation is a topic very important as much from the theoretical point of view [1, 2, 3, 17, 20, 23, 24, 25, 26, 27, 28, 29, 31] as well as from the view point of its applications; for example, in population models [10], and in modeling hydraulics [9].The usage of fuzzy differential equations is a natural way to model dynamical systems under possibilistic uncertainty [33]. The concept of differential equations in a fuzzy environment have been suggested by Kaleva [23] as a way of modeling uncertain and incompletely specified systems.
Formulation of the concept usually interprets the solution as a flow on some appropriate space of fuzzy sets and has been largely concerned with existence and uniqueness problems [9, 14, 23, 24, 31, 4].

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If an equation such as

$$x'(t) = f(t, x(t)), \quad x(0) = x_0 \tag{1.1}$$

is to be interpreted within a fuzzy context, a notation of differential of a fuzzy valued function of a real is required as well as some definition of what a solution means.
The concept of fuzzy derivative, was first introduced by Chang and Zadeh [12]. It was followed up by Dubios and Prade in [15] who defined and used the extension principle.
Other methods have been discussed by Puri and Ralescu in [30] and Goetschel and Voxman in [19].
A variety of methods, exact, approximate and purely numerical are available for the solution of fuzzy initial value problem. The first has usually taken the lines of Hukuhara differentiation [9, 14, 23, 24, 31].
Let ε^n be the space of all upper semi continuous (USC) normal convex fuzzy sets on $\mathbb{R}^n$, with compact support. Each level set of $u \in \varepsilon^n$ is a nonempty convex compact subset of $\mathbb{R}^n$. Derivatives of mappings $f : \mathbb{R} \to \varepsilon^n$ are defined in much the same way as those of set-valued functions. If u=v+w (equivalent to Minkowski sum of β level

sets at each β), $u - v := w$ is the Hukuhara difference. The fuzzy set $f'(t_0)$ is the Hukuhara derivative of f at t_0 if, for all small $\delta > 0$ the equations ($f(t_0+\delta)-f(t_0))/\delta, f(t_0)-f(t_0-\delta)/\delta$) exist and converge to $f'(t_0)$ as $\delta \to 0^+$. A solution to the fuzzy initial value problem (FIVP) $x'(t) = f(t, x(t)), x(0) = x_0 \in \varepsilon^n$, where $f : \mathbb{R} \times \varepsilon^n \to \varepsilon^n$, exists under appropriate conditions on f and defines a trajectory in ε^n.
Alternating approaches have been introduced by Buckly and Feurig [10]. The former treats crisp differential equations with fuzzy initial conditions to obtain fuzzy solutions, while the latter solves the same problem by fuzzifying via the Zadeh extension principle [12].
However these approaches suffers a grave disadvantage in so far as the solution has the property that diam $(x(t))$ is nondecreasing in t, that is, the solution is irreversible in possibilistic terms. Consequently, these formulations can not really reflect any of the rich behavior of ordinary differential equations such as Periodicity, Stability bifurcation and the like, and is ill-suited for modeling. Hallermeier in [21] suggested a different formulation of the FIVP based on a family of differential inclusions at each β-level, $0 \leq \beta \leq 1$,

$$x'(t) \in [f(t, x(t))]^\beta, \quad x(0) \in [x_0]^\beta \tag{1.2}$$

where now $[f(\cdot, \cdot)]^\beta : \mathbb{R} \times \mathbb{R}^n \to K_c^n$, the space of nonempty convex compact subsets of $\mathbb{R}^n$.
The idea is that the set of all such solutions $\Gamma_\beta(x_0, T)$ would be the β-level of a fuzzy set $\Gamma(x_0, T)$, in the sense that all attainable sets $A_\beta(x_0, t)$ to be the solution of the FIVP $x' = f(t, x(t)), x(0) = x_0$, thus captures both uncertainty and the rich properties of differential inclusions in one and the same technique. It has been shown that the solution set and attainability set are fuzzy sets under fairly relaxed conditions on f [5]. The previous approaches seem to be somewhat less general than (1), which also does have a fuzzy character in the formulation itself. The solutions obtained by Hallermeier are smaller than those provided by Buckly and Feurig, although they do not have the advantage of being fuzzy convex when f is a fuzzy convex valued function. But then, solution sets of set valued differential equations $x' \in f(t, x(t))$ are, in general, not convex even when f is compact convex set valued in spite of advantages of inclusion FIVP. Bede and Gal [8, 6, 7] introduce a more general definition of the derivative for fuzzy mappings enlarging the class of differentiable fuzzy mappings, and Chalco and Flores [11] solved FDEs, wich is used in the peresent work.
There are a lot of situations which differential inclusion naturally occur, but this concepts do not extended to FIVP. A certain number of papers have been appeared where attempts have been made to investigate differential inclusions with uncertainty about some of their components described in terms of fuzzy sets. Most of authors continued the idea from Hallermeier, but it is a correction of FIVPs' solving methods not really extension of crisp inclusion concepts. In other words if we consider real problem like oscillating system with combined dry and viscous damping, ... in the case of fuzzy initial value, they can not be modeled by existence model of differential equations because their initial value vary along fuzzy interval and interval can 't be determined exactly. As it is said, in the previous works authors used the concept of crisp inclusion to finding $r-soultion$ of FIVP after discretizing it to m crisp differential equation. To overcome this difficulty and generalizing the model to cover this kind of real problem, we suggest a new concept Fuzzy inclusions, which by it we mean that the problem that its initial value belongs to a fuzzy set. The origin of differential equations with a fuzzy right hand can be illustrated by the following example. Suppose, that there is a differential equation which models a real process:

$$x' = f(t, x, k) \tag{1.3}$$

where k is a vector formed by the parameters on the right hand side of equation. The vector k may often be completely unknown. Moreover, k may be vary according to an unknown low. If some set k of possible values of k can be defined, then it is convenient to replace 1.1 by the differential inclusion

$$x' \in f(t, x, k). \tag{1.4}$$

It may be happen that different points of k denote have an equal statues as possible samples of the values of k. Then, it is natural to regard k as a fuzzy set. If the function $f(t, x, .)$ is extended on the family of fuzzy sets in accordance with Zadeh's extension principle, we obtain a fuzzy set on the right hand side of 1.2. In this paper, we are going to solve FIs by differential transformation method and fuzzy partition. Intrinsically, differential transformation method evaluates the approximating solution by the finite Taylor series. The differential transformation method does not evaluate the derivative symbolically; instead, it calculates the relative derivatives by an iteration procedure described by the transformed equations obtained from the original equations using differential transformation. The concept of differential transformation was first proposed by Zhou [34] and it was applied to solve linear and nonlinear initial value problems in electric circuit analysis. The proposed method provides the Taylor's series expansion solution for the domain between any adjacent grid points. During the last 5 years, significant progress has been made in applications of the differential transformation approach for some linear and nonlinear initial value problems. In 1999, Chen and Ho [13] introduced two-dimensional differential transformation and applied it for solving partial differential equations. Jang et al. [22] introduced the concept of the differential transformation of fixed grid size and adaptive grid size mechanism to approximate solutions of initial-value problems. This paper is organized as follows: Section 2 contains the basic material to be used in the rest of paper, in Section 3 the proposed method for solving FDI is presented, and in the Section 4 proposed method are illustrated by numerical examples.

