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Research Article

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Applying fuzzy wavelet like operator to the numerical solution of linear fuzzy Fredholm integral equations and error analysis

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Abstract

In this paper, we propose a successive approximation method based on fuzzy wavelet like operator to approximate the solution of linear fuzzy Fredholm integral equations of the second kind with arbitrary kernels. We give the convergence conditions and an error estimate. Also, we investigate the numerical stability of the computed values with respect to small perturbations in the first iteration. Finally, to show the efficiency of the proposed method, we present some test problems, for which the exact solutions are known.

Keywords : Fuzzy Fredholm integral equation; Fuzzy wavelet like operator; Successive approximation method.

1 Introduction

Fuzzy linear integral equations arise frequently in physical problems as a result of the possibility of super-imposing the effects due to several reasons. The most important contribution of the theory of fuzzy integral equations consists in the solution of fuzzy initial and boundary value problems. Also, the theory of fuzzy Volterra integral equations makes it possible to solve an initial value problem for a linear fuzzy ordinary differential equation of an arbitrary order [39].
The concept of integration of fuzzy functions was introduced by Dubois and Prad [13] for the first time and then investigated by Goetschel and Voxman [22], Kaleva [26], Matloka [29], Nanda [30] and others. One numerical method for solving fuzzy integrals is presented in [8]. The fuzzy-Riemann integral and its numerical integration was investigated by Wu in [36]. Some applications of the fuzzy integral equations to control models with fuzzy uncertainties are presented in[12]. In [15], the authors gave one of the applications of fuzzy integral for solving fuzzy Fredholm integral equation of the second kind (FFIE-2). On of the main fuzzy equations, addressed by many researchers, is fuzzy Fredholm integral equation. Generally, the complexity of fuzzy integral equations hinders analytical solutions. Therefore, some numerical methods have been recently proposed to fuzzy fredholm integral equation. The iterative techniques are applied to FFIE-2 in [10, 16, 33, 38]. Friedman et al. [18] presented one successive approximations method for solving FFIE-2. Also, Friedman et al. [17] investigated numerical procedures for solving FFIE-2 using the embedding method. Babolian et al. [7] used the Adomian decomposition method (ADM) to solve FFIE-2. Abbssbandy et al. [1, 27] obtained the solution of fuzzy Fredholm integral equations by using the Nystrom

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method. Recently,the authors used Lagrange interpolation [6], divided and finite differences [32], Bernstein polynomials [14, 31],Chebyshev interpolation [9], Legendre wavelets [24], Splines interpolation [25],fuzzy Haar wavelets [35], and Galerkin type techniques [28]. Also, the authors of[36] proved the convergence of the method of successive approximations used to approximate the solution of nonlinear Hammerstein fuzzy integral equations.
Here, by using fuzzy wavelet like operator, we propose a numerical approach for solving linear (FFIF-2):

$$\tilde{F}(t) = \tilde{f}(t) \oplus \lambda \odot \int_a^b K(x,t) \odot \tilde{F}(x)dx, \quad (1.1)$$

where $\lambda > 0, K(x,t)$ is an arbitrary kernel function over the square $a \le x, t \le b$ and $\tilde{f}(t)$ is a fuzzy real valued function of t. Also, we present, the error estimation for approximating the solution of linear FFIF-2.
This paper includes the following parts: In Section 2, we review some elementary concepts of the fuzzy set theory and modulus of continuity. In Section 3, the algorithm is given. In Section 4, the convergence analysis is presented. In Section 5, the numerical stability with respect to the choice of the first iteration is demonstrated. In Section 6, we present two numerical examples for applicability of the proposed method to obtain numerical solution of linear FFIF-2 based on fuzzy wavelet like operator. Finally, Section 7 gives our concluding remarks.

2 Preliminaries

Definition 2.1 *[21] A fuzzy number is a function $u : \Re \longrightarrow [0,1]$. with the following properties:*

(i) *u is normal, i.e. $\exists x_0 \in \Re$ with $u(x_0) = 1$,*

(ii) *u is a convex fuzzy set,*

(iii) *u is upper semi-continuous on $\Re$,*

(iv) *$\overline{\{x \in \Re : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.*

The set of all fuzzy numbers is denoted by $\Re_F$.

