Science and Research Branch (IAU)

Numerical Solution of Volterra-Fredholm Integral Equations with The Help of Inverse and Direct Discrete Fuzzy Transforms and Collocation Technique

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Abstract

In this paper, a new approach to the numerical solution of Volterra- Fredholm integral equations by using expansion method based on the composition of the inverse and direct discrete fuzzy transforms (shortly F-transforms) in combination with the collocation technique is proposed. First, the unknown function is approximated by using the composition of the inverse and direct discrete F-transforms based on the fuzzy partition, then the Volterra- Fredholm integral equation is reduced to the linear system of equations. Moreover, the convergence theorem for the proposed method is given in terms of the modulus of continuity. Finally, illustrative examples are included to show the accuracy and the efficiency of the proposed method.

Keywords : Volterra-Fredholm integral equation; Basic function; Fuzzy transforms.

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1 Introduction

For solving Volterra-Fredholm integral equations, many methods with enough accuracy and efficiency have been used before by many researches [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Maleknejad and Fadaei Yami [11] solved the system of Volterra-Fredholm integral equations by Adomian decomposition method. Maleknejad and Hadizadeh [12] proposed a new computational method for this kind of equation. Kauthen in [10], used continuous time collocation method for Volterra-Fredholm integral equations. In [24], Yalsinbas developed numerical solution of nonlinear Volterra-Fredholm integral equations by using Taylor polynomials. Legendre wavelets also were applied for solving Volterra-Fredholm integral equations [25]. Maleknejad and Mahmodi in [13] applied Taylor polynomials for solving

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high-order Volterra-Fredholm integro-differential equations.

In classic mathematics, various kinds of transforms are used as powerful tools in the construction of approximation methods and especially for computing the numerical solution of integral equations. F-transform was proposed by Perfilieva and Chaldeeva [18] and studied in several papers [16, 17, 18, 20, 21, 22, 23]. Stepnikca and Valasek [23] have applied F-transforms to solve partial differential equations numerically. Recently, Ezzati and Mokhtari proposed a new approach to numerical solution of Fredholm integral equations of the second kind by using F-transform. Solving differential equations through the data analysis by the fuzzy approach is done by Martino, Loia and Seaaa [4]. They constructed a basic function corresponding to the given function in order to approximate it in the first step. Approximation properties of the F-transforms were studied in [2, 16, 20]. The success of these applications is due to the fact that F-transforms are capable to accurately approximate any continuous function.

We decide to show, how this technique can be used to obtain numerical solution of Volterra-Fredholm integral equations. In the structure of Volterra-Fredholm integral equation, we use inversion form of F-transform instead of precise representation of the original function. Then we form the linear system of equations. So, we can find its unknowns by this process. Here, we will apply the F-transform to approximate the unknown functions $f(x)$, which we have let as the solution of Volterra-Fredholm integral equations as follows:

$$f(x) = g(x) + \int_a^x k_1(x,t) f(t) \, dt + \int_a^b k_2(x,t) f(t) \, dt \, , \qquad x \in [a,b] \, . \qquad (1.1)$$

Some basic definitions for F-transforms are presented subsequently. Section 3 investigates the mathematical formulation of proposed method. In Section 4, we present the error analysis of proposed method. In Section 5, some numerical results are provided. Finally, Section 6 gives our concluding remarks.

2 Preliminaries

Consider a Volterra-Fredholm integral equation of the second kind defined in Eq. (1.1), in which the function $g(x)$ and kernel $k(x,t)$ are given and the function $f(x)$ is unknown. We take an interval $[a,b]$ as a universe. The fuzzy partition of the universe is given by fuzzy subsets of the universe $[a_i, b_i]$ determined by their membership function which must have the properties described in the following definition.

