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Sensitivity and Stability Radius in Data Envelopment Analysis

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Abstract

One important issue in DEA which has been studied by many DEA researchers is the sensitivity analysis of a specific DMU_o, the unit under evaluation.
Moreover, we know that in most models of DEA, the best DMUs have the efficiency score of unity. In some realistic situations, the performance of some inefficient DMUs is similar to that of efficient ones.
In this paper, we define a new efficiency category, namely '$quasi-efficient$'. Then, we develop a procedure for performing a sensitivity analysis of the efficient and quasi-efficient decision making units.
The procedure yields an exact '$stability\ radius$' within which data variations will not alter a DMU's classification from efficient or quasi-efficient to inefficient status (or vice versa).
Keywords : Data Envelopment Analysis, Sensitivity, Stability, Efficiency, Quasi-efficient.

1 Introduction

Data envelopment analysis (DEA), which was introduced by Charnes et al.[6] (CCR) and extended by Banker et al.[3] (BCC), is a useful method to evaluate the relative efficiency of multiple-input and multiple-output units based on observed data.
The sensitivity analysis has received much attention in recent years from researches, and so many researches have been carried out in this regard. Sensitivity analysis in DEA has been deliberated on from various points of view.
The first DEA sensitivity analysis paper by Charnes et al.[5] examined change in a single output. This was followed by a sensitivity analysis article by Charnes and Neralic [8] in which sufficient conditions for preserving efficiency are determined. Another type of DEA sensitivity analysis is based on the super-efficiency DEA approach in which the DMU under evaluation is not included in the reference set[1, 15]. Charnes et al.[7, 9]

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developed a super-efficiency DEA sensitivity analysis technique for the situation where simultaneous proportional change is assumed in all inputs and outputs for a specific DMU under consideration. This data variation condition is relaxed in Zhu [16] and Seiford[15] to a situation where inputs or outputs can be changed individually and the largest stability region that encompasses that of Charnes et al.[7] is obtained.
Especially, some valuable researches have been on sensitivity analysis of extreme efficient units that lead to reaching the stability regions of these units. The first attempt was made to reach the input and output stability region for extreme efficient units by Seiford and Zhu[14]. These regions are those within which variations of inputs or outputs cause no change in the DMU class. In other words, after any kind of interior variation, the extreme efficient unit under evaluation remains efficient. Jahanshahloo et al.[13] proposed a method that requires a less complex computational process and overcomes some difficulties in the previous method.
The DEA sensitivity analysis methods we have just reviewed are all developed for the situation where data variations are only applied to the efficient DMU under evaluation and the data for the remaining DMUs are assumed fixed.
In this paper, we develop a procedure for performing a sensitivity analysis of the efficient and quasi-efficient DMUs.
The procedure yields an exact '*stability radius*' within which data variations will not alter a DMU's classification from efficient or quasi-efficient to inefficient status (or vice versa).
The current paper proceeds as follows. Section 2 discusses the basic DEA models. Section 3 develops our proposed method for finding the '*stability radius*', and section 4 provides a numerical example. Finally, conclusions are given in section 5.

2 DEA Background

Data Envelopment Analysis (DEA) is a technique that has been used widely in the supply chain management literature. This non-parametric, multi-factor approach enhances our ability to capture the multi-dimensionality of performance discussed earlier. More formally, DEA is a mathematical programming technique for measuring the relative efficiency of decision making units (DMUs), where each DMU has a set of inputs used to produce a set of outputs [2].
Consider $DMU_j, (j = 1, ..., n)$, where each DMU consumes m inputs to produce s outputs. Suppose the observed input and output vectors of DMU_j are $X_j = (x_{1j}, ..., x_{mj})$ and $Y_j = (y_{1j}, ..., y_{sj})$, respectively, and let $X_j \geq 0$, $X_j \neq 0$, $Y_j \geq 0$, and $Y_j \neq 0$.
The production possibility set T_c is defined as:

$$T_c = \Big\{(X, Y) \mid \ X \geq \sum_{j=1}^{n} \lambda_j X_j, \ Y \leq \sum_{j=1}^{n} \lambda_j Y_j \ , \ \lambda_j \geq 0, \quad j = 1, \ldots, n\Big\}.$$