Definition 2.2 *[19] Suppose that $u \in \Re_F$. The r-level set of u is denoted by $[u]^r = [u_-^{(r)}, u_+^{(r)}]$ and defined by $[u]^r = \{x \in \Re; u(x) \ge r\}$, where $0 < r \le 1$. Also, $[u]^0$ is called the support of u and it is given as $[u]^0 = \overline{\{x \in \Re; u(x) > 0\}}$. It follows that the level sets of u are closed and bounded intervals in $\Re$.*
It is well-known that the addition and multiplication operations of real numbers can be extended to $\Re_F$. In other words, for $u,\ v \in \Re_F$ and $\lambda \in \Re$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, [\lambda \odot u]^r = \lambda[u]^r, \forall r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\Re$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\Re$. We use the same symbol $\sum$ both for the sum of real numbers and for the sum $\oplus$ (when the terms are fuzzy numbers).

Definition 2.3 *[19] An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \le r \le 1$, which satisfy the following requirements:*

(i) *$\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0,1]$,*

(ii) *$\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0,1]$,*

(iii) $\underline{u}(r) \le \overline{u}(r), \quad 0 \le r \le 1.$

The addition and scaler multiplication of fuzzy numbers in $\Re_F$ are defined as follows:

(i) $u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)),$

(ii) $(\lambda \odot u) = \begin{cases} (\lambda\underline{u}(r), \lambda\overline{u}(r)) & \lambda \ge 0, \\ (\lambda\overline{u}(r), \lambda\underline{u}(r)) & \lambda < 0. \end{cases}$

Definition 2.4 *[20] For arbitrary fuzzy numbers $u,\ v$, the quantity*

$$D(u,v) = \sup_{r\in[0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\}$$

is the distance between u and v. It is proved that $(\Re_F, D)$ is a complete metric space with the properties ([23, 34]).

(i) $D(u \oplus w, v \oplus w) = D(u,v)\ \forall\ u,v,w \in \Re_F,$

(ii) $D(k \odot u, k \odot v) = |k|\, D(u,v)\ \forall\ u,v \in \Re_F \quad \forall k \in R,$

(iii) $D(u \oplus v, w \oplus e) \le D(u,w) + D(v,e) \quad \forall\ u,v,w,e \in \Re_F.$

Definition 2.5 *[20] Let $f, g : [a, b] \to \Re_F$, be fuzzy real number valued functions. The uniform distance between f, g is defined by*

$$D^*(f, g) = \sup\{D(f(x), g(x))|\ x \in [a, b]\}.$$

Definition 2.6 *[20] Let $f : [a, b] \to \Re_F$. f is fuzzy-Riemann integrable to $J \in \Re_F$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v]; \xi\}$ of $[a, b]$ with the norms $\Delta(p) < \delta$, we have*

$$D(\sum_P {}^*(v - u) \odot f(\xi), I) < \varepsilon,$$

where $\sum^*$ denotes the fuzzy summation. In this case it is denoted by $I = (FR)\int_a^b f(x)dx$.

Definition 2.7 *[3] A fuzzy real number valued function $f : \Re \to \Re_F$ is said to be continuous in $x_0 \in \Re$, if for each $\varepsilon > 0$ there is $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$, whenever $x \in \Re$ and $|x - x_0| < \delta$. We say that f is fuzzy continuous on $\Re$ if f is continuous at each $x_0 \in \Re$, and denote the space of all such functions by $C_F(\Re)$.*

Theorem 2.1 *[2] If $f, g : [a, b] \subseteq \Re \to \Re_F$ are fuzzy continuous functions, then the function $F : [a, b] \to \Re_+$ defined by $F(x) = D(f(x), g(x))$ is continuous on $[a, b]$, and*

$$D\left((FR)\int_a^b f(x)\, dx, (FR)\int_a^b g(x)\, dx\right) \le \int_a^b D(f(x), g(x))\, dx.$$

Definition 2.8 *[3] Let $f : \Re \to \Re_F$. One call f a uniformly continuous fuzzy real number valued function, if and only if for any $\epsilon > 0$ there exists $\delta > 0$ whenever $|x - y| \le \delta; x, y \in \Re$, implies that $D(f(x), f(y)) \le \epsilon$. One denotes it as $f \in C_F^U(\Re)$.*

Definition 2.9 *[20, 4] Let $f : \Re \to \Re_F$ be a bounded function, then function*

$$\omega_{\Re}(f, .) : \Re_+ \cup \{0\} \to \Re_+,$$

.

$$\omega_{\Re}(f, \delta) = \sup\{D(f(x), f(y))|\ x, y \in \Re,$$

$|x - y| \le \delta\}$, where $\Re_+$ is the set of positive real numbers, is called the modulus of continuity of f on $\Re$.
Some properties of the modulus of continuity are presented below:

Theorem 2.2 *[4] The following properties holds:*

(1) $D(f(x), f(y)) \le \omega_{[a,b]}(f, |x - y|)$ *for any* $x, y \in [a, b]$,

(2) $\omega_{[a,b]}(f, \delta)$ *is increasing function of* δ,

(3) $\omega_{[a,b]}(f, 0) = 0$,

(4) $\omega_{[a,b]}(f, \delta_1 + \delta_2) \le \omega_{[a,b]}(f, \delta_1) + \omega_{[a,b]}(f, \delta_2)$ *for any* $\delta_1, \delta_2 \ge 0$, *and* $f : \Re \to \Re_F$,

(5) $\omega_{[a,b]}(f, n\delta) \le n\omega_{[a,b]}(f, \delta)$ *for any* $\delta \ge 0$, $n \in N$, *and* $f : \Re \to \Re_F$,

(6) $\omega_{[a,b]}(f, \lambda\delta) \le [\lambda]\omega_{[a,b]}(f, \delta) \le (\lambda + 1)\omega_{[a,b]}(f, \delta)$ *for any* $\delta, \lambda \ge 0$, *where* $[.]$ *is the ceiling of the number, any* $f : \Re \to \Re_F$.

(7) *If* $[c, d] \subseteq [a, b]$ *then* $\omega_{[c,d]}(f, \delta) \le \omega_{[a,b]}(f, \delta)$.

In [5], the following theorem is proved.

Theorem 2.3 *[4] Let $f \in C_F(\Re)$ and the scaling function $\varphi(x)$ a real-valued bounded function with $supp\varphi(x) \subseteq [-a, a], 0 < a < +\infty, \varphi(x) \ge 0$, such that*

$$\sum_{j=-\infty}^{+\infty} \varphi(x - j) = 1$$

on $\Re$. For $k \in Z, x \in R$, put

$$(B_k f)(x) := \sum_{j=-\infty}^{+\infty}{}^* f(\frac{j}{2^k}) \odot \varphi(2^k x - j),$$

which is a fuzzy-wavelet-like operator. Then

$$D((B_k f)(x), f(x)) \le \omega_R(f, \frac{a}{2^k}),$$

$$D^*((B_k f), f) \le \omega_R(f, \frac{a}{2^k}),$$

for all $x \in \Re$ and $k \in Z$. If $f \in C_F^U(\Re)$, then as $k \to +\infty$ one gets $\omega_{\Re}(f, \frac{a}{2^k}) \to 0$ and $\lim_{k \to +\infty} B_k f = f$, pointwise and uniformly with rates.

3 The algorithm

Suppose that for all $x \in [a,b]$ there exist $j \in Z$ such that $2^m x - j \in [-a,a]$. Now apply fuzzy wavelet like operator in the computation of the terms of the sequence of successive approximations

$$\tilde{F}_0(t) = \tilde{f}(t),$$

$$\tilde{F}_k(t) = \tilde{f}(t) \oplus \lambda \odot \int_a^b K(x,t) \odot \tilde{F}_{k-1}(x)dx,$$

$k \geq 1$. Moreover we suppose that scaling function is as follow:

$$\varphi(x) = \begin{cases} 1 & \frac{-1}{2} \leq x \leq \frac{1}{2}, \\ 0 & o.w. \end{cases}$$

It obtains the following iterative algorithm:
Step 0: There are introduced the a, b, m, ϵ and functions $\tilde{f}(t), \lambda, K(x,t)$.
Step 1:

$$\tilde{Y}_{0,m}(t) = \tilde{f}(t),$$

Step 2: (the first iterative Step):
For $k = 1$ compute

$$\tilde{Y}_{k,m}(t) = \tilde{f}(t) \oplus \lambda \odot$$

$$\int_a^b K(x,t) \odot \sum_{j=-\infty}^{\infty} \tilde{Y}_{k-1,m}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx,$$

Step 3: (the generic iterative Step):
By induction for $k \in N, k \geq 2$, we use Step 2.
Step 4:(a condition if "do-while" type):If

$$D^*(\tilde{Y}_{k,m}, \tilde{Y}_{k-1,m}) < \epsilon,$$

and

$$D^*(\tilde{Y}_{k,m}, \tilde{f}) < \epsilon,$$

then we stop to this "k" and computed at this last iterative retain the values $\tilde{Y}_{k,m}(t,r)$ computed $|\tilde{Y}_{k,m}(t,r) - \tilde{f}(t,r)|$ for $r \in [0,1]$ and $t = 0.5$.
Step 5: Print "k". STOP.