2 Preliminaries

There are various definitions for the concept of fuzzy numbers ([15, 18]) A nonempty subset A of R is called convex if and only if $(1-k)x + ky \in A$ for every $x, y \in A$ and $k \in [0,1]$. By $p_k(R)$, we denote the family of all nonempty compact convex subsets of R.
There are various definitions for the concept of fuzzy numbers ([15, 18])

Definition 2.1 *A fuzzy number is a function $u : R \to [0,1]$ satisfying the following properties:*
(i) u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$,
(ii) u is a convex fuzzy set (i.e. $u(\lambda x+(1-\lambda)y) \geq \min\{u(x), u(y)\}\forall x, y \in R, \lambda \in [0,1]$),
(iii) u is upper semi-continuous on R,
(iv) $\overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy real numbers is denoted by E. Obviously $R \subset E$. Here $R \subset E$ is understood as $R = \{\chi_x : \chi$ is usual real number$\}$. For $0 < r \leq 1$, denote $[u]_r = \{x \in R; u(x) \geq r\}$ and $[u]_0 = \overline{\{x \in R; u(x) > 0\}}$. Then it is well-known that for any $r \in [0,1]$, $[u]_r$ is a bounded closed interval. For $u, v \in E$, and $\lambda \in R$, where sum $u + v$ and the product $\lambda.u$ are defined by $[u + v]_r = [u]_r + [v]_r$, $[\lambda.u]_r = \lambda[u]_r$, $\forall r \in [0,1]$, where $[u]_r + [v]_r = \{x + y : x \in [u]_r, y \in [v]_r\}$ means the conventional addition of two intervals (subsets) of R and $\lambda[u]_r = \{\lambda x : x \in [u]_r\}$ means the conventional product between a scalar and a subset of R (see e.g. [15, 32]).

Another definition for a fuzzy number is as follows:

Definition 2.2 *An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0,1]$.*

2. *$\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0,1]$.*

3. *$\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.*

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. We recall that for $a < b < c$, $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(r) = a+(b-c)r$ and $\overline{u}(r) = c-(c-b)r$ are the endpoints of the r-level sets, for all $r \in [0,1]$. Here $\underline{u}(r) = \overline{u}(r) = b$ and it is denoted by $[u]_1$. For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ we define addition and multiplication by k as

1. $\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r))$,

2. $\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r))$,

3. $\underline{(ku)}(r) = k\underline{u}(r), \overline{(ku)}(r) = k\overline{u}(r), \quad k \geq 0$,

4. $\underline{(ku)}(r) = k\overline{u}(r), \overline{(ku)}(r) = k\underline{u}(r), \quad k < 0$.

Definition 2.3 *Let E be a set of all fuzzy numbers, we say that $\widetilde{f}(x)$ is a fuzzy valued function if $\widetilde{f} : \Re \to E$*

In this paper, we follow [1] and represent an arbitrary fuzzy number with compact support by a pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \le r \le 1$. Also, we use the Hausdorff distance between fuzzy numbers. This fuzzy number space as shown in [7] can be embedded into Banach space $B = \overline{c}[0,1] \times \overline{c}[0,1]$ where the metric is usually defined as follows: Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $r-$level sets. Since the $r-$cuts of fuzzy numbers are always closed and bounded, the intervals are written as $\widetilde{u}[r] = [\underline{u}(r), \overline{u}(r)]$, for all r. We denote by ω the set of all nonempty compact subsets of R and by ω_c the subsets of ω consisting of nonempty convex compact sets. Recall that

$$\rho(x, A) = \min_{a \in A} \|x - a\|$$

is the distance of a point $x \in$ R from $A \in \omega$ and the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \omega$ is defined as

$$\rho(A, B) = \max_{a \in A} \rho(a, B).$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$. The *Hausdorff metric* d_H on ω is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The metric d_H is defined on E as

$$d_\infty(\widetilde{u}, \widetilde{v}) = \sup\{d_H(\widetilde{u}[r], \widetilde{v}[r]) : 0 \le r \le 1\},$$

$$\widetilde{u}, \widetilde{v} \in E.$$

for arbitrary $(u, v) \in \overline{c}[0,1] \times \overline{c}[0, \quad 1]$. The following properties are well-known. (see e.g. [18, 32])
(i)$d_\infty(u + w, v + w) = d_\infty(u, v), \quad \forall u, v, w \in E,$
(ii)$d_\infty(k.u, k.v) = |k| d_\infty(u, v), \quad \forall k \in R, u, v \in E,$
(iii)$d_\infty(u + v, w + e) \le d_\infty(u, w) + D(v, e), \quad \forall u, v, w, e \in E,$
(iv) For any $\lambda \in R$ *and any* $u, v \in E$, *we have* $\lambda.(u + v) = \lambda.u + \lambda.v;$
(v) For any $\lambda, \mu \in R$ *and any* $u \in E$, *we have* $\lambda.(\mu.u) = (\lambda.\mu).u;$

Theorem 2.1 *(i) if we define* $\widetilde{0} = \chi_0$, *then* $\widetilde{0} \in R_f$ *is a neutral element with respect to addition, i.e.* $u + \widetilde{0} = \widetilde{0} + u = u$, *for all* $u \in E$.
(ii) With respect to $\widetilde{0}$, *none of* $u \in E \setminus R$, *has opposite in* E.
(iii) For any $a, b \in R$ *with* $a, b \ge 0$ *or* $a, b \le 0$ *and any* $u \in E$, *we have* $(a + b).u = a.u + b.u;$ *however,this relation dose not necessarily hold for any* $a, b \in R$,*in general.*

Remark 2.1 $d_\infty(u, 0) = d_\infty(0, u) = \| u \|$.

Definition 2.4 *Let* $\widetilde{f}(x)$ *be a fuzzy valued function on* $[a, b]$. *Suppose that* $\underline{f}(x, r)$ *and* $\overline{f}(x, r)$ *are improper Riemman-integrable on* $[a, b]$ *then we say that* $\widetilde{f}(x)$ *is improper on* $[a, b]$, *furthermore,*
$\underline{(\int_a^b \widetilde{f}(x, r)dt)} = \int_a^b \underline{f}(x, r)dt,$
$\overline{(\int_a^b \widetilde{f}(x, r)dx)} = \int_a^b \overline{f}(x, r)dx.$

Definition 2.5 *Consider* $x, y \in E$. *If there exists* $z \in E$ *such that* $x = y + z$, *then* z *is called the H-difference of* x *and* y *and it is denoted by* $x \ominus y$.

In this paper, the sign "$\ominus$" always stands for H-difference and note that $x \ominus y \ne x + (-y)$. Let us recall the definition of strongly generalized differentiability introduced in [2, 3].