Definition 2.1. *[16] Let $x_1 < \ldots < x_n$ be fixed nodes within $[a,b]$ such that $x_1 = a, x_n = b$ and $n \geq 2$. We say that fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions $A_1(x), \ldots, A_n(x)$ defined on $[a,b]$, form a fuzzy partition of $[a,b]$ if they fulfill the following conditions for $k = 1, \ldots, n$:*

(a) $A_k : [a,b] \longrightarrow [0,1]$, $A_k(x_k) = 1$;

(b) $A_k(x) = 0$ *if* $x \notin (x_{k-1}, x_{k+1})$ *where for the uniformity of denotation, we put $x_0 = a$, $x_{n+1} = b$;*

(c) $A_k(x)$ *is continuous;*

(d) $A_k(x)$, $k = 2, \ldots, n$, *strictly increases on $[x_{k-1}, x_k]$ and $A_k(x)$, $k = 1, \ldots, n-1$, strictly decreases on $[x_k, \; x_{k+1}]$;*

(e) *for all $x \in [a, b]$*

$$\sum_{k=1}^{n} A_k(x) = 1.$$

The membership functions $A_1, \ldots, A_n$ are called basic functions.

Definition 2.2 (17). *Let $A_1, \ldots, A_n$ be basic functions which form a fuzzy partition of $[a, b]$ and f be any function from $C([a, b])$. We say that the n-tuple of real numbers $[F_1, \ldots, F_n]$ given by*

$$F_k = \frac{\int_a^b f(x) A_k(x) dx}{\int_a^b A_k(x) dx}$$

is the (integral) F-transform of f with respect to $A_1, \ldots, A_n$. Let us remark that this definition is correct because for each $k = 1, \ldots, n$ the product $f A_k$ is an integrable function on $[a, b]$.

Denote the F-transform of a function f with respect to $A_1, \ldots, A_n$ by $F_n[f]$. Then, according to Definition 2.2, we can write

$$F_n[f] = [F_1, \ldots, F_n].$$

The elements $F_1, \ldots, F_n$ are called components of the F-transform.

Lemma 2.1. *Let f be a continuous function on $[a, b]$ and $A_1, \ldots, A_n, n \geq 3$ be basic functions which form a uniform fuzzy partition of $[a, b]$, but function f be twice continuously differentiable in (a, b). Then for each $k = 1, \ldots, n$*

$$F_k = f(x_k) + O(h^2).$$

Proof. *For proof, see [16].*

Definition 2.3. *[16] Let $A_1, \ldots, A_n$ be basic functions which form a fuzzy partition of $[a, b]$ and f be a function from $C([a, b])$. Let $F_n[f] = [F_1, \ldots, F_n]$ be the integral F-transform of f with respect to $A_1, \ldots, A_n$. Then the function*

$$f_{F,n}(x) = \sum_{k=1}^{n} F_k A_k(x),$$

is called the inverse F-transform.

The following theorem shows that the inverse F-transform, $f_{F,n}$, can approximate the original continuous function f with an arbitrary precision.

Lemma 2.2. *Let f be a continuous function on $[a, b]$. Then for any $\varepsilon > 0$ there exists n_ε and a fuzzy partition $A_1, \ldots, A_{n\varepsilon}$ of $[a, b]$ such that for all $x \in [a, b]$*

$$|f(x) - f_{F,n_\varepsilon}(x)| \leq \varepsilon,$$

where $f_{F,n_\varepsilon}(x)$ is the inverse F-transform of f with respect to the fuzzy partition $A_1, \ldots, A_{n\varepsilon}$.

Proof. *For proof, see [16].*

Definition 2.4. *[16] Suppose that function f is given at nodes $p_1, \ldots, p_l \in [a, b]$ and $A_1, \ldots, A_n$, $n < l$, be basic functions which form a fuzzy partition of $[a, b]$. We say that the n-tuple of real numbers $[F_1, \ldots, F_n]$ is the discrete F-transform of f with respect to $A_1, \ldots, A_n$ if*

$$F_k = \frac{\sum_{j=1}^{l} f(p_j) A_k(p_j)}{\sum_{j=1}^{l} A_k(p_j)}.$$

In the discrete case, we define the inverse F-transform only at nodes where the original function is given.

Through the paper, we suppose $l = n$.