4 The convergence analysis

Definition 4.1 *We say that the algorithm of successive approximations applied to the integral equation (1) is numerically stable with respect to the choice of the first iteration iff there exist a natural number $k \geq 1, m \in Z$ and two constants $K_1, K_2 > 0$ such that*

$$D^*(\tilde{F}, \tilde{Y}_{k,m}) \leq K_1 \epsilon_1^k + K_2 \epsilon_2^m, for \epsilon_1 > 0, \epsilon_2 > 0.$$

Theorem 4.1 *[22, 26] Let $K(x,t)$ be continuous for $a \leq x,t \leq b$ and $\tilde{f}(t)$ a fuzzy continuous function. If $\lambda < \frac{1}{M(b-a)}$, where $M =$* $\max|K(x,t)|$, *$a \leq x,t \leq b$, then the iterative procedure*

$$\tilde{F}_0(t) = \tilde{f}(t)$$

$$\tilde{F}_k(t) = \tilde{f}(t) \oplus \lambda \odot \int_a^b K(x,t) \odot \tilde{F}_{k-1}(x)dx,$$

$k \geq 1$ converges to the unique solution of above fuzzy integral equation. Specifically,

$$\sup_{a \leq t \leq b} D(\tilde{F}(t), \tilde{F}_k(t))$$

$$\leq \frac{L^k}{1-L} \sup_{a \leq t \leq b} D(\tilde{F}_1(t), \tilde{F}_0(t)),$$

where $L = \lambda M(b-a)$.

Theorem 4.2 *Under the hypotheses of Theorem 4.1, we consider the following iterative procedure*

$$\tilde{Y}_{0,m}(t) = \tilde{f}(t),$$

$$\tilde{Y}_{k,m}(t) = \tilde{f}(t) \oplus \lambda \odot \int_a^b K(x,t) \odot$$

$$\sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx, k \geq 1, m \in Z$$

where $\tilde{Y}_{k,m} \in C(\Re)$, $k \geq 0, m \in Z$ the scaling function $\varphi(x)$ a real valued bounded function with $supp\varphi(x) \subseteq [-a,a], 0 < a < +\infty, \varphi(x) \geq 0, x \in \Re$ such that

$$\sum_{j=-\infty}^{j=+\infty} \varphi(x-j) \equiv 1$$

on $\Re$. Then

$$D(\tilde{F}(t), \tilde{Y}_{k,m}(t)) \leq \frac{L^k}{1-L} \sup_{a \leq t \leq b} D(\tilde{F}_1(t), \tilde{F}_0(t))$$

$$+D(\tilde{F}_0(t), \tilde{Y}_{m,0}(t)) + \frac{L}{1-L} \omega(\tilde{Y}_{max}, \frac{a}{2^m})),$$

where

$$\omega(\tilde{Y}_{max}, \frac{a}{2^m}) = max\{\omega(\tilde{Y}_{m,0}, \frac{a}{2^m}),$$

$$\omega(\tilde{Y}_{m,1}, \frac{a}{2^m}), \omega(\tilde{Y}_{m,2}, \frac{a}{2^m}), \cdots, \omega(\tilde{Y}_{m,k}, \frac{a}{2^m})\},$$

Table 1: Numerical results on the level sets for Example 6.1

k=2, m=4	r-level	$\vert\underline{Y}_{k,m}-\underline{f}\vert$	$\vert\overline{Y}_{k,m}-\overline{f}\vert$	k=4, m=16	$\vert\underline{Y}_{k,m}-\underline{f}\vert$	$\vert\overline{Y}_{k,m}-\overline{f}\vert$
	0.0	0.000000000	0.460913000		0.000000000	0.150909000
	0.1	0.023045700	0.437868000		0.007545470	0.143364000
	0.2	0.046091300	0.414822000		0.015090900	0.135818000
	0.3	0.069137000	0.391776000		0.022636400	0.128273000
	0.4	0.092182600	0.368731000		0.030181900	0.120727000
	0.5	0.115228000	0.345685000		0.037727300	0.113182000
	0.6	0.138276000	0.322639000		0.045272800	0.105637000
	0.7	0.161320000	0.299594000		0.052818300	0.098091100
	0.8	0.184365000	0.276548000		0.060363700	0.090545600
	0.9	0.207411000	0.253502000		0.067909200	0.083000100
	1.0	0.230457000	0.230457000		0.075454700	0.075454700

