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Fuzzy Galerkin Method for Solving Fredholm Integral Equations with Error Analysis

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Abstract

In this paper, the classic Galergin method for solving integral equations of the second kind is extended to fuzzy Galerkin method. Moreover, the error analysis, particularly, error estimate, stability and convergence of the extended method are studied.

Keywords : Fredholm integral equation; Fuzzy Galerkin method; error analysis.

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1 Introduction

In this paper the classic Galerkin method for linear fuzzy Fredholm integral equations of the second kind is extended and the error estimate of this method is given. Also, convergence of the mentioned method is studied.

In [31], Zadeh introduced the concept of a fuzzy set. His pioneering work generated applications in a wide range of areas. In [16], Dubois and Prade considered a certain type of fuzzy-value function and defined the integral of such a function using the extension principle. Definitions in terms of the extension principle are usually cumbersome to use in applications. In [16], Dubois and Prade considered a restricted class of fuzzy-value functions and stated a more practical definition of the integral of this type of function, which they showed to be equivalent to the extension principle for this class of functions.

The fuzzy integral equations theory is well developed. In the existence of the solution of fuzzy integral equations, the Ascoli's theorem or metric fixed point theorems are used. For the existence and uniqueness, the main tool is the Banach fixed point principle. Such results can be found in [8, 9, 10, 14, 19, 20, 21, 23, 25, 26]

Numerical methods for fuzzy integral equations can be found in [1, 2, 6, 11, 13, 17, 18, 24, 29]. These methods use quadrature formulas (for linear fuzzy Fredholm integral equations, see [17, 18]) and Adomian decomposition (see [1]). For instance, in [17, 18] an iterative numerical method, using the trapezoidal quadrature rule for linear fuzzy

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Fredholm integral equations is given. In these two papers, the convergence of the method was proved, but without the error estimate. In the recent paper [13], the author developed a numerical method for nonlinear fuzzy Fredholm integral equations, based on Henstock integral of Lipschitzian fuzzy-number-valued fuctions. He also presented the error estimate of the method and the corresponding algorithm. Results on the fuzzy Henstock integral theory can be found in [3, 30] and for the metric spaces theory in fuzzy context see [15, 28].

In what follows, the results from [3, 30] (about fuzzy Henstock and Riemann integral) are used, obtaining a new numerical method for linear fuzzy Fredholm integral equations of the second kind. Moreover, the error estimate of the proposed method is presented and it's convergence is discussed. Finally, some numerical examples are given to show the applicability of fuzzy Galerkin method.

2 Preliminaries

The following definitions are needed:

Definition 2.1. *(see [3, 30]). Let* $\mu : R \to [0, 1]$ *with the following properties:*

(i) *is normal, i.e.,* $\exists x_0 \in R$*;* $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in R$, $\forall \lambda \in [0, 1]$ *(μ is called a convex fuzzy subset).*

(iii) *μ is upper semicontinuous on R, i.e.,* $\forall x_0 \in R$ *and* $\forall \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0)$: $\mu(x) \leq \mu(x_0) + \varepsilon$, $\forall x \in V(x_0)$.

(iv) *The set* $\overline{supp(\mu)}$ *is compact in R (where* $supp(\mu) := \{x \in R; \mu(x) > 0\}$*).*

We call μ a *fuzzy real number.* Denote the set of all μ with $R_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in R_{\mathcal{F}}$, for any $x_0 \in R$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 \leq r \leq 1$ and $\mu \in R_{\mathcal{F}}$ define $[\mu]^r := \{x \in R : \mu(x) \geq r\}$ and

$$[\mu]^0 := \overline{\{x \in R : \mu(x) > 0\}}.$$

Then it is well known that for each $r \in [0, 1]$, $[\mu]^r$ is a closed and bounded interval of R, ([3]). For $u, v \in R_{\mathcal{F}}$ and $\lambda \in R$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda[u]^r, \qquad \forall r \in [0, 1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of R) and $\lambda[u]^r$ means the usual product between a scalar and a subset of R (see, e.g., [30]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in R$, $\forall r \in [0, 1]$. For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define

$$D : R_{\mathcal{F}} \times R_{\mathcal{F}} \to R_+$$

by

$$\begin{aligned} D(u,v) : &= \sup_{r\in[0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\} \\ &= \sup_{r\in[0,1]} \text{Hausdorff distance}([u]^r, [v]^r), \end{aligned}$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in R_{\mathcal{F}}$. We have that D is a metric on $R_{\mathcal{F}}$. Then $(R_{\mathcal{F}}, D)$ is a complete metric space, see [3, 30], with the properties