Definition 2.6 *(see [8]). Let* $f : (a, b) \to E$ *and* $x_0 \in (a, b)$. *We say that* f *is strongly generalized differentiable at* x_0 *(Bede differentiability), if there exists an element* $f'(x_0) \in E$, *such that*
(i) for all $h > 0$ *sufficiently small,* $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ *and the limits(in the metric* d_∞*)*
$\lim_{h \searrow 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0)$
or
(ii) for all $h > 0$ *sufficiently small,* $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ *and the limits(in the metric* d_∞*)*
$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0+h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0-h) \ominus f(x_0)}{-h} = f'(x_0)$
or
(iii) for all $h > 0$ *sufficiently small,* $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ *and the limits(in the metric* d_∞*)*
$\lim_{h \searrow 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0-h) \ominus f(x_0)}{-h} = f'(x_0)$
or
(iv) for all $h > 0$ *sufficiently small,* $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0) \ominus f(x_0 - h)$ *and the limits(in the metric* d_∞*)*

$\lim_{h\searrow 0} \frac{f(x_0)\ominus f(x_0+h)}{-h} = \lim_{h\searrow 0} \frac{f(x_0)\ominus f(x_0-h)}{h} = f'(x_0)$
(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively)

Proposition 2.1 *If $\widetilde{f} : (a,b) \to E$ is a continuous fuzzy valued function then $g(x) = \int_a^x f(t)dt$ is differentiable with derivative $\widetilde{g}'(x) = \widetilde{f}(x)$ (see [15]).*

In the special case when $\widetilde{f}$ is a fuzzy-valued function, we have the following result.

Theorem 2.2 . *Let $\widetilde{f} : R \to E$ be a function and denote $\widetilde{f}(t) = (\underline{f}(t,r), \overline{f}(t,r))$, for each $r \in [0,1]$. Then*
(1) If $\widetilde{f}$ is differentiable in the first form (i), then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and
$\widetilde{f}'(t) = (\underline{f}'(t,r), \overline{f}'(t,r))$.
(2) If $\widetilde{f}$ is differentiable in the second form (ii), then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and
$\widetilde{f}'(t) = (\overline{f}'(t,r), \underline{f}'(t,r))$.

Lemma 2.1 *(see [7]) For $x_0 \in R$, the fuzzy differential equation $\widetilde{y}' = f(x,\widetilde{y})$, $\widetilde{y}(x_0) = \widetilde{y}_0 \in E$ where $\widetilde{f} : R \times E \longrightarrow E$ is supposed to be continuous, is equivalent to one of the integral equations:*
$\widetilde{y}(x) = \widetilde{y}_0 + \int_{x_0}^x f(t,\widetilde{y}(t))dt, \qquad \forall x \in [x_0,x_1]$
or
$\widetilde{y}(0) = \widetilde{y}(x) + (-1).\int_{x_0}^x f(t,\widetilde{y}(t))dt, \qquad \forall x \in [x_0,x_1]$
on some interval $(x_0,x_1) \subset R$, depending on the strong differentiability considered, (i) or (ii), respectively.
Here the equivalence between two equations means that any solution of an equation is a solution for the other one,too

Remark 2.2 *(see [7]). In the case of strongly generalized differentiability, to the fuzzy differential equation $y' = f(x,y)$ we may attach two different integral equations, while in the case of differentiability in the sense of the Definition of H-differentiable, we may attach only one. The second integral equation in Lemma (2.1) can be written in the form $\widetilde{y}(x) = \widetilde{y}_0 \ominus (-1).\int_{x_0}^x f(t,\widetilde{y}(t))dt$.*

The following theorems concern the existence of solutions of a fuzzy initial-value problem under generalized differentiability (see [7]).

Theorem 2.3 *Let us suppose that the following conditions hold: (a)Let $R_0 = [x_0, x_0+p] \times \overline{B}(y_0,q)$, $p,q > 0$, $y_0 \in E$, where $\overline{B}(y_0,q) = \{y \in E : d_\infty(y,y_0) \leq q\}$ denote a closed ball in E and let $f : R_0 \longrightarrow E$ be a continuous function such that $d_\infty(\widetilde{0}, f(x,y)) = \|f(x,y)\| \leq M$ for all $(x,y) \in R_0$ (b) Let $g : [x_0,x_0+p] \times [0,q] \longrightarrow E$, such that $g(x,0) \equiv 0$ and $0 \leq g(x,u) \leq M_1$, $\forall x \in [x_0,x_0+p]$, $0 \leq u \leq q$, such that $g(x,u)$ is non-decreasing in u and g is such that the initial-value problem $u'(x) = g(x,u(x)), u(x_0) = 0$ has only the solution $u(x) \equiv 0$ on $[x_0,x_0+p]$. (c) We have $d_\infty(f(x,y), f(x,z)) \leq g(x, d_\infty(y,z))$, $\forall (x,y),(x,z) \in R_0$ and $d_\infty(y,z) \leq q$. (d) There exists $d > 0$ such that for $x \in [x_0, x_0+d]$ the sequence $\overline{y}_n : [x_0,x_0+d] \longrightarrow E$ given by $\overline{y}_0(x) = y_0$, $\overline{y}_{n+1}(x) = y_0 \ominus (-1).\int_{x_0}^x f(t,\overline{y}_n)dt$ is defined for any $n \in N$. Then the fuzzy initial -value problem*

$$\begin{cases} y' = f(x,y), \\ y(x_0) = y_0 \end{cases}$$

has two solutions (one (i)-differentiable and the other one (ii)-differentiable) $y, \hat{y} : [x_0, x_0+r] \longrightarrow B(y_0,q)$ where $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ and the successive iterations

$$y_0(x) = y_0, y_{n+1}(x) = y_0 + \int_{x_0}^x f(t, y_n(t))dt$$

and

$$\hat{y}_0(x) = y_0, \hat{y}_{n+1}(x) = y_0 \ominus (-1).\int_{x_0}^x f(t, \hat{y}_n(t))dt$$

converge to these two solutions respectively.

According to theorem 2.3, we restrict our attention to functions which are (i)- or (ii)-differentiable on their domain except for a finite number of points (see also [7]).
The following lemma gives a sufficient condition for the existence of the H-difference of two triangular fuzzy numbers.

Lemma 2.2 *(see [7]) Let $u,v \in E$ be such that $u(1) - \underline{u}(0) > 0$, $\overline{u}(0) - u(1) > 0$ and $len(v) = (\overline{v}(0) - \underline{v}(0)) \leq \min\{u(1) - \underline{u}(0), \overline{u}(0) - u(1)\}$. Then the H-difference $u \ominus v$ exists.*

The following corollary gives simple sufficient condition for the existence of fuzzy differential equations under strongly generalized differentiability.