Table 2: Numerical results on the level sets for Example 6.1

k=6,m=64	r-level	$\vert\underline{Y}_{k,m}-\underline{f}\vert$	$\vert\overline{Y}_{k,m}-\overline{f}\vert$	k=8,m=256	$\vert\underline{Y}_{k,m}-\underline{f}\vert$	$\vert\overline{Y}_{k,m}-\overline{f}\vert$
	0.0	0.000000000	0.080381700		0.000000000	0.010791500
	0.1	0.005855430	0.076362600		0.000539573	0.010251900
	0.2	0.004822860	0.072343500		0.001079150	0.009712320
	0.3	0.003682420	0.068324400		0.001618720	0.009172750
	0.4	0.002422790	0.064305300		0.002158290	0.008633180
	0.5	0.001031470	0.060286300		0.002697870	0.008093600
	0.6	0.000505347	0.056267200		0.003237440	0.007554030
	0.7	0.002202910	0.052248100		0.003777010	0.007014460
	0.8	0.004078110	0.048229000		0.004316590	0.006852580
	0.9	0.006149550	0.044209900		0.004856160	0.005935310
	1.0	0.008437820	0.040190800		0.005395730	0.005395730

Table 3: Numerical results on the level sets for Example 6.2

k=2,m=4	r-level	$\vert\underline{Y}_{k,m}-\underline{f}\vert$	$\vert\overline{Y}_{k,m}-\overline{f}\vert$	k=4,m=16	$\vert\underline{Y}_{k,m}-\underline{f}\vert$	$\vert\overline{Y}_{k,m}-\overline{f}\vert$
	0.0	0.000000000	0.080045200		0.000000000	0.024050900
	0.1	0.004002260	0.076043000		0.001202550	0.022848400
	0.2	0.008004520	0.072040700		0.002405090	0.021645900
	0.3	0.012006800	0.068038400		0.003607640	0.020443300
	0.4	0.016009000	0.064036200		0.004810190	0.019240800
	0.5	0.020011300	0.060033900		0.006127400	0.018038200
	0.6	0.024013600	0.056317000		0.007215280	0.016835700
	0.7	0.028015800	0.052029400		0.008417830	0.015633100
	0.8	0.032018100	0.048027100		0.009620380	0.014430600
	0.9	0.036020300	0.044024900		0.010822900	0.013228000
	1.0	0.040022600	0.040022600		0.012025500	0.012025500

and

$$D^*(\tilde{F},\tilde{Y}_{k,m}) \leq$$

$$\frac{L^k}{1-L}D^*(\tilde{F}_1,\tilde{F}_0)+\frac{L}{1-L}\omega(\tilde{Y}_{max},\frac{a}{2^m})).$$

If $\tilde{Y}_{k,m} \in C_F^U(\Re)$, *then as* $m \to +\infty$ *we get* $\omega(\tilde{Y}_{max},\frac{a}{2^m}) \to 0$, *pointwise and uniformly with rates.*

Proof.

$$D(\tilde{F}_k(t),\tilde{Y}_{k,m}(t)) =$$

$$D(\tilde{f}(t)\oplus\lambda\odot\int_a^b K(x,t)\odot\tilde{F}_{k-1}(x)dx,\tilde{f}(t)$$

$$\oplus\lambda\odot\int_a^b K(x,t)\odot\sum_{j=-\infty}^{\infty}\tilde{Y}_{m,k-1}(\frac{j}{2^m})\odot\varphi(2^m x-j)dx)$$

Table 4: Numerical results on the level sets for Example 6.2

k=6,m=64	r-level	$\lvert \underline{Y}_{k,m} - \underline{f} \rvert$	$\lvert \overline{Y}_{k,m} - \overline{f} \rvert$	k=8,m=256	$\lvert \underline{Y}_{k,m} - \underline{f} \rvert$	$\lvert \overline{Y}_{k,m} - \overline{f} \rvert$
	0.0	0.000000000	0.006444650		0.000000000	0.001648600
	0.1	0.000322233	0.006122420		0.000082430	0.001566170
	0.2	0.000644465	0.005800190		0.000164860	0.001483740
	0.3	0.000966698	0.005477960		0.000247290	0.001401310
	0.4	0.001288930	0.005155720		0.000329720	0.001318880
	0.5	0.001611160	0.004833490		0.000412149	0.001236450
	0.6	0.001933400	0.004511260		0.000494579	0.001154020
	0.7	0.002255630	0.004189020		0.000577009	0.001071590
	0.8	0.002577860	0.003866790		0.000659439	0.000989159
	0.9	0.002900090	0.003544560		0.000741869	0.000906729
	1.0	0.003222330	0.003222330		0.000824299	0.000824299

$$= D(\lambda \odot \int_a^b K(x,t) \odot \tilde{F}_{k-1}(x)dx, \lambda \odot \int_a^b K(x,t) \odot$$