Definition 2.5. *[16] Let function f be given at nodes $p_1, \ldots, p_l \in [a, b]$ and $F_n[f] = [F_1, \ldots, F_n]$ be the discrete F-transform of f w.r.t. $A_1, \ldots, A_n$. Then the function*

$$f_n^F(p_j) = \sum_{k=1}^{n} F_k A_k(p_j),$$

defined at the same nodes, is the inverse discrete F-transform.

Definition 2.6. *[1] Let $f : [a, b] \rightarrow R$ be a bounded function. Then*

$$\omega(f, \delta) : [0, \infty) \rightarrow [0, \infty)$$

defined by

$$\omega(f, \delta) = \sup \{ | f(x) - f(y) | : \quad x, y \in [a, b], \ |x - y| \le \delta \}$$

is called the first order modulus of smoothness of f. If f is continuous then $\omega(f, \delta)$ is called the uniform modulus of continuity.

Theorem 2.1. *[1] The following properties hold true:*
(i) $|f(x) - f(y)| \le \omega(f, d(x, y))$ for any $x, y \in [a, b]$;
(ii) $\omega(f, \delta)$ is continuous in δ;
(iii) $\omega(f, \delta) = 0$;
(iv) $\omega(f, \delta_1 + \delta_2) \le \omega(f, \delta_1) + \omega(f, \delta_2)$ for any $\delta_1, \delta_2 \in [0, \infty)$;
(v) $\omega(f, n\delta) \le n\omega(f, \delta)$ for any $\delta \in [0, \infty)$ and $n \in N$;
(vi) $\omega(f, \lambda\delta) \le (\lambda + 1)\omega(f, \delta)$ for any $\delta, \lambda \in [0, \infty)$;
(vii) If f is continuous then $\lim_{\delta \to 0} \omega(f, \delta) = 0$.

Let n and k be arbitrarily fixed. The composition between the direct and inverse F-transform can be expressed as [2]

$$F_{n,k}(x) = \sum_{i=1}^{k} A_i(x) \frac{\sum_{j=1}^{n} A_i(x_j) f(x_j)}{\sum_{j=1}^{n} A_i(x_j)},$$

and $f(x)$ can be rewritten as

$$f(x) = \sum_{i=1}^{k} A_i(x) \frac{\sum_{j=1}^{n} A_i(x_j) f(x)}{\sum_{j=1}^{n} A_i(x_j)}.$$

Theorem 2.2. *Let $y_1, \ldots, y_k \in [a, b]$ be a partition of the interval $[a, b]$ having the norm δ. If $A = \{A_1, \ldots, A_k\}$ is a fuzzy partition with small support, then the following error estimate holds true:*

$$| F_{n,k}(x) - f(x) | \leq r\omega(f, \delta), \qquad \forall x \in [a, b],$$

where $F_{n,k}(x)$ denotes the composition of the inverse and direct discrete F- transform.
Proof. *For proof, see [2].*

3   Proposed method

In this section, we consider Eq. (1.1). For solving this kind of integral equations numerically, the unknown function, $f(x)$, is approximated by $F_{n,n}(x)$. By substituting $F_{n,n}(x)$ instead of the unknown function, $f(x)$, inside of integrals, we have:

$$f(x) \cong g(x) + \int_a^x k_1(x, t) F_{n,n}(t)\, dt + \int_a^b k_2(x, t) F_{n,n}(t)\, dt,$$

where $A_i(x), \ i = 1, \ldots, n$, are introduced in the previous section. Therefore, we have:

$$f(x) \cong g(x) + \int_a^x k_1(x, t) \left(\sum_{i=1}^n \left(\frac{\sum_{k=1}^n f(x_k) A_i(x_k)}{\sum_{k=1}^n A_i(x_k)} \right) A_i(t) \right) dt +$$

$$\int_a^b k_2(x, t) \left(\sum_{i=1}^n \left(\frac{\sum_{k=1}^n f(x_k) A_i(x_k)}{\sum_{k=1}^n A_i(x_k)} \right) A_i(t) \right) dt,$$

$$f(x) \cong g(x) + \sum_{k=1}^n f(x_k) \left(\sum_{i=1}^n \frac{A_i(x_k)}{\sum_{k=1}^n A_i(x_k)} \int_a^x k_1(x, t) A_i(t)\, dt \right) +$$

$$\sum_{k=1}^n f(x_k) \left(\sum_{i=1}^n \frac{A_i(x_k)}{\sum_{k=1}^n A_i(x_k)} \int_a^b k_2(x, t) A_i(t)\, dt \right).$$