(i) $D(u \oplus w, v \oplus w) = D(u,v), \quad \forall u, v, w \in R_{\mathcal{F}}$,

(ii) $D(k \odot u, k \odot v) = |k| D(u,v), \quad \forall u, v \in R_{\mathcal{F}},\ \forall k \in R$,

(iii) $D(u \oplus v, w \oplus e) \le D(u,w) + D(v,e), \quad \forall u, v, w, e \in R_{\mathcal{F}}$.

Let $f, g : R \to R_{\mathcal{F}}$ be *fuzzy number valued functions*. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x\in R} D(f(x), g(x)).$$

If we denote $\|u\|_{\mathcal{F}} := D(u, \widetilde{o})$, $\forall u \in R_{\mathcal{F}}$, then $\|.\|_{\mathcal{F}}$ has the properties of a usual norm on $R_{\mathcal{F}}$, i.e.,

$$\begin{gathered} \|u\|_{\mathcal{F}} = 0 \text{ iff } u = \widetilde{o},\ \|\lambda \odot u\|_{\mathcal{F}} = |\lambda|.\|u\|_{\mathcal{F}}, \\ \|u \oplus v\|_{\mathcal{F}} \le \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}},\ \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \le D(u,v). \end{gathered}$$

A subset $K \subseteq R_{\mathcal{F}}$ is called *fuzzy bounded*, iff $D(u,v) \le M$, $M > 0$, $\forall u, v \in K$.

Definition 2.2. *(see [30]). Let $x, y \in R_{\mathcal{F}}$. If there exists a $z \in R_{\mathcal{F}}$ such that $x = y \oplus z$, then we call z the H-difference of x and y, denoted by $z := x - y$.*

Definition 2.3. *([3]). Let $f : [a,b] \to R_{\mathcal{F}}$. We say that f is Fuzzy- Riemann integrable to $I \in R_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v]; \xi\}$ of $[a,b]$ with the norms $\triangle(P) < \delta$, we have*

$$D\left(\sum_P^{*} (v-u) \odot f(\xi), I\right) < \varepsilon,$$

where Σ^* denotes the fuzzy summation. We choose to write

$$I := (FR)\int_a^b f(x)dx.$$

We also call an f as above (FR)-integrable.

Theorem 2.1. *([3]). If $f, g : [c,d] \to R_{\mathcal{F}}$ are (FR)-integrable fuzzy functions, and α, β are real numbers, then*

$$(FR)\int_c^d (\alpha f(x) \oplus \beta g(x))dx = \alpha (FR)\int_c^d f(x)dx \oplus \beta (FR)\int_c^d g(x)dx.$$

3 Extension: Fuzzy Galerkin Method

This section deals with solving fuzzy integral equations using Galerkin method. To this end, Consider the fuzzy Fredholm integral equation

$$[x(t)]^{(r)} = \left[y(t) \oplus (FR) \int_D k(t,s) \odot x(s) ds \right]^{(r)}, \qquad t \in D. \tag{3.1}$$

To solve (3.1), a finite dimensional family of fuzzy functions is chosen, which is believed to contain a fuzzy function $[\widetilde{x}(s)]^{(r)}$ close to the true fuzzy solution $[x(s)]^{(r)}$.

The desired numerical fuzzy solution $[\widetilde{x}(s)]^{(r)}$ is selected by having it satisfy (3.1) fuzzy approximately. The most popular of these are fuzzy collocation and Galerkin methods, and we want to extend the classic Galerkin method to fuzzy Galerkin method. When this method is formulated in an abstract framework using fuzzy operator, it makes essential use of fuzzy projection operator. Since the error analysis and convergence are most easily carried out within such a fuzzy operator framework, the method is referred to as fuzzy projection method.