Corollary 2.1 *Let* $\widetilde{f} : R_0 \longrightarrow E$ *where* $R_0 = [x_0, x_0+p] \times (\overline{B}(\widetilde{y}_0, q) \cap E)$, *and* $\widetilde{y}_0 \in E$ *such that* $y(0,1) - \underline{y}(0,0)$ *and* $\overline{y}(0,0) - y(0,1)$. *Let* $m = \min\{y(0,1) - \underline{y}(0,0), \overline{y}(0,0) - y(0,1)\}$. *Under the assumptions (a)-(c) of Theorem (2.3), the fuzzy initial-value problem*

$$\begin{cases} \widetilde{y}' = f(x, \widetilde{y}), \\ \widetilde{y}(x_0) = \widetilde{y}_0 \end{cases}$$

has two solutions $y, \overline{y} : [x_0, x_0 + r] \longrightarrow B(\widetilde{y}_0, q)$ *where* $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, \frac{m}{2M}\}$ *and the successive iterations in (2.3) converge to these two solutions.*

Definition 2.7 *Let* $f : (a,b) \to E$ *and* $x_0 \in (a,b)$. *We define the* $n-th$*order differential of* f *as follow: We say that* f *is strongly generalized differentiable of the* $n-th$*order at* x_0. *If there exists an element* $f^{(s)}(x_0) \in E, \quad \forall s = 1 \ldots n$, *such that*

(i) for all $h > 0$ *sufficiently small,* $\exists f^{(s-1)}(x_0 + h) \ominus f^{(s-1)}(x_0)$, $\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 - h)$ *and the limits(in the metric* d_∞*)*

$$\lim_{h\searrow 0} \frac{f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)}{h} = \lim_{h\searrow 0} \frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)}{h} = f^{(s)}(x_0)$$

or

(ii) for all $h > 0$ *sufficiently small,* $\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 + h)$, $\exists f^{(s-1)}(x_0 - h) \ominus f^{(s-1)}(x_0)$ *and the limits(in the metric* d_∞*)*

$$\lim_{h\searrow 0} \frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h)}{-h} = \lim_{h\searrow 0} \frac{f^{(s-1)}(x_0-h)\ominus f(x_0)}{-h} = f^{(s)}(x_0)$$

or

(iii) for all $h > 0$ *sufficiently small,* $\exists f^{(s-1)}(x_0 + h) \ominus f^{(s-1)}(x_0)$, $\exists f^{(s-1)}(x_0 - h) \ominus f^{(s-1)}(x_0)$ *and the limits(in the metric* d_∞*)*

$$\lim_{h\searrow 0} \frac{f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)}{h} = \lim_{h\searrow 0} \frac{f^{(s-1)}(x_0-h)\ominus f^{(s-1)}(x_0)}{-h} = f^{(s)}(x_0)$$

or

(iv) for all $h > 0$ *sufficiently small,* $\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 + h)$, $\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 - h)$ *and the limits(in the metric* d_∞*)*

$$\lim_{h\searrow 0} \frac{f^{(k-1)}(x_0)\ominus f^{(s-1)}(x_0+h)}{-h} = \lim_{h\searrow 0} \frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)}{h} = f^{(s)}(x_0)$$

(h *and* $-h$ *at denominators mean* $\frac{1}{h}$ *and* $\frac{-1}{h}$, *respectively* $\forall s = 1 \ldots n$*)*

Remark 2.3 *Note that by the above definition a fuzzy -valued function is i-differentiable (or ii-differentiable) of order* n *if* $f^{(s)}$ *for* $s = 1 \ldots n$ *is i-differentiable (or ii-differentiable). It is possible that the different orders have different kind of (i or ii) differentiability, but we do not consider this kind of function in this paper.*

Following [1] we define a first-order fuzzy differentiable equation by

$$x' = f(t, x(t))$$

where $x(t) = (\underline{x}(t,r), \overline{x}(t,r))$ is a fuzzy function of t. $f(t, x(t))$ is a fuzzy -valued function and the fuzzy variable $x'(t)$ is the defined derivative of $X(t,r)$. If an initial value $x(t_0) = x_0$ is given, we obtain a fuzzy Cauchy problem of the first order

$$x' = f(t, X(t,r)), \quad x(t_0) = x_0 \tag{2.5}$$

So , if we consider derivative form i or ii we may replace the FIVP by the equivalent system

$$\begin{array}{ll} \underline{x}'(t,r) = h(t, \underline{x}(t,r), \overline{x}(t,r)) & \underline{x}(t_0,r) = \underline{x}_0(r) \\ \overline{x}'(t,r) = g(t, \underline{x}(t,r), \overline{x}(t,r)) & \overline{x}(t_0,r) = \overline{x}_0(r) \end{array} \tag{2.6}$$

for $r \in [0,1]$.

or

$$\begin{array}{ll} \underline{x}'(t,r) = g(t, \underline{x}(t,r), \overline{x}(t,r)) & , \underline{x}(t_0,r) = \underline{x}_0(r) \\ \overline{x}'(t,r) = h(t, \underline{x}(t,r), \overline{x}(t,r)) & , \overline{x}(t_0,r) = \overline{x}_0(r) \end{array} \tag{2.7}$$

for $r \in [0,1]$,For every prefixed r, the system represents an ordinary Cauchy problem for which any converging classical numerical procedure can be applied. In the next section a differential transformation method is proposed for solving this problem.

The set of all r-solution to (2.6) is denoted $\Gamma(r)$ and the r-reachable set $\Gamma(r,t)$ is defined as $\Gamma(r,t) = \{x(t,r) : x(r) \in \Gamma(r)\}$. For each selection of $X(0,r)$ from fuzzy set $[\gamma_0(r), \gamma_1(r)]$, we have one FIVP in parametric form which should be solved.

3 Extension of DTM to solve Fuzzy Differential inclusions

In this section we are going to propose a new numerical method for computing approximation of the set of all solutions to a FDI. Let approximate $\Gamma(r)$ by choosing m distinct initial value from $[\gamma_0(r), \gamma_1(r)]$. So if we find a fuzzy partition with m elements and solved then, we can obtained an approximation from $\Gamma(r)$. Whit out loss of generality, assume $m = 5$ then from which can be obtained by applying the nearest symmetric triangular defuzzification procedure from two extreme values. We consider two alternatives :

- Two extreme values are real.
- Two extreme values are fuzzy.

Real case:
Step 1: Given the extreme values γ_0,γ_1 we define the fuzzy number medium $x^{(1)}(r)$ as

$$\begin{aligned} \underline{x}^1(r) &= 1 - \tfrac{\gamma_0+\gamma_1}{2} r \\ \overline{x}^1(r) &= \tfrac{\gamma_0+\gamma_1}{2} r \end{aligned}$$

Step 2: Defuzzify γ_0 and $x^{(1)}(r)$ to obtain the lower medium $x^{2,1}(r)$ for which

$$\begin{aligned} X_0 &= 1/4 \int_0^1 [\underline{x}^1(r) + \overline{x}^1(r) + 2\gamma_0] dr \\ \sigma &= 3/4 \int_0^1 [\overline{x}^1(r) - \underline{x}^1(r) - 2\gamma_0](1-r) dr \end{aligned}$$

Thus $\underline{x}^{2,1}(r) = x_0 - \sigma + \sigma r$ and $\overline{x}^{2,1}(r) = x_0 + \sigma - \sigma r$. We now defuzzify $x^{(1)}(r)$ and γ_1 to get upper medium $x^{2,3}(r)$, we compute x_0 and σ as it is computed for $x^{2,1}(r)$.