$$\sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx) = |\lambda|$$

$$D(\int_a^b K(x,t) \odot \tilde{F}_{k-1}(x)dx, \int_a^b K(x,t) \odot$$

$$\sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx)$$

$$\leq |\lambda| M D(\int_a^b \sum_{j=-\infty}^{\infty} \tilde{F}_{k-1}(x) \odot \varphi(2^m x - j)dx,$$

$$\int_a^b \sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(x) \odot \varphi(2^m x - j)dx) +$$

$$|\lambda| M D(\int_a^b \sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(x) \odot \varphi(2^m x - j)dx,$$

$$\int_a^b \sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx)$$

$$\leq |\lambda| M \int_a^b D(\tilde{F}_{k-1}(x), \tilde{Y}_{m,k-1}(x))dx \odot$$

$$\sum_{j=-\infty}^{\infty} \varphi(2^m x - j) + |\lambda| M$$

$$\int_a^b D(\sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(x) \odot \varphi(2^m x - j),$$

$$\sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j))dx$$

$$\leq |\lambda| M(b-a) D^*(\tilde{F}_{k-1}, \tilde{Y}_{m,k-1}) +$$

$$|\lambda| M(b-a)\omega(\tilde{Y}_{m,k-1}, \frac{a}{2^m}) \quad (***)$$

Since

$$D(\sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(x) \odot \varphi(2^m x - j),$$

$$\sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j))$$

$$\leq \sum_{j=-\infty}^{\infty} \varphi(2^m x - j) D(\tilde{Y}_{m,k-1}(x), \tilde{Y}_{m,k-1}(\frac{j}{2^m}))$$

$$\leq \sum_{(2^m x - j) \in [-a,a]} \varphi(2^m x - j)\omega(\tilde{Y}_{m,k-1}, |x - \frac{j}{2^m}|)$$

$$\leq \omega(\tilde{Y}_{m,k-1}, \frac{a}{2^m}),$$

we conclude that

$$(***) \leq L D^*(\tilde{F}_{k-1}, \tilde{Y}_{m,k-1}) + L\omega(\tilde{Y}_{m,k-1}, \frac{a}{2^m})$$

$$\Rightarrow D^*(\tilde{F}_k, \tilde{Y}_{m,k}) \leq L D^*(\tilde{F}_{k-1}, \tilde{Y}_{m,k-1})$$

$$+ L\omega(\tilde{Y}_{m,k-1}, \frac{a}{2^m})$$

Then

$$D^*(\tilde{F}_{k-1}, \tilde{Y}_{m,k-1}) \leq L D^*(\tilde{F}_{k-2}, \tilde{Y}_{m,k-2})$$

$$+ L\omega(\tilde{Y}_{m,k-2}, \frac{a}{2^m}).$$

Hence, we have

$$D^*(\tilde{F}_k, \tilde{Y}_{m,k}) \leq L^2 D^*(\tilde{F}_{k-2}, \tilde{Y}_{m,k-2})$$

$$+ L^2\omega(\tilde{Y}_{m,k-2}, \frac{a}{2^m}) + L\omega(\tilde{Y}_{m,k-1}, \frac{a}{2^m})$$

$$\vdots$$

$$\leq L^k D^*(\tilde{F}_0, \tilde{Y}_{m,0}) + L\omega(\tilde{Y}_{m,k-1}, \frac{a}{2^m}))+$$

$$L^2\omega(\tilde{Y}_{m,k-2}, \frac{a}{2^m})) + \cdots + L^k\omega(\tilde{Y}_{m,0}, \frac{a}{2^m})$$

So,

$$D^*(\tilde{F}_k, \tilde{Y}_{m,k}) \leq L^k D^*(\tilde{F}_0, \tilde{Y}_{m,0}) +$$

$$L\omega(\tilde{Y}_{m,k-1}, \frac{a}{2^m}) + L^2\omega(\tilde{Y}_{m,k-2}, \frac{a}{2^m}) + \cdots +$$

$$L^k\omega(\tilde{Y}_{m,0}, \frac{a}{2^m})$$

$$D^*(\tilde{F}_k, \tilde{Y}_{m,k}) \leq L^k D^*(\tilde{F}_0, \tilde{Y}_{m,0}) +$$

$$(L + L^2 + L^3 + \cdots + L^k)\omega(\tilde{Y}_{max}, \frac{a}{2^m})$$

$$\leq L^k D^*(\tilde{F}_0, \tilde{Y}_{m,0}) + \frac{L(1-L^k)}{1-L}\omega(\tilde{Y}_{max}, \frac{a}{2^m}),$$