3.1 General theory

Let $r \in [0,1]$. Then it follows that

$$\begin{aligned}
\left[(FR) \int_D k(t,s) \odot x(s) ds \right]^r &= \left[\int_D (k(t,s) \odot x(s))_-^{(r)} ds, \int_D (k(t,s) \odot x(s))_+^{(r)} ds \right] \\
&= \left[\int_D k(t,s) x_-^{(r)}(s) ds, \int_D k(t,s) x_+^{(r)}(s) ds \right] \\
&= (FR) \int_D k(t,s) x^{(r)}(s) ds,
\end{aligned}$$

and

$$\left[\int_D k(t,s) x(s) ds \right]_\pm^r = \int_D k(t,s) x_\pm^{(r)}(s) ds.$$

Thus

$$x_\pm^{(r)}(t) = y_\pm^{(r)}(t) + \int_D k(t,s) x_\pm^{(r)}(s) ds,$$

and therefore

$$\left[x_-^{(r)}(t), x_+^{(r)}(t) \right] = \left[y_-^{(r)}(t) + \int_D k(t,s) x_-^{(r)}(s) ds, y_+^{(r)}(t) + \int_D k(t,s) x_+^{(r)}(s) ds \right].$$

So we get

$$\begin{aligned}
[x(t)]^{(r)} &= \left[x_-^{(r)}(t), x_+^{(r)}(t) \right] \\
&= \left[y_-^{(r)}(t) + \int_D k(t,s) x_-^{(r)}(s) ds, y_+^{(r)}(t) + \int_D k(t,s) x_+^{(r)}(s) ds \right] \\
&= y^{(r)}(t) + \left[(FR) \int_D k(t,s) \odot x(s) ds \right]^{(r)}, \ r \in [0,1] \\
&= \left[y(t) \oplus (FR) \int_D k(t,s) \odot x(s) ds \right]^{(r)}, \ r \in [0,1].
\end{aligned}$$

We write the fuzzy integral equation (3.1) in the operator form

$$(I-K)x^{(r)}=y^{(r)},$$

and the fuzzy operator K is assumed to be compact on a fuzzy Banach space $X_{\mathcal{F}}$ to $X_{\mathcal{F}}$. The most popular choices are $C_{\mathcal{F}}(D)$ and $L^2_{\mathcal{F}}(D)$.

In fact, this framework is needed to understand the behavior of fuzzy Galerkin method for solving fuzzy integral equation (3.1), including solving equations regarding convergence, numerical stability and error estimates.

In practice, we choose a sequence of finite dimensional fuzzy subspace $X_{\mathcal{F}} \subset X,\ n \geq 1$, with $X_{\mathcal{F}}$ having dimension d_n. Let $X_{\mathcal{F}}$ have a basis $\{\varphi_i\}_{i=1}^{d}$, with $d \equiv d_n$ for notational simplicity. We seek a fuzzy function $x_n^{(r)} \in X_{\mathcal{F}}$, and it can be written as

$$[x_n(t)]^{(r)}=\sum_{j=1}^{d*} c_j^{(r)} \odot [\varphi_j(t)]\,, \qquad t \in D, \tag{3.2}$$

where Σ^* denotes the fuzzy summation. This is substituted into (3.1), and the coefficient $\left\{c_i^{(r)}\right\}_{i=1}^{d}$ is determined by forcing the equation to be almost fuzzy exact in Galerkin sense. Now, we introduce fuzzy *residual* in the approximation of equation (3.1), when $[x(t)]^{(r)} \approx [x_n(t)]^{(r)}$.

Theorem 3.1. *Let $x^{(r)} \in C_F(D)$, then*

$$[x_n(t)]^{(r)}=\left[R_n(t) \oplus y(t) \oplus (FR)\int_D k(t,s)\odot x_n(s)ds\right]^{(r)}.$$

Proof: Let $r \in [0,1]$. We have $[x_n(t)]^{(r)}=\left[x_{n-}^{(r)}(t), x_{n+}^{(r)}(t)\right]$ and considering what was discussed before,

$$x_{n\pm}^{(r)}(t)=R_{n\pm}^{(r)}(t)-\int_D k(t,s)x_{n\pm}^{(r)}(s)ds-y^{(r)}(t),$$

so

$$\begin{aligned}
[x_n(t)]^{(r)} &= \left[x_{n-}^{(r)}(t), x_{n+}^{(r)}(t)\right] \\
&= \left[R_{n-}^{(r)}(t), R_{n+}^{(r)}(t)\right]-\left[\int_D k(t,s)x_{n-}^{(r)}(s)ds, \int_D k(t,s)x_{n+}^{(r)}(s)ds\right]-\left[y_-^{(r)}(t), y_+^{(r)}(t)\right] \\
&= [R_n(t)]^{(r)} \oplus \left[(FR)\int_D k(t,s)\odot x_n(s)ds\right]^{(r)}+[y(t)]^{(r)}.
\end{aligned}$$