Step 3: Update the medium by defuzzifying the lower medium $x^{2,1}(r)$ and upper medium $x^{2,3}(r)$. The result is the medium $x^{2,2}(r)$ centered at

$$\begin{aligned} x_0 &= 1/4 \\ &\int_0^1 [\underline{x}^{2,1}(r) + \overline{x}^{2,1}(r) + \underline{x}^{2,3}(r) + \overline{x}^{2,3}(r)] dr \\ \sigma &= 3/4 \\ &\int_0^1 [\overline{x}^{2,1}(r) - \underline{x}^{2,1}(r) - \underline{x}^{2,3}(r) + \overline{x}^{2,3}(r)] \\ &(1-r) dr \end{aligned}$$

We thus obtain a fuzzy partition with five elements. The procedure can be extended easily to obtain a fuzzy partition with arbitrary elements. So it is sufficient to solve these m equations [16].
Fuzzy extreme values'case: In this case we need only to repeat Step 3 for generating sufficient number of fuzzy number.
Let show elements of a fuzzy partition by $\{y_0^1(r), \ldots, y_0^m(r)\}$. Then we should solve

$$y'(t,r) = F(t, y(t), r), \quad y_0(r) = y_0^i(r), \quad (3.8)$$
$$i = 1, \ldots, m$$

So we have m fuzzy differential equations, we can solve them in one of the following method.

Crisp inclusion point of view In this case, we should solve m fuzzy differential equations using the concept of crisp inclusion and then solve t crisp differential equation by one of the existence methods

One of the fuzzy derivative concept In this case we should try to transform m fuzzy differential equations to crisp differential equation by using one of the fuzzy derivative.

In the first case as described in introduction we should use the idea which is proposed to solve the fuzzy initial value problem by diamond; it worked by the many of authors but if we consider the second point of view we need a method to solve the fuzzy differential equation using one of the fuzzy derivative.In this part of paper we want to extend differential transformation method to solving fuzzy initial value problem based on lateral Hukuhara derivative which is shown that do not have the disadvantage of Sikala's derivative [5].

3.1 The Differential transformation method

Definition 3.1 *If $x(t,r)$ is strongly generalized differentiable of order k in the time domain T then let If f is differentiable in first form (i)*

$$\begin{aligned} &\overline{\varphi}(t,k,r) = \tfrac{d^k(\overline{x}(t,r))}{dt^k} \quad \forall t \in T \\ &\overline{X}_i(k,r) = \overline{\varphi}(t_i,k,r) = \tfrac{d^k(\overline{x}(t,r))}{dt^k}]_{t=t_i} \quad \forall k \in K \end{aligned}$$

$$\begin{aligned} &\underline{\varphi}(t,k,r) = \tfrac{d^k(\underline{x}(t,r))}{dt^k} \quad \forall t \in T \\ &\underline{X}_i(k,r) = \underline{\varphi}(t_i,k) = \tfrac{d^k(\underline{x}(t,r))}{dt^k}]_{t=t_i} \quad \forall k \in K \end{aligned}$$

If f is differentiable in second form (ii)

$$\begin{aligned} &\overline{\varphi}(t,k,r) = \tfrac{d^k(\underline{x}(t,r))}{dt^k} \quad \forall t \in T \\ &\overline{X}_i(k,r) = \underline{\varphi}(t_i,k,r) = \tfrac{d^k(\overline{x}(t,r))}{dt^k}]_{t=t_i} \quad k \textit{ is odd} \end{aligned}$$

$$\begin{aligned} &\underline{\varphi}(t,k,r) = \tfrac{d^k(\overline{x}(t,r))}{dt^k} \quad \forall t \in T \\ &\underline{X}_i(k,r) = \overline{\varphi}(t_i,k) = \tfrac{d^k(\overline{x}(t,r))}{dt^k}]_{t=t_i} \quad k \textit{ is odd} \end{aligned}$$

where $\underline{X}(k,r)$ *and* $\overline{X}(k,r)$ *are called the lower and the upper spectrum of* $x(t,r)$ *at* $t = t_i$ *in the domain* K*, respectively.*

If k is even then φ is considered as it considered in the first form(i) . So $x(t,r)$ can be represented as

$$\overline{x}(t,r) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \overline{X}(k,r).$$

$$\underline{x}(t,r) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \underline{X}(k,r).$$

or

$$\underline{x}(t,r) = \sum_{k=1,\ odd}^{\infty} \frac{(t-t_i)^k}{k!} \overline{X}(k,r) + \sum_{k=0,\ even}^{\infty} \frac{(t-t_i)^k}{k!} \underline{X}(k,r).$$

$$\overline{x}(t,r) = \sum_{k=1,\ odd}^{\infty} \frac{(t-t_i)^k}{k!} \underline{X}(k,r) + \sum_{k=0,\ even}^{\infty} \frac{(t-t_i)^k}{k!} \overline{X}(k,r).$$

The above set of equations is known as the inverse transformation of $X(k)$ if $X(k)$ is defined as

$$\underline{X}(k,r) = M(k)[\frac{d^k(\underline{q(t)x(t,r)})}{dt^k}]_{t=0}, \quad k = 0, 1, 2, \ldots, \infty$$

$$\overline{X}(k,r) = M(k)[\frac{d^k(\overline{q(t)x(t,r)})}{dt^k}]_{t=0}, \quad k = 0, 1, 2, \ldots, \infty$$

or

$$\overline{X}(k,r) = M(k)[\frac{d^k(\underline{q(t)x(t,r)})}{dt^k}]_{t=0}, \quad k = 1, 3, 5, \ldots, \infty$$

$$\underline{X}(k,r) = M(k)[\frac{d^k(\overline{q(t)x(t,r)})}{dt^k}]_{t=0}, \quad , k = 0, 2, 4, \ldots, \infty$$

Then the function $x(t,r)$ can be described as

$$\underline{x}(t,r) = \frac{1}{q(t)} \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{X}(k,r)}{M(k)}.$$

$$\overline{x}(t,r) = \frac{1}{q(t)} \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\overline{X}(k,r)}{M(k)}.$$

or

$$\overline{x}(t,r) = \frac{1}{q(t)}(\sum_{k=1,\ odd}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{X}(k,r)}{M(k)} + \sum_{k=0,\ even}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\overline{X}(k,r)}{M(k)}).$$

$$\underline{x}(t,r) = \frac{1}{q(t)}(\sum_{k=1,\ odd}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\overline{X}(k,r)}{M(k)} + \sum_{k=0,\ even}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{X}(k,r)}{M(k)}).$$

where $M(K) > 0$ and $q(t) > 0$. $M(k)$ is called the weighting factor and $q(t)$ is regarded as a kernel corresponding to $X(t,r)$. If $M(k) = 1$ and $q(t) = 1$, then (5) can be treated as a special case of (7). In this paper, the transformation with $M(k) = \frac{H^k}{k!}$ and $q(t) = 1$ is applied, where H is the time horizon of interest. Then If f is differentiable in first form (i)

$$\underline{X}(k,r) = \frac{H^k}{k!} \frac{d^k \underline{x}(t,r)}{dt^k}.$$

$$\overline{X}(k,r) = \frac{H^k}{k!} \frac{d^k \overline{x}(t,r)}{dt^k}.$$

If f is differentiable in second form (ii)

$$\overline{X}(k,r) = \frac{H^k}{k!} \frac{d^k \underline{x}(t,r)}{dt^k}. \quad k \ is \ odd$$

$$\underline{X}(k,r) = \frac{H^k}{k!} \frac{d^k \overline{x}(t,r)}{dt^k}. \quad k \ is \ odd$$