By the stated definition, the CCR model is as follows:

$$\begin{aligned} Min \quad & \theta \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, && i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, && r = 1, \ldots, s \\ & \lambda_j \geq 0, && j = 1, \ldots, n \end{aligned} \tag{2.1}$$

Moreover, the production possibility set T_v is defined as:

$$T_v = \Big\{(X,Y) \mid \; X \geq \sum_{j=1}^{n} \lambda_j X_j, \; Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \; \sum_{j=1}^{n} \lambda_j = 1, \; \lambda_j \geq 0, \quad j = 1, \ldots, n\Big\}.$$

By the above definition, the BCC model is:

$$\begin{array}{lll} Min & \theta & \\ s.t. & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, & i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, & r = 1, \ldots, s \\ & \sum_{j=1}^{n} \lambda_j = 1 & \\ & \lambda_j \geq 0, & j = 1, \ldots, n \end{array} \tag{2.2}$$

Furthermore, the multiplier form of the BCC model is as follows:

BCC model

$$\begin{array}{lll} Max & \sum_{r=1}^{s} u_r y_{ro} + u_o & \\ s.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leq 0 & , j = 1, \cdots, n \\ & \sum_{i=1}^{m} v_i x_{io} = 1 & \\ & v_i \geq 0 & , i = 1, ..., m \\ & u_r \geq 0 & , r = 1, ..., s \end{array} \tag{2.3}$$

In what follows, two sensitivity analysis models for efficient and inefficient units are reconsidered as a reminder. Having identified efficient and inefficient DMUs in a DEA analysis, one may want to know how sensitive these identifications are, for possible variations in the data. The basic idea is to use concepts such as "distance" or "norm" (= length of a vector), as defined in the mathematical literature dealing with metric spaces, and use these concepts to determine "radii of stability", within which data variations will not alter a DMU's classification from efficient to inefficient status (or vice versa) [10].
A new avenue for sensitivity analysis was opened by charnes et al [9]. The proposed model for finding the '*stability radius*' of efficient DMUs is as follows:

$$\begin{array}{lll} Min & \delta & \\ s.t. & \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} + s_i^- - \delta = x_{io} & , i = 1, \cdots, m \\ & \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} - s_r^+ + \delta = y_{ro} & , r = 1, \cdots, s \\ & \sum_{j=1, j\neq o}^{n} \lambda_j = 1 & \end{array} \tag{2.4}$$

Here, all variables are constrained to be non-negative.
Moreover, δ^* represents the '*stability radius*' within which data variations will not alter

a DMU's classification from efficient to inefficient status (or vice versa).

Similarly the model proposed for finding the '*stability radius*' of inefficient DMUs by Charnes et al. [7] is as follows:

$$\begin{aligned} &Max \quad \delta \\ &s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- + \delta = x_{io} \qquad , i = 1, \cdots, m \\ &\qquad \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ - \delta = y_{ro} \qquad , r = 1, \cdots, s \\ &\qquad \sum_{j=1}^{n} \lambda_j = 1 \end{aligned} \tag{2.5}$$

Here, again, all variables are constrained to be non-negative.
The above formulations pertain to an inefficient DMU, which continues to be inefficient for all data alterations which yield improvements from x_{io} to $x_{io} - \delta^*$ and from y_{ro} to $y_{ro} + \delta^*$.
This means that no reclassification to efficient status will occur within the open set defined by the value of $\delta^* > 0$,[10].