Using H-difference, we have

$$[R_n(t)]^{(r)}=[x_n(t)]^{(r)}-\left[(FR)\int_D k(t,s)\odot x_n(s)ds\right]^{(r)}-[y(t)]^{(r)}.$$

or

$$[R_n]^{(r)}=(I-K)\odot[x_n]^{(r)}-y^{(r)}. \tag{3.3}$$

It is hoped and expected that the resulting fuzzy function $[x_n(t)]^{(r)}$ will be a good approximation of the true fuzzy solution $[x(t)]^{(r)}$. To do this via fuzzy Galerkin method, let

$X_F = L^2_F(D)$, and let $< .,. >_F$ denote the fuzzy inner product for X_F. Require $R_n^{(r)}$ to satisfy

$$< R_n^{(r)}, \varphi_i >_F = \widetilde{0}, \quad i = 1, 2, \cdots, d, \; r \in [0, 1], \tag{3.4}$$

where $< R_n^{(r)}, \varphi_i >_F = \left[(FR) \int_D R_n(t) \odot \varphi_i(t) dt\right]^{(r)}$. The left side is the Fourier coefficient of $R_n^{(r)}$ associated with φ_i. If $\{\varphi_i\}_{i=1}^d$ are the leading members of an orthonormal fuzzy family $\phi = \{\varphi_i\}_{i=1}^\infty$ that is complete in X_F, then (3.4) requires the leading terms to be zero in the Fourier expansion of $R_n^{(r)}$ with respect to ϕ.

To find $x_n^{(r)}$, apply (3.4) to (3.3), we have

$$\left\langle [R_n(t)]^{(r)}, \varphi_i(t) \right\rangle = \left\langle [x_n(t)]^{(r)} - (FR) \int_D k(t,s) \odot [x_n(s)]^{(r)} ds - [y(t)]^{(r)}, \varphi_i(t) \right\rangle = \widetilde{0},$$

This yields the fuzzy linear system

$$\sum_{j=1}^{d*} c_j^{(r)} \odot \{\langle \varphi_j, \varphi_i \rangle - \langle K\varphi_j, \varphi_i \rangle\} = \left\langle y^{(r)}, \varphi_i \right\rangle, \; i = 1, 2, \cdots, d, \tag{3.5}$$

or

$$\sum_{j=1}^{d*} c_j^{(r)} \odot \left\{ (FR) \int_D \varphi_j(t)\varphi_i(t) dt - (FR) \iint_D k(t,s)\varphi_i(t)\varphi_j(s) ds\, dt \right\}$$
$$= (FR) \int_D [y(t)]^{(r)} \odot \varphi_i(t) dt.$$

This is fuzzy Galerkin's method for obtaining an approximate solution to (3.1).

4 Error Estimation and Convergence

In this section, error analysis and convergence conditions of the fuzzy Galerkin method are discussed. We begin with the following definition.

Definition 4.1. *Let X_F and Y_F be fuzzy Banach spaces and let $L_F : X_F \to Y_F$ be an injective bounded fuzzy linear operator. Let $X_{F_n} \subset X_F$ and $Y_{F_n} \subset Y_F$ be two sequences of fuzzy subspaces with $dim X_{F_n} = dim Y_{F_n} = n$ and let $P_{F_n} : X_{F_n} \to Y_{F_n}$ be fuzzy projection operators. The fuzzy projection method, generated by X_{F_n} and P_{F_n}, approximates the fuzzy equation*

$$L_F x^{(r)} = y^{(r)}, \tag{4.6}$$

by the fuzzy projected equation

$$P_{F_n} L_F x_n^{(r)} = P_{F_n} y^{(r)}. \tag{4.7}$$

This fuzzy projection method is called fuzzy convergent for the fuzzy operator L_F if there exists an index N such that for each $y^{(r)} \in L_F(X)$ the fuzzy approximating equation $P_{F_n} L_F x_n^{(r)} = P_{F_n} y^{(r)}$ has a unique solution $x_n^{(r)} \in X_{F_n}$ for all $n \geq N$ and if this fuzzy solutions converge $x_n^{(r)} \to x^{(r)}$, $n \to \infty$, to the unique fuzzy solution $x^{(r)}$ of $L_F x^{(r)} = y^{(r)}$.