If k is even then φ is considered as it considered in the first form(i). Using the differential transformation, a differential equation in the domain of interest can be transformed to an algebraic equation in the domain K and $X(t,r)$ can be obtained by the finite-term Taylor series plus a reminder, as

$$\begin{aligned} \overline{x}(t,r) &= \frac{1}{q(t)} \sum_{k=0}^{n} \frac{(t-t_0)^k}{k!} \frac{\overline{X}(k,r)}{M(k)} + R_{n+1}(t) \\ &= \sum_{k=0}^{\infty} (\frac{t-t_0}{H})^k \overline{X}(k,r) + R_{n+1}(t) \end{aligned}$$

$$\begin{aligned} \underline{x}(t,r) &= \frac{1}{q(t)} \sum_{k=0}^{n} \frac{(t-t_0)^k}{k!} \frac{\underline{X}(k,r)}{M(k)} + R_{n+1}(t) \\ &= \sum_{k=0}^{n} (\frac{t-t_0}{H})^k \underline{X}(k,r) + R_{n+1}(t) \end{aligned}$$

or

$$\begin{aligned}
\underline{x}(t,r) &= \tfrac{1}{q(t)}\big(\textstyle\sum_{k=1,\ odd}^{\infty} \frac{(t-t_0)^k}{k!}\frac{\overline{X}(k,r)}{M(k)} \\
&+ \textstyle\sum_{k=0,\ even}^{\infty} \frac{(t-t_0)^k}{k!}\frac{\underline{X}(k,r)}{M(k)}\big) + R_{n+1}(t) \\
&= \textstyle\sum_{k=1,\ odd}^{\infty} (\frac{t-t_0}{H})^k \overline{X}(k,r) \\
&+ \textstyle\sum_{k=0,\ even}^{\infty} (\frac{t-t_0}{H})^k \underline{X}(k,r) + R_{n+1}(t)
\end{aligned}$$

$$\begin{aligned}
\overline{x}(t,r) &= \tfrac{1}{q(t)}\big(\textstyle\sum_{k=1,\ odd}^{\infty} \frac{(t-t_0)^k}{k!}\frac{\underline{X}(k,r)}{M(k)} \\
&+ \tfrac{1}{q(t)}\big(\textstyle\sum_{k=0,\ even}^{\infty} \frac{(t-t_0)^k}{k!}\frac{\overline{X}(k,r)}{M(k)}\big) + R_{n+1}(t) \\
&= \textstyle\sum_{k=1,\ odd}^{\infty} (\frac{t-t_0}{H})^k \underline{X}(k,r) \\
&+ \textstyle\sum_{k=0,\ even}^{\infty} (\frac{t-t_0}{H})^k \overline{X}(k,r) + R_{n+1}(t)
\end{aligned}$$

The objective of this section is to find the solution of (1.1) at the equally spaced grid points $[t_0, t_1, \ldots, t_N]$ where $t_i = a + ih$ for each $i = 0, 1, \ldots, N$ and $h = \frac{(b-a)}{N}$ the domain interest $[a, b]$ is divided in to N sub-domains and the approximation function in each sub-domain are $x_i(t,r), i = 0, 1, \ldots, N-1$ respectively. Taking the differential transformation of (1.2) or (1.3), the transformed equation describes the relationship between the spectrum $x(t,r)$ as

$$\begin{aligned}
(k+1)\underline{X}(k+1,r) &= H(t, \underline{X}(k,r), \overline{X}(k,r)) \\
(k+1)\overline{X}(k+1,r) &= G(t, \underline{X}(k,r), \overline{X}(k,r))
\end{aligned}$$

Or

$$(k+1)\underline{X}(k+1,r) = G(t, \underline{X}(k,r), \overline{X}(k,r))$$

$$(k+1)\overline{X}(k+1,r) = H(t, \underline{X}(k,r), \overline{X}(k,r))$$

$$k \text{ is odd}$$

$$(k+1)\underline{X}(k+1,r) = H(t, \underline{X}(k,r), \overline{X}(k,r))$$

$$(k+1)\overline{X}(k+1,r) = G(t, \underline{X}(k,r), \overline{X}(k,r))$$

$$k \text{ is even}$$

where $H(.)$ denotes the transformed function of $h(t, \underline{x}(t,r), \overline{x}(t,r))$ and $G(.)$ denotes the transformed function of $g(t, \underline{x}(t,r), \overline{x}(t,r))$. From the initial conditions the following can be obtained

$$\underline{X}(0,r) = \underline{x}_0(r), \quad \overline{X}(0,r) = \overline{x}_0(r).$$

In the first sub-domain, $\underline{x}(t,r)$, $\overline{x}(t,r)$ can be described by $\underline{x}_0(t,r)$ and $\overline{x}_0(t,r)$ respectively. They can be represented in terms of their nth order Taylor Polynomial with respect to a, that is

$$\underline{x}_0(t,r) = \underline{X}_0(0,r) + \underline{X}_0(1,r)(t-a)$$
$$+ \underline{X}_0(2,r)(t-a)^2 + \ldots + \underline{X}_0(n,r)(t-a)^n \quad (3.9)$$
$$\overline{x}_0(t,r) = \overline{X}_0(0,r) + \overline{X}_0(1,r)(t-a)$$
$$+ \overline{X}_0(2,r)(t-a)^2 + \ldots + \overline{X}_0(n,r)(t-a)^n \quad (3.10)$$

where the subscript 0 denotes that the Taylor Polynomial is expanded to $t_0 = a$. Once the Taylor Polynomial is obtained $x(t_1, r)$ can be evaluated as

$$\begin{aligned}
\underline{x}(t_1,r) &= \underline{X}_0(0,r) + \underline{X}_0(1,r)(t_1-a) \\
&+ \underline{X}_0(2,r)(t_1-a)^2 + \ldots \\
&+ \underline{X}_0(n,r)(t_1-a)^n \\
&= \underline{X}_0(0,r) + \underline{X}_0(1,r)h + X_0(2,r)h^2 \\
&+ \ldots + \underline{X}_0(n,r)h^n = \textstyle\sum_{j=0}^{n} \underline{X}_0(j,r)h^j \\
\overline{x}(t_1,r) &= \overline{X}_0(0,r) + \overline{X}_0(1,r)(t_1-a) \\
&+ \overline{X}_0(2,r)(t_1-a)^2 + \ldots \\
&+ \overline{X}_0(n,r)(t_1-a)^n \\
&= \overline{X}_0(0,r) + \overline{X}_0(1,r)h + \overline{X}_0(2,r)h^2 \\
&+ \ldots + \overline{X}_0(n,r)h^n = \textstyle\sum_{j=0}^{n} \overline{X}_0(j,r)h^j
\end{aligned}$$