Since

$$0 < L < 1 \Rightarrow 0 < L^k < 1 \Rightarrow D^*(\tilde{F}_k, \tilde{Y}_{m,k})$$

$$\leq D^*(\tilde{F}_0, \tilde{Y}_{m,0}) + \frac{L}{1-L}\omega(\tilde{Y}_{max}, \frac{a}{2^m}).$$

Using properties of metrics space, we have:

$$D(\tilde{F}(t), \tilde{Y}_{k,m}(t)) \leq$$

$$D(\tilde{F}(t), \tilde{F}_k(t)) + D(\tilde{F}_k(t), \tilde{Y}_{k,m}(t))$$

Then

$$D^*(\tilde{F}, \tilde{Y}_{k,m}) \leq D^*(\tilde{F}, \tilde{F}_k) + D^*(\tilde{F}_k, \tilde{Y}_{k,m}).$$

Hence

$$D^*(\tilde{F}, \tilde{Y}_{k,m}) \leq \frac{L^k}{1-L} D^*(\tilde{F}_1, \tilde{F}_0) +$$

$$D^*(\tilde{F}_0, \tilde{Y}_{m,0}) + \frac{L}{1-L}\omega(\tilde{Y}_{max}, \frac{a}{2^m}))$$

$$= \frac{L^k}{1-L} D^*(\tilde{F}_1, \tilde{F}_0) + \frac{L}{1-L}\omega(\tilde{Y}_{max}, \frac{a}{2^m})).$$

Remark 4.1 *Since $0 < L < 1$, it follows that $lim_{k\to\infty} L^k = 0$. In addition, from Theorem 15 we have when $m \to +\infty$ then $\omega(\tilde{Y}_{max}, \frac{a}{2^m}) \to o$ So,*

$$\lim_{k\to\infty, m\to\infty} D^*(\tilde{F}, \tilde{Y}_{k,m}) = 0.$$

that shows the convergence of the method.

5 The numerical stability analysis

In order to investigate the numerical stability of the computed values with respect to small perturbations in the firs iteration we consider another first iteration term $G_0 \in C_F(\Re)$ such that there exists $\epsilon > 0$ for which $D^*(F_0, G_0) < \epsilon$, for all $t \in [a, b]$. The new sequence of successive approximation is:

$$\tilde{G}_{k,m}(t) = \tilde{f}(t) \oplus \lambda \odot \int_a^b K(x,t) \odot$$

$$\sum_{j=-\infty}^{\infty} \tilde{G}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx,$$

$$k \geq 1, m \in Z$$

we redefine the new numerical iterative algorithm as follows:

$$\overline{\tilde{Y}}_{0,m}(t) = \tilde{G}_{0,m}(t)$$

$$\overline{\tilde{Y}}_{k,m}(t) = \tilde{f}(t) \oplus \lambda \odot \int_a^b K(x,t)$$

$$\odot \sum_{j=-\infty}^{\infty} \overline{\tilde{Y}}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx,$$

$$k \geq 1, m \in Z$$

Definition 5.1 *We say that the method of successive approximation applied for solving the fuzzy linear FFIF-2 is numerically stable with respect to the choice of the first iteration term iff for each $\epsilon > 0$, such that $D^*(\tilde{Y}_{k,m}, \overline{\tilde{Y}}_{k,m}) < \epsilon$.*

In order to obtain the numerical stability using given iterative procedure $\tilde{Y}_{k,m}(t), \overline{\tilde{Y}}_{k,m}(t)$ for $k \geq 1, m \in Z$ and $t \in [a, b]$ we have

$$D(\tilde{Y}_{k,m}(t), \overline{\tilde{Y}}_{k,m}(t)) \leq$$

$$D(\tilde{f}(t) \oplus \lambda \odot \int_a^b K(x,t) \odot \sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(\frac{j}{2^m})$$

$$\odot \varphi(2^m x - j)dx, , \tilde{f}(t) \oplus \lambda \odot \int_a^b K(x,t) \odot$$

$$\sum_{j=-\infty}^{\infty} \overline{\tilde{Y}}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx)$$

$$\leq D(\tilde{f}(t), \tilde{f}(t)) + D(\lambda \odot \int_a^b K(x,t)$$

$$\odot \sum_{j=-\infty}^{\infty} \tilde{Y}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx,$$

$$\lambda \odot \int_a^b K(x,t) \odot \sum_{j=-\infty}^{\infty} \overline{\tilde{Y}}_{m,k-1}(\frac{j}{2^m}) \odot \varphi(2^m x - j)dx)$$

$$\leq |\lambda| M \int_a^b D(\tilde{Y}_{m,k-1}(\frac{j}{2^m}), \overline{\tilde{Y}}_{m,k-1}(\frac{j}{2^m}))dx$$

$$\leq |\lambda| M(b-a) D(\tilde{Y}_{m,k-1}(\frac{j}{2^m}), \overline{\tilde{Y}}_{m,k-1}(\frac{j}{2^m}))$$