In terms of fuzzy operators, fuzzy convergence of the fuzzy projection method means that for all $n \geq N$ the finite dimensional fuzzy operators $L_{F_n} = P_{F_n} L_F : X_{F_n} \to Y_{F_n}$ are invertible and that fuzzy piontwise convergence $L_F^{-1} P_{F_n} L_F x^{(r)} = x^{(r)}$, $n \to \infty$, holds for all $x^{(r)} \in X_F$. In general, we can expect fuzzy convergence only if the fuzzy subspaces X_{F_n} possess the density property

$$\inf_{\widetilde{x}^{(r)} \in X_{F_n}} \|\widetilde{x}^{(r)} - x^{(r)}\|_F \to 0, \quad n \to \infty, \quad \forall x^{(r)} \in X_F, \ \widetilde{x}^{(r)} \in X_{F_n}, \tag{4.8}$$

for all $x^{(r)} \in X_F$, [3]. Therefore, in the subsequent analysis we will always assume that this condition is fulfilled.

Since $L_{F_n} = P_{F_n} L_F$ is a linear fuzzy operator between two finite dimensional fuzzy spaces, carrying out the fuzzy projection method is reduced to solving a finite dimensional fuzzy linear system. Here, we first proceed with a general convergence and error analysis.

Theorem 4.1. *The fuzzy projection method convergence if and only if there exist an index N and a positive constant M such that for all $n \geq N$ the finite dimensional fuzzy operators*

$$L_{F_n} = P_{F_n} L_F : X_{F_n} \to X_{F_n},$$

are invertible and the fuzzy operators $L_{F_n}^{-1} P_{F_n} L_F : X_{F_n} \to X_{F_n}$ are uniformly fuzzy bounded

$$\|L_{F_n}^{-1} P_{F_n} L_F\|_F \leq M. \tag{4.9}$$

There holds an error estimate

$$\|x_n^{(r)} - x^{(r)}\|_F \leq (1+M) \inf_{\widetilde{x}^{(r)} \in X_{F_n}} \|\widetilde{x}^{(r)} - x^{(r)}\|_F. \tag{4.10}$$

Proof: Provided the fuzzy projection method converges, the uniform fuzzy boundedness (4.9) is a consequence of the uniform fuzzy bounded principle, [3, 28]. Conversely, if the assumptions of the theorem are fulfilled we can write

$$x_n^{(r)} - x^{(r)} = (L_{F_n}^{-1} P_{F_n} L_F - I) x^{(r)}.$$

Since for all $\widetilde{x}^{(r)} \in X_{F_n}$, trivially, there holds $L_{F_n}^{-1} P_{F_n} L_F \widetilde{x}^{(r)} = \widetilde{x}^{(r)}$, we have

$$x_n^{(r)} - x^{(r)} = (L_{F_n}^{-1} P_{F_n} L_F - I)(x^{(r)} - \widetilde{x}^{(r)}).$$

Hence, we have the error estimate (4.10) and, with the aid of the density (4.8), the convergence follows.

We now state and extend the main stability property of the classic projection method for fuzzy projection method.

Theorem 4.2. *Assume that $S_F : X_F \to Y_F$ is a bounded fuzzy linear operator with a bounded fuzzy inverse $S_F^{-1} : Y_F \to X_F$ and that the fuzzy projection method is convergent for S_F. Let $L_F : X_F \to Y_F$ be a bounded fuzzy linear operator satisfying either*

(1) $\|L_F\|_F$ is sufficiently small or

(2) L_F is compact and $S_F - L_F$ is injective.

Then the fuzzy projection method also converges for $S_F - L_F$.