The first value, $x_0(t_1, r)$ of the first sub-domain is the initial value of the second sub-domain, i.e. $x_1(t_1, r) = x_0(t_1, r)$. In a similar manner $x(t_2, r)$ can be represented as

$$\begin{aligned}
\underline{x}(t_2,r) &\approx \underline{x}_1(t_2,r) \\
&= \underline{X}_1(0,r) + \underline{X}_1(1,r)(t_2-t_1) \\
&+ \underline{X}_1(2,r)(t_2-t_1)^2 + \ldots \\
&+ \underline{X}_1(n,r)(t_2-t_1)^n \\
&= \underline{X}_1(0,r) + \underline{X}_1(1,r)h + \underline{X}_1(2,r)h^2 \\
&+ \ldots + \underline{X}_1(n,r)h^n \\
&= \textstyle\sum_{j=0}^{n} \underline{X}_1(j,r)h^j
\end{aligned}$$

Table 1: Data Set.

Functional form	Differential Transform
$y(x) = u(x) \pm v(x)$	$Y(k) = u(k)(k)$
$y(x) = \alpha w(x)$	$Y(k) = \alpha w(k)$
$y(x) = d^m z(x)/dx^m$	$Y(k) = \frac{(m+k)!}{k!} z(k+m)$
$y(x) = u(x).v(x)$	$Y(k) = \sum_{i=0}^{k} u(l)v(k-l)$
$y(x) = x^m$	$Y(k) = \delta(k-m)$
$y(x) = exp(\lambda x)$	$Y(k) = \lambda^k/k!$
$y(x) = (1+x)^m$	$Y(k) = \frac{m(m-1)\cdots(m-k-1)}{k!}$
$y(x) = \sin(\omega x + \alpha)$	$Y(k) = \frac{\omega^k}{k!} \sin(\pi \frac{k}{2!} + \alpha)$
$y(x) = \cos(\omega x + \alpha)$	$Y(k) = \frac{\omega^k}{k!} \cos(\pi \frac{k}{2!} + \alpha)$

$$\begin{aligned} \overline{x}(t_2, r) &\approx \overline{x}_1(t_2, r) \\ &= \overline{X}_1(0, r) + \overline{X}_1(1, r)(t_2 - t_1) \\ &+ \overline{X}_1(2, r)(t_2 - t_1)^2 + \dots \\ &+ \overline{X}_1(n, r)(t_2 - t_1)^n \\ &= \overline{X}_1(0, r) + \overline{X}_1(1, r)h \\ &+ \overline{X}_1(2, r)h^2 + \dots + \overline{X}_1(n, r)h^n \\ &= \textstyle\sum_{j=0}^{n} \overline{X}_1(j, r)h^j \end{aligned}$$

Hence, the solution on the grid points (t_{i+1}) can be obtained as follows:

$$\begin{aligned} \underline{x}(t_{i+1}, r) &\cong \underline{x}_i(t_{i+1}, r) \\ &= \underline{X}_i(0, r) + \underline{X}_i(1, r)(t_{i+1} - t_i) \\ &+ \underline{X}_i(2, r)(t_{i+1} - t_i)^2 + \dots \\ &+ \underline{X}_i(n, r)(t_{i+1} - t_i)^n \\ &= \textstyle\sum_{j=0}^{n} \underline{X}_i(j, r)h^j \end{aligned}$$

$$\begin{aligned} \overline{x}(t_{i+1}, r) &\cong \overline{x}_i(t_{i+1}, r) \\ &= \overline{X}_i(0, r) + \overline{X}_i(1, r)(t_{i+1} - t_i) \\ &+ \overline{X}_i(2, r)(t_{i+1} - t_i)^2 + \dots \\ &+ \overline{X}_i(n, r)(t_{i+1} - t_i)^n \\ &= \textstyle\sum_{j=0}^{n} \overline{X}_i(j, r)h^j \end{aligned}$$

From definition (3.1), it can be easily proven that the transformation function has basic mathematical operations shown in Table (1).

Theorem 3.1 *[11] If f is nondecreasing with respect to the second argument then, using the (i) or (ii) differentiability, the fuzzy solution of the FIVP and the solution via differential inclusions are identical.*

4 Numerical example

Consider $y'(t) = y(t) - t^2 + 1, \quad 0 \le t \le 2$, in the case that its initial value belongs to a fuzzy interval as $[0, 1]$. By the proposed algorithm partition P is obtained.

$P = \{0, y_0^1(r), y_0^2(r), y_0^3(r), 1\}$ where

$y_0^1(r) : (r/4, 1/2 - r/4)$,
$y_0^2(r) : (1/2 + r/4, 1 - r/4)$,

$y_0^3(r) : (1/4 + r/4, 3/4 - r/4)$.

Then we should have 5 FIVP but in this case since the extreme values are crisp we have 3 FIVP and 2 crisp initial value problem.

FIVP

$$y'(t) = y(t) - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = y_0^i$$
$$i = 1, \cdots, 3$$

crisp initial value problem

$$y'(t) = y(t) - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0$$

$$y'(t) = y(t) - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 1$$

by solving them we have The exact solution of each problem is

$$\begin{aligned} \underline{y}(t,r) &= (-2r)e^t + (t+1)^2 \\ \overline{y}(t,r) &= (2+2r)e^t + (t+1)^2 \end{aligned}$$

$$\begin{aligned} \underline{y}(t,r) &= (-1/2 - r/4)e^t + (t+1)^2 \\ \overline{y}(t,r) &= (1 - r/4)e^t + (t+1)^2 \\ \underline{y}(t,r) &= (-r/4)e^t + (t+1)^2 \\ \overline{y}(t,r) &= (-1/2 + r/4)e^t + (t+1)^2 \\ \underline{y}(t,r) &= (-1/4 - r/4)e^t + (t+1)^2 \\ \overline{y}(t,r) &= (-3/4 + r/4)e^t + (t+1)^2 \end{aligned}$$

$$\begin{array}{ll} y(t,r) = (t+1)^2 - e^t & y(0) = 0 \\ y(t,r) = (t+1)^2 & y(0) = 1 \end{array}$$

Let $N = 10$ and $h = 0.2$, the differential equation of a system between t_i and t_{i+1} can be represented as

$$y'(t^*) = y(t^*) - t^{*2} - 2t_i t^* + (1 - t_i^2) \qquad (*)$$

where $t^* = t - t_i$. Since we know that $y(0) \in E$, then $y'(t) \in E$ and so on. If we show the equation $(*)$ in parametric form, we have

$$\overline{y'}(t^*) = \overline{y}(t^*) - t^* - 2t_i t^* + (1 - t_i)^2 \qquad (4.11)$$

$$\underline{y'}(t^*) = \underline{y}(t^*) - t^* - 2t_i t^* + (1 - t_i)^2 \qquad (4.12)$$

Taking the differential transformation of (12) and (13), it can be obtained that

$$\begin{aligned} \overline{Y}_i(k+1,r) = [&\overline{Y}_i(k,r) - \delta(k-2) - 2t_i(\delta(k-1)) \\ &+(1 - t_i^2)\delta(k)]/(k+1) \qquad (**) \end{aligned}$$

$$\begin{aligned} \underline{Y}_i(k+1,r) = [&\underline{Y}_i(k,r) - \delta(k-2) - 2t_i(\delta(k-1)) \\ &+(1 - t_i^2)\delta(k+1)]/(k+1) \qquad (***) \end{aligned}$$

with $\overline{Y}_0() = \overline{y}_0^i$ and $\underline{Y}_0(0) = \underline{y}_0^i$. The approximate of $\underline{y}(t,r)$ and $\overline{y}(t,r)$ on the grid point can be obtained by $(*)$, $(**)$ and $(***)$ for each i; i.e there are a Taylor series corresponds to each i as ξ_i then the set of solution is $\Gamma(r) = \{\xi_i | i = 1 \cdots 5\}$.