3 Proposed Model

In some realistic situations, we know the performance of some inefficient DMUs is similar to that of efficient DMUs. This similarity leads us to suggesting a new definition.
For this purpose, first consider the following data set. The data are summarized in Table 1 and are illustrated in Figure 1:

Table 1
Data

DMUs	A	B	C	D	E	F	G	H	I	J	K	L	M	N
Input	1	2	4	7	1.5	2.1	4	6	7	1.8	6.5	4	5	7
Output	1.5	3	5	7	2	2.7	4.5	6	6.5	2.2	6	2	3.5	4.5

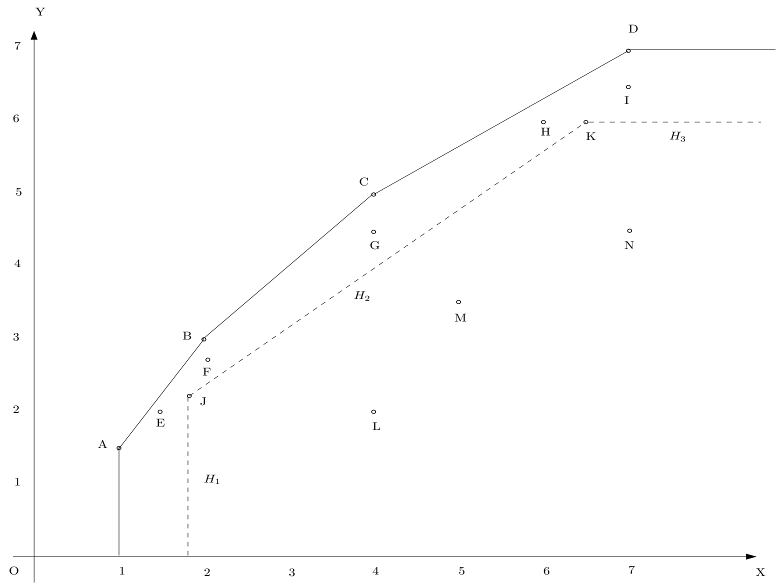

Fig. 1. Data set in T_v and the new frontier

Fig. 1 exhibits 14 DMUs, each with one input and one output. By evaluating these DMUs by model (2), we find out that DMUs A,B,C and D are efficient. In addition, the performance of units E,F,G,H,I,J,K is similar to that of the efficient DMUs. So, we introduce a new definition. To do so, we focus on the inefficient DMUs. We assume DMU_o is an arbitrary inefficient DMU and we sort out the inefficient DMUs according to efficiency scores as follows:

(a). If $\theta_o^* \leq \alpha$ (α is the efficiency score determined by the conditions of the situation), DMU_o is called completely inefficient.

(b). If the efficiency score of DMU_o is close to 1($\theta_o^* \rightarrow 1$), DMU_o is called quasi-efficient.

Since the focus is on the stability of the classification of DMUs into efficient and inefficient performers, efficient and quasi-efficient DMUs are considered to belong to the same class. The basic idea is to use concepts such as '*distance*' or '*norm*' (=length of a vector), and to employ these concepts to determine the '*stability radius*' within which data variations will not alter a DMU's classification from efficient or quasi-efficient to completely inefficient status (or vice versa).

In order to help our development, we classify the set of n DMUs into three classes:

(i) class Ω_1 contains all efficient DMUs;

(ii) class Ω_2 contains quasi-efficient DMUs ($\theta_o^* \rightarrow 1$);

(iii) class Ω_3 contains completely inefficient DMUs ($\theta_o^* \leq \alpha$).

In what follows, we apply the following procedure:

Step 1: Use model (2.2) to determine all efficient, quasi-efficient and completely inefficient DMUs.

Next, remove efficient DMUs.

Step 2: Solve model (2.2) for the remaining DMUs. If the efficiency scores of all quasi-efficient DMUs equal one, then stop, and go to step (4). Otherwise, exclude the quasi-efficient DMUs whose efficiency scores become one, and go to step (3).