Proof: The fuzzy operator S_F satisfies the conditions of theorem (4.1), that is, the fuzzy operators $S_{F_n} = P_{F_n} S_F$ are invertible for all sufficiently large n and satisfy $\|S_{F_n}^{-1} P_{F_n} S_F\|_F \leq M$ with some constant M. Since S has a bounded fuzzy inverse, the pointwise fuzzy convergence $S_F^{-1} P_{F_n} S_F \to I, n \to \infty$, on X_{F_n} implies pointwise fuzzy convergence $S_F^{-1} P_{F_n} \to S_F^{-1}, n \to \infty$, on Y_{F_n}. We will show that for sufficiently large n, the inverse fuzzy operators of $I - S_F^{-1} P_{F_n} L_F : X_{F_n} \to X_{F_n}$ exist and are uniformly fuzzy bounded if (1) or (2) is satisfied.

(1) We apply the uniform fuzzy boundedness principle to the pointwise fuzzy convergent sequence $(S_{F_n}^{-1} P_{F_n})$. Then the inverse fuzzy operators $(I - S_F^{-1} P_{F_n} L_F)^{-1}$ exist and are uniformly fuzzy bounded for all sufficiently large n, provided L_F satisfies $\sup_{n \in N} \|S_{F_n}^{-1} P_{F_n}\|_F \|L_F\|_F < 1$.

(2) Since $S_F^{-1} L_F$ is compact, by the Riesz fuzzy theory, $I - S^{-1}{}_F L_F : X_F \to X_F$ has a bounded fuzzy inverse. From the pointwise fuzzy convergence of the sequence $(S_{F_n}^{-1} P_{F_n})$ and the fuzzy compactness of L_F, by theorem (4.1), we derive norm fuzzy convergence $\|S_F^{-1} L_F - S_{F_n}^{-1} P_{F_n} L_F\|_F \to 0,\ n \to \infty$. Therefore the inverse fuzzy operators $(I - S_{F_n}^{-1} P_{F_n} L_F)^{-1}$ exist and are uniformly fuzzy bounded for all sufficiently large n.

Note that $(I - S_{F_n}^{-1} P_{F_n} L_F)^{-1}$ maps X_{F_n} into itself. We abbreviate $\widetilde{S_F} = S_F - L_F$ and $\widetilde{S_{F_n}} = P_{F_n} \widetilde{S_F}$. Then $\widetilde{S_{F_n}} = S_{F_n}(I - S_{F_n}^{-1} P_{F_n} L_F) : X_{F_n} \to Y_{F_n}$ is fuzzy invertible for sufficiently large n with the inverse given by

$$\widetilde{S_{F_n}}^{-1} = (I - S_{F_n}^{-1} P_{F_n} L_F)^{-1} S_{F_n}^{-1}.$$

From $\widetilde{S_{F_n}}^{-1} P_{F_n} \widetilde{S_F} = (I - S_{F_n}^{-1} P_{F_n} L_F)^{-1} S_{F_n}^{-1} P_{F_n} S_F (I - S_F^{-1} L_F)$ we estimate

$$\|\widetilde{S_{F_n}}^{-1} P_{F_n} \widetilde{S_F}\|_F \leq \|(I - S_{F_n}^{-1} P_{F_n} L_F)^{-1}\|_F \|I - S_F^{-1} L_F\|_F M,$$

and observe that the condition (4.9) is satisfied for $\widetilde{S_F}$.

For an fuzzy equation of the second kind

$$x^{(r)} - L_F x^{(r)} = y^{(r)}, \tag{4.11}$$

with a bounded fuzzy linear operator $L_F : X_F \to X_F$ we need only a sequence of fuzzy subspaces $X_n \subset X$ and fuzzy projection operators $P_{F_n} : X_F \to X_{F_n}$. Then the fuzzy projection method assumes the form

$$x_n^{(r)} - P_{F_n} L_F x_n^{(r)} = P_{F_n} y^{(r)}. \tag{4.12}$$

Note that each solution $x_n^{(r)} \in X_F$ to (4.12) automatically belongs to X_{F_n}. When L is fuzzy compact, from theorem (4.2) we have the following convergence property.

Corollary 4.1. *Let $L_F : X_F \to X_F$ be fuzzy compact, $I - L_F$ be fuzzy injective and let the fuzzy projections $P_{F_n} : X_F \to X_F$ converge pointwise $P_{F_n} x^{(r)} \to x^{(r)},\ n \to \infty$, for all $x^{(r)} \in X_F$. Then the fuzzy projection method for $I - L_F$ converges.*

Proof: Apply the second part of theorem (4.2) for $S_F = I$ and identify $X_F = Y_F$ and $X_{F_n} = Y_{F_n}$.