Remark 4.1 *Note that this problem have 2 solutions By theorem 3, but We considered only one solution. Based on application different form of ODE can be considered for more detail about it you can see [8].*

5 Conclusion

A new concept, Fuzzy inclusions, and a new method, DTM, are introduced in this paper. During this work we do not use the crisp inclusion concept to find $r-soulution$ of FIVP, it can be done easily after generating sufficient FIVP by fuzzy partition,instead it a new method to solving a FIVP is proposed which is an analytical-numerical method and give us an appropriate accuracy related to other methods. The Concept of Fuzzy inclusion due to its properties can play an important role in mechanics and physics problems.

References

[1] S. Abbasbandy, T. Allahviranloo, Oscar Lopez-Pouso, Juan J. Nieto, *Numerical Methods for Fuzzy Differential Inclusions*, Journal of Computer and Mathematics With Applications 48 (2004) 1633-1641.

[2] S. Abbasbandy, T. Allahviranloo, *Numerical Solutions of Fuzzy Differential Equations By Taylor Method*, Journal of Computational Methods in Applied Mathematics 2 (2002) 113-124.

[3] T. Allahviranloo, N. Ahmady, E.Ahmady, *Numerical solution of fuzzy differential equations by predictorcorrector method Information Sciences*, 177 (2007) 1633-1647

[4] T. Allahviranloo, N. A. Kiani, M. Barkhordari, *Toward the existence and uniqueness of second-order fuzzy differential equations*, Information Sciences 179 (2009) 1207-1215.

[5] V. A. Baidosov, *Fuzzy differential inculusion*, PMM U.S.S.R. 54 (1990) 8-13.

[6] B. Bede, S. G. Gal, *Almost Periodic fuzzy-number valued functions*, Fuzzy Sets and Systems 147 (2004) 385-403.

[7] B. Bede, S. G. Gal, *Generalizations of differentiablity of fuzzy number valued function with application to fuzzy differential equations*, Fuzzy Sets and Systems 151 (2005) 581-599.

[8] B. Bede, I. J. Rudas, L. Attila, *First order linear fuzzy differential equations under generalized differentiability, Information Sciences 177 (2007) 3627-3635.*

[9] *A. Bencsik, B Bede, J. Tar, J. Fodor, Fuzzy Differential Equations in modeling hydraulic differential servo cylinders, In :Third Romanian-Hungarian join Symposium on applied computational intelligence(SACI), Timisoara, Romania 2006.*

[10] *J. Buckley, Thomas Feuring, Fuzzy differential equations, Fuzzy Sets and Systems 110 (2000) 43-54.*

[11] *Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations, Chaos, solitons and Fractals (2006) 1016-1043.*

[12] *S. L. Chang, L. A. Zadeh, On fuzzy mapping and control, IEEE Trans, Systems Man Cybernet. 2 (1972) 30-34.*

[13] *C. K. Chen, S. H. Ho, Solving partial differential equations by two-dimensional differential transform method, Applied Math. Comput. 106 (1999) 171-179.*

[14] *W. Congxin, S. Shiji, Exitance theorem to the cuchy problem of fuzzy Differential Equations under compactance-tyape conditions, Information Science 108 (1993) 123-134.*

[15] *D. Dubois, H. Prade, Towards fuzzy differential calculus: Part 3, differentiation, Fuzzy Sets and Systems 8 (1982) 225-233.*

[16] *M. Delago, A. Vila, W. Voxman, On a cononical representation of fuzzy number, Fuzzy Sets Syst. 93 (1998) 125-135.*

[17] *W. Fei, Existence and uniqueness of solution for fuzzy random differential equations with non-Lipschitz coefficients, Information Sciences 20 (2007) 4329-4337.*

[18] *S. G. Gal, Approximation theory in fuzzy setting, in: G.A. Anastassiou (Ed.), Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman Hall CRC Press, (2000) 617-666.*

[19] *R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy sets and Systems 18 (1986) 31-43.*

[20] *M. Ghanbari, Numerical solution of fuzzy initial value problems under generalized differentiability by HPM, Int. J. Industrial Mathematics 1 (2009) 19-39*

[21] *E. Houllermeier, An approach to modeling and simulation of uncertain dynamical systems, Int. J. Uncertainty, Fuzziness Knowledge-based Systems 5 (1997) 117137.*

[22] *M. J. Jang, C. L. Chen, Y. C. Liy, On solving the initial-value problems using the differential transformation method, Applied Math. Comput. 115 (2000) 145-160.*

[23] *O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301-317.*

[24] *O. Kaleva, The Cuachy problem for Fuzzy differential equations, Fuzzy Sets and Systems 35 (1990) 389-396.*

[25] *R. R. Lopez, Comparison results for fuzzy differential equations, Information Sciences 178 (2008) 1756-1779*

[26] *M. Ma, M. Friedman, A. Kandel, Numerical Solutions of fuzzy differential equations, Fuzzy Sets and Systems 105 (1999) 133-138.*

[27] *M. T. Mizukoshi, L. C. Barros, Y. Chalco-Cano, H. Romn-Flores, R. C. Bassanezi, Fuzzy differential equations and the extension principle, Information Sciences 177 (2007) 3627-3635*

[28] *Y. Nejatbakhsh, M. Khezerloo, Extrapolation Method for Fuzzy differential Inclusions, Int. J. Industrial Mathematics 3 (2009) 235-247*

[29] *G. Papaschinopoulos, G. Stefanidou, P. Efraimidi Existence, Uniqueness and asymptotic behavior of the solutions of a fuzzy differential equation with piecewise constant argument, Information Sciences 177 (2007) 3855-3870.*

[30] *M. L. Puri, D. A. Ralescu, Differentials of fuzzy functions, J. math. Analysis. Appl. 91 (1983) 552-558.*

[31] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319-330.

[32] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions I, Fuzzy Sets and Systems 120 (2001) 523-532.

[33] L. Zadeh, Toward a generalized theory of uncertainty (GTU) an outline, Information Sciences 172 (2005) 140.

[34] J. K. Zhou, Differential transformation and its applications for Electrical Circuits, Huarjung University press, (1986).

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