Step 3: Solve model (2.2) again for the remaining DMUs. If the efficiency scores of all remaining quasi-efficient DMUs equal one, then stop, and go to step (4). Otherwise, omit

the quasi-efficient DMUs whose efficiency scores equal one, and repeat step (3).
Step 4: Determine the quasi-efficient DMUs whose efficiency scores equalled one in the previous steps (whether it be step 2 or step 3).
Let $\Omega_4 = \{DMU_1, DMU_2, ..., DMU_h\}$ be the set of these DMUs. By applying the BCC multiplier model to the members of Ω_4, the new frontier is constructed.
Step5: Let

$$\begin{cases} \Omega = (\Omega_1 \bigcup \Omega_2)/\Omega_4 = \{DMU_1, DMU_2, ..., DMU_l\} \\ \Omega' = (\Omega_3 \bigcup \Omega_4) = \{DMU_{j1}, DMU_{j2}, ..., DMU_{je}\} \end{cases}$$

Next, add each member of Ω to Ω' one by one, which is done as follows:
$\Gamma_1 = \{DMU_{j1}, DMU_{j2}, ..., DMU_{je}, DMU_1\}$
$\Gamma_2 = \{DMU_{j1}, DMU_{j2}, ..., DMU_{je}, DMU_2\}$
...
$\Gamma_l = \{DMU_{j1}, DMU_{j2}, ..., DMU_{je}, DMU_l\}$
Then, we use model (2.4) for Γ_i, for each $i \in \{1, 2, ..., l\}$, and we obtain the stability radii for $DMU_1, DMU_2, ..., DMU_l$ $(\delta_1^*, \delta_2^*, ..., \delta_l^*)$.

4 Example

Recall the above mentioned example. First, we apply model (2.2).
The results are summarized in Table 2:

Table 2
Results of step 1.

DMUs	A	B	C	D	E	F	G	H	I	J	K	L	M	N
θ^*	1	1	1	1	0.889	0.857	0.875	0.917	0.893	0.815	0.846	0.334	0.500	0.500

By assuming $\alpha = 0.7$, it can be seen that units A,B,C,D are efficient and units E,F,G,H,I,J,K are quasi-efficient. Moreover, L,M,N are completely inefficient. So we set:
$\Omega_1 = \{A, B, C, D\}$.
$\Omega_2 = \{E, F, G, H, I, J, K\}$.
$\Omega_3 = \{L, M, N\}$.
Next, we remove A,B,C,D (efficient DMUs) and use model (2.2) for the remaining DMUs.
The results are summarized in Table 3:

Table 3
Results of step 2.

DMUs	E	F	G	H	I	J	K	L	M	N
θ^*	1	1	1	1	1	0.929	0.923	0.375	0.589	0.571

Then, we omit E,F,G,H,I and apply model (2.2) for the remaining DMUs. The results are summarized in Table 4:

Table 4
Results of step 3.

DMUs	J	K	L	M	N
θ^*	1	1	0.450	0.681	0.663

Let $\Omega_4 = \{J, K\}$.
The BCC multiplier model can be applied for to members of Ω_4 and the supporting hyperplanes are found in step (4). These hyperplanes are as follows:
$H_1 = \{(x, y)|x = 1.8\}$
$H_2 = \{(x, y)|y - 0.808x - 0.75 = 0\}$
$H_3 = \{(x, y)|y = 6\}$
Finally, let $\Omega = (\Omega_1 \bigcup \Omega_2)/\Omega_4 = \{A, B, C, D, E, F, G, H, I\}$ and $\Omega' = \Omega_3 \bigcup \Omega_4 = \{L, M, N, J, K\}$.
Next, we add each member of Ω to Ω' one by one as follows:
$\Gamma_1 = \{L, M, N, J, K, A\}$
$\Gamma_2 = \{L, M, N, J, K, B\}$
...
$\Gamma_9 = \{L, M, N, J, K, I\}$
Then, we use model (2.4) for Γ_i, for each $i \in \{1, 2, ..., 9\}$, and obtain the stability radii for A,B,C,D,E,F,G,H,I ($\delta^*_A, \delta^*_B, ..., \delta^*_I$).
The results are summarized in Table 5:

Table 5
Results of step 5.