5 Examples

In this section, using two examples, the proposed fuzzy Galerkin method for solving fuzzy Fredholm integral equations of the second kind is illustrated.

Example 5.1. *Suppose*

$$x^{(r)}(t) = y^{(r)}(t) + \int_{-1}^{1} stx^{(r)}(s)\,ds, \tag{5.13}$$

where

$$y^{(r)}(t) = [r^2 + r, 4 - r^3 - r]t, \qquad r \in [0, 1].$$

We choose here three linearly independent functions $\phi_1(t) = 1$, $\phi_2(t) = t$ *and* $\phi_3(t) = t^2$. *So the approximate solution from (3.2) is*

$$x_3^{(r)}(t) = c_1^{(r)} + c_2^{(r)}t + c_3^{(r)}t^2. \tag{5.14}$$

If we substitute (5.14) in (3.3), then we have

$$R_2^{(r)}(t) = c_1^{(r)} + c_2^{(r)}t + c_3^{(r)}t^2 - y^{(r)}(t) - \int_{-1}^{1} st\left(c_1^{(r)} + c_2^{(r)}s + c_3^{(r)}s^2\right) ds.$$

Using (3.3), then fuzzy Galerkin method gives

$$\left\langle R_2^{(r)}(t), \phi_k(t)\right\rangle = \tilde{0}, \quad k = 1, 2, 3,$$

which is equivalent to the following fuzzy linear system of equations

$$\int_{-1}^{1}\left[c_1^{(r)} + c_2^{(r)}t + c_3^{(r)}t^2 - \int_{-1}^{1} st\left(c_1^{(r)} + c_2^{(r)}s + c_3^{(r)}s^2\right) ds\right] dt = \int_{-1}^{1} y^{(r)}(t)\,dt,$$

$$\int_{-1}^{1} t\left[c_1^{(r)} + c_2^{(r)}t + c_3^{(r)}t^2 - \int_{-1}^{1} st\left(c_1^{(r)} + c_2^{(r)}s + c_3^{(r)}s^2\right) ds\right] dt = \int_{-1}^{1} ty^{(r)}(t)\,dt,$$

$$\int_{-1}^{1} t^2\left[c_1^{(r)} + c_2^{(r)}t + c_3^{(r)}t^2 - \int_{-1}^{1} st\left(c_1^{(r)} + c_2^{(r)}s + c_3^{(r)}s^2\right) ds\right] dt = \int_{-1}^{1} t^2y^{(r)}(t)\,dt.$$

It is clear that

$$\int_{-1}^{1} s(c_1^{(r)} + c_2^{(r)}s + c_3^{(r)}s^2)ds = \frac{2}{3}c_2^{(r)}.$$

Therefore, we have

$$\int_{-1}^{1}\left(c_1^{(r)} + \frac{1}{3}c_2^{(r)}t + c_3^{(r)}t^2\right) dt = \int_{-1}^{1} y^{(r)}(t)dt,$$

$$\int_{-1}^{1} t\left(c_1^{(r)} + \frac{1}{3}c_2^{(r)}t + c_3^{(r)}t^2\right) dt = \int_{-1}^{1} ty^{(r)}(t)dt,$$

$$\int_{-1}^{1} t^2\left(c_1^{(r)} + \frac{1}{3}c_2^{(r)}t + c_3^{(r)}t^2\right) dt = \int_{-1}^{1} t^2y^{(r)}(t)dt.$$

Since $y^{(r)}(t) = [r^2 + r, 4 - r^3 - r]t$, *then*

$$\begin{aligned} 2c_1^{(r)} + \quad & \frac{2}{3}c_3^{(r)} = 0, \\ \frac{2}{9}c_2^{(r)} \quad & = \frac{2}{3}[r^2 - r, 4 - r^3 - r], \\ \frac{2}{3}c_1^{(r)} + \quad & \frac{2}{5}c_3^{(r)} = 0. \end{aligned}$$

The solution is $c_1^{(r)} = c_2^{(r)} = 0$ *and* $c_2^{(r)} = 3[r^2 + r, 4 - r^3 - r]$. *Thus, the fuzzy solution of (5.13) is* $x_2^{(r)}(t) = 3t[r^2 + r, 4 - r^3 - r]$. *It can be verified that the exact fuzzy solution of the given fuzzy integral equation (5.13) is* $x^{(r)}(t) = x_2^{(r)}(t)$. *It should be noted, however, that such agreement is not possible when we consider another basis. Furthermore, if we let* $r = 1$, *the exact solution of (5.13) is again* $x(t) = x_2(t) = 6t$.