If we set $x = x_j, \ j = 1, \ldots, n$, then we conclude the linear system of equations as follows:

$$f(x_j) \cong g(x_j) + \sum_{k=1}^n f(x_k) \left(\sum_{i=1}^n \frac{A_i(x_k)}{\sum_{k=1}^n A_i(x_k)} \int_a^{x_j} k_1(x_j, t) A_i(t)\, dt \right)$$

$$+ \sum_{k=1}^n f(x_k) \left(\sum_{i=1}^n \frac{A_i(x_k)}{\sum_{k=1}^n A_i(x_k)} \int_a^b k_2(x_j, t) A_i(t)\, dt \right).$$

Equivalently, it can be solved the following linear system of equations:

$$BX = y, \tag{3.2}$$

where $B = (b_{ij}), \ i, j = 1, \ldots, n,$

$$b_{ii} = 1 - \sum_{m=1}^n \frac{A_m(x_i)}{\sum_{k=1}^n (A_m(x_k))} \left(\int_a^{x_i} k_1(x_i, t) A_m(t)\, dt - \int_a^b k_2(x_i, t) A_m(t)\, dt \right).$$

$$b_{ij} = -\sum_{m=1}^{n} \frac{A_m(x_j)}{\sum_{k=1}^{n}(A_m(x_k))} \left(\int_a^{x_i} k_1(x_i,t) A_m(t)\, dt \int_a^b k_2(x_i,t) A_m(t)\, dt \right) \ , \ i \neq j,$$

$$X = [f(x_1), f(x_2), \ldots, f(x_n)]^t, y = [g(x_1), g(x_2), \ldots, g(x_n)]^t.$$

By solving the system $BX = y$, we can obtain $f(x_i), i = 1, \ldots, n$. Substituting $f(x_i)$, $i = 1, \ldots, n$, in F_k defined in Definition 2.4, we can compute $F_{n,n}(x) = \sum_{i=1}^{n} F_i A_i(x)$ as an approximation of $f(x)$. For solving the system $BX = y$, it is clear that matrix B must be invertible. So, we give the following theorem.

Theorem 3.1. *Let B be an $n \times n$ matrix as introduced in Eq. (3.2), and let $M_1 = \sup_{x,t\in[a,b]} |k_1(x,t)|$, $M_2 = \sup_{x,t\in[a,b]} |k_2(x,t)|$. If*

$$M_1 + M_2 < \frac{1}{(b-a)},$$

then B is invertible.
Proof: *We prove that $\|I - B\|_\infty < 1$. We have*

$$\|I - B\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |b_{ij}|$$

$$= \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} \left| \sum_{m=1}^{n} \frac{A_m(x_j)}{\sum_{k=1}^{n}(A_m(x_k))} \left(\int_a^{x_i} k_1(x_i,t) A_m(t)\, dt - \int_a^b k_2(x_i,t) A_m(t)\, dt \right) \right| \right)$$

$$\leq \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} \sum_{m=1}^{n} \frac{|A_m(x_j)|}{|\sum_{k=1}^{n}(A_m(x_k))|} \left(\int_a^{x_i} |k_1(x_i,t)| A_m(t)\, dt + \int_a^b |k_2(x_i,t)| A_m(t)\, dt \right) \right)$$

$$\leq (b-a)(M_1 + M_2) < 1.$$

Now, by using Neuman series, we conclude that $(I - (I - B))^{-1} = B^{-1}$ exists. This completes the proof.