DMUs	A	B	C	D	E	F	G	H	I
δ^*	0.8000	0.3529	0.5647	1.0000	0.3000	0.1424	0.2882	0.2235	0.5000

5 Conclusions

One research issue which has received widespread attention in the rapidly growing field of DEA is the sensitivity of the results of an analysis to perturbations in the data.
In some real situations, the performance of some inefficient DMUs is similar to efficient ones.
In such cases, as we know, inefficient units are divided into two categories. The inefficient units whose performance is completely poor and those whose performance is closer to efficient units and whose efficiency scores are closer to one. So, it is necessary to distinguish between these units and completely inefficient units. A new definition is hence presented in this paper for these units and they are termed '*quasi − efficient*' units. Furthermore, efficient and '*quasi − efficient*' units are considered to belong in the efficient category (In this category, the score of all units is not necessarily one) and the sensitivity analysis of these units in relation to completely inefficient units is discussed.
The procedure results in an exact '*stability radius*' within which data variations will not alter a *DMU*'s classification from efficient or quasi-efficient to completely inefficient status (or vice versa).

References

[1] Andersen, P., Petersen, N.C., 1993. A pdrocedure for ranking efficient units in data envelopment analysis. Management Science 39 (10), 1261-1264.

[2] Anthony Ross, Kathryn Ernstberger, 2006. Benchmarking the IT Productivity Paradox: Recent evidence from the manufacturing sector. Mathematical and Computer Modelling 44, 30-42.

[3] Banker, R.D., Charnes, A., Cooper, W.W., 1984. Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management Science 30 (9), 1078-1092.

[4] Charnes, A., Cooper, W.W., Golany, B., Seiford, L. Stutz, J.,1985b. Foundation of data envelopment analysis for Pareto-Koopmans efficient empirical production functions. Journal of Econometrics 30, 91-107.

[5] Charnes, A., Cooper W.W. Lewin A.Y. Morey R.C. and Rousseau J.(1985). Sensitivity and stability analysis in DEA. *Ann Opns Res* 2: 139-156.

[6] Charnes, A., Cooper, W.W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European Journal of Operational Research 2, 429-444.

[7] Charnes, A., Haag S, Jaska, P., and Semple, J. (1992). Sensitivity of efficiency classifications in the additive model of data envelopment analysis. *Int J Syst Sci* 23: 789-798.

[8] Charnes, A., and Neralic, L. (1990).Sensitivity analysis of the additive model in data envelopment analysis. European Journal of Operational Research 48: 332-341

[9] Charnes, A., Rousseau, J., and Semple, J. (1996). Sensitivity and stability of efficiency classifications in data envelopment analysis.Journal of Productivity Analysis 7: 5-18.

[10] Cooper, W.W., Shanling Li, Seiford,L.M.,Tone,K.,Thrall,R.M,Zhu,J., 2001. Sensitivity and Stability Analysis in DEA: Some Recent Developments. Journal of Productivity Analysis 15,217-246.

[11] Cooper, W.W., Seiford, L., Tone, K., 2002. Data envelopment analysis a comprehensive text with Models applications, references and DEA solver software. Third Printing. Kluwer Academic Publishers.

[12] Fare, R., Grosskopf, S., Lovell, C.A.K., 1994. Production frontiers, Cambridge University Press, New York.

[13] Jahanshahloo, G.R., Hosseinzadeh Lotfi,F., Shoja,N., Sanei,M.,Tohidi,G.,2005. Sensitivity and Stability analysis in DEA. Applied Mathematics and Computation,169, 897-904.

[14] Seiford, Lawrence M.,Zhu,J.,1998. Stability regions for maintaining efficiency in data envelopment analysis. European Journal of Operational Research 108, 127-139.

[15] Seiford, Lawrence M., Zhu, J.,1999. Infeasibility of super efficiency data envelopment analysis models. INFOR 37:174-187

[16] Zhu, J. (1996). Robustness of the efficient DMUs in data envelopment analysis. European Journal of Operational Research9: 451-460