Example 5.2. *Choosing suitable basis is very important. If we consider the previous example again, but set* $\phi_1(t) = 1$, $\phi_2(t) = \sin t$ *and* $\phi_3(t) = \cos t$, *then repeating the computations with these functions as basis gives*

$$x_2^{(r)}(t) = 3.2997 \sin(t)[r^2 + r, 4 - r^3 - r].$$

Example 5.3. *Consider*

$$x^{(r)}(t) = y^{(r)}(t) + \int_0^{2\pi} \frac{\sin(t)\sin(0.5s)}{10} x^{(r)}(s)\, ds, \tag{5.15}$$

where

$$y^{(r)}(t) = \sin(0.5t)\left[\frac{2(r^2 + r) + 13(4 - r^3 - r)}{15}, \frac{13(r^2 + r) + 2(4 - r^3 - r}{15}\right].$$

The exact fuzzy solution of (5.15) is

$$x^{(r)}(t) = \sin(0.5t)\left[r^2 + r, 4 - r^3 - r\right].$$

Here the suitable choices of basis functions are $\phi_1(t) = 1$, $\phi_2(t) = \sin(0.5t)$ *and* $\phi_3(t) = \cos(0.5t)$. *So, the fuzzy approximate solution has the following form*

$$x_2^{(r)}(t) = c_1^{(r)} + c_2^{(r)} \sin(0.5t) + c_3^{(r)} \cos(0.5t). \tag{5.16}$$

If we substitute (5.16) in (3.3), then we have

$$\begin{aligned} R_2^{(r)}(t) \quad &= c_1^{(r)} + c_2^{(r)} \sin(0.5t) + c_3^{(r)} \cos(0.5t) - y^{(r)}(t) \\ &- \int_0^{2\pi} \tfrac{\sin(t)\sin(0.5s)}{10}\left(c_1^{(r)} + c_2^{(r)} \sin(0.5s) + c_3^{(r)} \cos(0.5s)\right) ds. \end{aligned}$$

Using (3.3), then fuzzy Galerkin method gives

$$\left\langle R_2^{(r)}(t), \phi_k(t)\right\rangle = \tilde{0}, \quad k = 1, 2, 3,$$

which is equivalent to the following fuzzy linear system of equations

$$\int_0^{2\pi} 1.\left[x_2^{(r)}(t) - \int_0^{2\pi} \frac{\sin(t)\sin(0.5s)}{10} x_2^{(r)}(s)\,ds\right] dt = \int_0^{2\pi} 1.y^{(r)}(t)\,dt,$$

$$\int_0^{2\pi} \sin(0.5t)\left[x_2^{(r)}(t) - \int_0^{2\pi} \frac{\sin(t)\sin(0.5s)}{10} x_2^{(r)}(s)\,ds\right] dt = \int_0^{2\pi} \sin(0.5t)y^{(r)}(t)\,dt,$$

$$\int_0^{2\pi} \cos(0.5t)\left[x_2^{(r)}(t) - \int_0^{2\pi} \frac{\sin(t)\sin(0.5s)}{10} x_2^{(r)}(s)\,ds\right] dt = \int_0^{2\pi} \cos(0.5t)y^{(r)}(t)\,dt.$$

Solving this system, gives $c_1^{(r)} = c_3^{(r)} = 0$ *and* $c_2^{(r)} = [r^2 + r, 4 - r^3 - r]$. *Therefore* $x_2^{(r)}(t) = \sin(0.5t)c_2^{(r)} = \sin(0.5t)[r^2 + r, 4 - r^3 - r]$ *which is the exact fuzzy solution of the given integral equation.*

6 Conclusion

In this paper, the classic Galergin method for solving integral equations of the second kind was improved to fuzzy Galerkin method. Also, the error analysis, namely, error estimate, stability and convergence of the extended method were discussed and some results were established. To support the applicability of the proposed method, some examples were carried out.

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