4   Error analysis

Now, we present the convergence theorem, that justifies proposed method for approximating solution of Eq. (1.1). Let $(C([a,b]), \| \ . \ \|)$ be the space of all continuous functions on interval $[a,b]$ with the norm $\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$. We suppose that $g(x) \neq 0$, and $M_1 = \sup_{x,t\in[a,b]} |k_1(x,t)|$, $M_2 = \sup_{x,t\in[a,b]} |k_2(x,t)|$. With these conditions, we have the following theorem.

Theorem 4.1. *Let $y_1, \ldots, y_n \in [a,b]$ be a partition of the interval $[a,b]$ having the norm δ. Also, let $f(x)$ be an exact solution of Eq. (1.1) which is twice continuously differentiable*

in (a, b). Let $F_{n,n}(x)$ be the approximate solution of Eq. (1.1). Then the absolute error of approximate solution can be estimated as follows:

$$|f(x) - F_{n,n}(x)| \leq r(M_1 + M_2)(b-a)\omega(f, \delta).$$

Proof: *Clearly, we have*

$$|f(x) - F_{n,n}(x)| =$$

$$|\int_a^x k_1(x, t) f(t) dt + \int_a^b k_2(x, t) f(t) dt - \int_a^x k_1(x, t) F_{n,n}(t) dt - \int_a^b k_2(x, t) F_{n,n}(t) dt|$$

$$= |\int_a^x k_1(x, t)(f(t) - F_{n,n}(t)) dt + \int_a^b k_2(x, t)(f(t) - F_{n,n}(t)) dt$$

$$\leq \int_a^x k_1(x, t) \left(\sum_{i=1}^k A_i(t) \frac{\sum_{j=1}^n A_i(x_j)|f(t) - f(x_j)|}{\sum_{j=1}^n A_i(x_j)} \right) dt +$$

$$\int_a^b k_2(x, t) \left(\sum_{i=1}^k A_i(t) \frac{\sum_{j=1}^n A_i(x_j)|f(t) - f(x_j)|}{\sum_{j=1}^n A_i(x_j)} \right) dt.$$

We observe that the product $A_i(t) A_i(x_j) \neq 0$ if and only if $t, x_j \in [y_i, y_{i+r}]$, and in this case we have $|t - x_j| \leq r\delta$. By using Theorem 2.2, we conclude that:

$$\left| f(x) - f_n^F(x) \right| \leq \int_a^x |k_1(x, t)| \ r\omega(f, \delta) \ dt + \int_a^b |k_2(x, t)| \ r\omega(f, \delta) \ dt$$

$$\leq r(M_1 + M_2)(b-a)\omega(f, \delta).$$

5 Numerical examples

In order to show the accuracy of the proposed method, we present two examples. Software "Mathematica7" is applied for computing results.

Example 5.1. *[8] Consider the equation*

$$f(x) = x - 2e^x + e^{-x} + 1 + \int_0^x y e^x f(y) \, dy + \int_0^1 e^{(x+y)} f(y) \, dy.$$

The exact solution of this equation is as follows:

$$f(x) = e^{-x}.$$

For comparing numerical and exact solutions, see Table1.

Example 5.2. *[8] Consider the Volterra-Fredholm integral equation:*

$$f(x) = \cos(x)\left(\frac{1}{2}x - \frac{1}{4}\right) + \frac{1}{4}\cos(2 - x) + \int_0^x \sin(x - t) f(t) \, dt + \int_0^1 \cos(x - t) f(t) \, dt,$$

The exact solution of this equation is as follows:

$$f(x) = \sin(x).$$

To compare numerical and exact solutions, see Table1.

Table 1.
Absolute errors for Examples 5.1 and 5.2.

n	Example 5.1	Example 5.2
10	1.215E-3	1.127E-2
20	2.741E-4	2.523E-3
30	1.178E-4	1.082E-3
40	6.514E-5	5.987E-4

6   Conclusions

We used the composition of inverse and direct discrete F-transform to approximate the numerical solution of Volterra-Fredholm integral equations. The properties of basic functions and the F-transform provided us the possibility of reducing Volterra- Fredholm integral equation to a system of linear equations. The numerical and exact solutions are compared by considering the absolute error in two examples. The results show that the proposed approach can be a suitable method for solving Volterra-Fredholm integral equations numerically.

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