Then

$$D(\tilde{Y}_{m,k-1}(t), \overline{\tilde{Y}}_{m,k-1}(t)) \leq$$

$$(|\lambda| M(b-a))^2 D(\tilde{Y}_{m,k-2}(\frac{j}{2^m}), \overline{\tilde{Y}}_{m,k-2}(\frac{j}{2^m}))$$

$$\vdots$$

$$D(\tilde{Y}_{k,m}(t), \overline{\tilde{Y}}_{k,m}(t)) \leq$$

$$(|\lambda| M(b-a))^k D(\tilde{Y}_{m,0}(\frac{j}{2^m}), \overline{\tilde{Y}}_{m,0}(\frac{j}{2^m}))$$

$$D^*(\tilde{Y}_{k,m}, \overline{\tilde{Y}}_{k,m}) \leq (|\lambda| M(b-a))^k D^*(\tilde{Y}_{m,0}, \overline{\tilde{Y}}_{m,0})$$

regarding to

$$D^*(\tilde{Y}_{m,0}, \overline{\tilde{Y}}_{m,0}) = D^*(\tilde{F}_0, \tilde{G}_{0,m}) < \epsilon$$

we deduce that

$$D^*(\tilde{Y}_{k,m}, \overline{\tilde{Y}}_{k,m}) \leq \epsilon(|\lambda| M(b-a))^k$$

and since $(|\lambda| M(b-a)) < 1$, we conclude that the stability of the numerical method is proved. Indeed we have

$$lim_{k\to\infty} D^*(\tilde{Y}_{k,m}, \overline{\tilde{Y}}_{k,m}) = 0.$$

6 Numerical Examples

In this section, we apply the proposed method in Section 3 for solving the fuzzy linear FFIE-2 in some examples . We compare numerical results with exact solutions. Also, we apply the following scaling function

$$\varphi(x) = \begin{cases} 1 & \frac{-1}{2} \leq x \leq \frac{1}{2}, \\ 0 & o.w. \end{cases}$$

Example 6.1 *Consider the following fuzzy Fredholm integral equation*

$$\tilde{F}(t) = \tilde{f}(t) \oplus (FR) \int_0^1 k(x,t) \odot \tilde{F}(x)dx,$$

$$\tilde{f}(t) = (\underline{f}(t,r), \overline{f}(t,r) =$$
$$((t+1)r - 7r(1+t^2)/12,$$
$$(t+1)(2-r) - 7(2-r)(1+t^2)/12),$$
$$t, r \in [0,1],$$
$$k(x,t) = t^2/(1+x^2), \qquad t, x \in [0,1]$$

by exact solution $\tilde{F}(t) = (\underline{F}(t,r), \overline{F}(t,r)) = ((t+1)r, (t+1)(2-r)), t, r \in [0,1]$. *By using proposed algorithm in Section 3, we present approximate solution to this example in* $t = 0.5$ *for different values of k ,m in Tables 1, 2.*

Example 6.2 *Consider the following fuzzy Fredholm integral equation*

$$\tilde{F}(t) = \tilde{f}(t) \oplus (FR) \int_0^1 k(x,t) \odot \tilde{F}(x)dx,$$

$$\tilde{f}(t) = (f_-(t,r), f^-(t,r) =$$
$$(tr - 5/52r - t^2r/26,$$
$$2t - rt - t^2/13 + t^2r/26 - 10/52 + 5r/42),$$
$$t, r \in [0,1],$$
$$k(x,t) = (t^2 + x^2 + 2)/13, \qquad t, x \in [0,1]$$

by exact solution

$$\tilde{F}(t) = (F_-(t,r), F^-(t,r)) = (tr, t(2-r)),$$
$$t, r \in [0,1].$$

By using proposed algorithm in Section 3, we present approximate solution to this example in $t = 0.5$ for different values of k ,m in Tables 3, 4.

7 Conclusion

In this paper, we proposed a successive approximation method to solve linear FFIE-2 with arbitrary kernels based on fuzzy wavelet like operator. Also, we have developed an iterative algorithm based on fuzzy wavelet like operator. In Theorem 4.1, by presenting the convergence conditions, we obtained an error estimate in terms of uniform and partial modulus of continuity with respect to the choice of the first iteration. Also, the numerical stability with respect to the choice of the first iteration is demonstrated. Finally, some numerical examples are give to show the validity of the presented algorithm.

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