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Research Article

A new network simplex algorithm to reduce consecutive degenerate pivots and prevent stalling

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Abstract

It is well known that in operations research, degeneracy can cause a cycle in a network simplex algorithm which can be prevented by maintaining strong feasible bases in each pivot. Also, in a network consists of n arcs and m nodes, not considering any new conditions on the entering variable, the upper bound of consecutive degenerate pivots is equal $\binom{n-m+k}{k}$ where k is the number of degenerate arcs in the basis. As well as, the network simplex algorithm may stall if it goes through some long consecutive degenerate pivot. Through conditions such as (LRC) and (LRS) upon entering variable rules, this upper bound can be reduced to mn and m^2 respectively. In this current paper we first suggest a new algorithm for anti–stalling in which a new condition is provided to the entering variable and then show that through this algorithm there are at most k consecutive degenerate pivots.

Keywords : Network flow problem; Network simplex algorithm; Degeneracy; Strong feasible basis; Stalling.

1 Introduction

The primal simplex algorithm for the minimum cost flow problem is known as the network simplex algorithm. If there would be at least one zero arc in one feasible basis solution then it can be concluded that this basis solution is degenerate. The degeneracy may cause a cycling in a network of n arcs and m nodes. Although Cummingham [3] showed that the cycling can be prevented considering strongly basis, but the algorithm may be forced to go through a long sequence of consecutive degenerate pivots, which is called stalling. There exist several articles related to this point which could reduce the upper bound of consecutive degenerate pivots to mn, m^2. Goldfarb [5] proposed one anti stalling pivot rule by at most $k(k+1)/2$ consecutive degenerate pivots where k is the number of degenerate arcs in the basis. Also Ahuja [1] identified a sequence of entering variables using a negative cost augmenting cycle which is guaranteed after at most k consecutive degenerate pivots in which a non degenerate pivot can be achieved. In this article, two definitions have been presented for a valid cycle, but the equivalence between them has not been proved. In this paper, a new algorithm by a new entering variable rule is proposed. This rule maintains a strong feasible basis and ensures that the algorithm goes through at most k consecutive degenerate pivots. We can classify the advantages of this algorithm as follows: this new algorithm is similar to (LRC) and (LRS) enter-

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ing variable rules and it does not require negative cost augmenting cycle to identify entering variables and it can guarantee that after at most k consecutive degenerate pivots we can achieve a non-degenerate pivot.

2 Network simplex algorithm for the minimum cost flow

This Section reviews some basic definitions presented in Ahuja [1]. Let $G = (N, A)$ be a directed network defined by a set N of n nodes and a set A by m directed arcs. Each arc $(i, j) \in A$ is associated with the cost c_{ij} denoting the cost per unit flow on that arc. Each node $i \in N$ is associated with number $b(i)$ representing its supply/demand. A path in $GN, A)$ is a sequence of nodes and arcs $i_1, (i_1, i_2), i_2, (i_2, i_3)...(i_{r-1}, i_r)i_r$ satisfying the property that either $(i_k, i_{k+1}) \in A$ or $(i_{k+1}, i_k) \in A$, for each k, $l \le k \le r-1$, and all nodes visited are distinct. For simplicity's sake, we refer to a path by a sequence of nodes i_1 to i_r. A directed path is a sequence of nodes i_1 to i_r such that $(i_k, i_{k+1}) \in A$ for each k, $l \le k \le r-1$. A directed cycle is the directed path $i_1, ..., i_r$ together with the arc (i_r, i_1), and a cycle is a path together with arc (i_r, i_1) or (i_1, i_r).

Considering directed arc (i, j), we call node i as from node and node j as end-node for this arc. A network is considered connected if the network contains at least one path between any two nodes. A tree is a connected graph that includes no cycle. Subgraph B is a spanning tree of G if B is a tree of G containing all of its nodes. In a spanning tree, there is a unique path between any pair of nodes. The minimum cost flow problem can be formulated as the following optimization problem:

$$Min \sum_{(i,j)\in A} c_{ij}x_{ij} \qquad (1a)$$

$$s.t \sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b_i \quad \forall i \in N (1b)$$

$$x_{ij} \ge 0 \quad \forall (i,j) \in A \qquad (1c)$$

A basic solution for the minimum cost flow problem is to partition the set of arcs into two parts: the set B of basic arcs, the set L of nonbasic arcs in their lower bounds. B is referred to as a basis. A basis structure (B, L) is called feasible if by setting $x_{ij} = 0$ for each arc $(i, j) \in L$, problem $(1a)$ includes a feasible flow x satisfying $(1b)$ and $(1c)$. It is assumed that the basis is hanging from a specific node called the root node (the root node can be selected arbitrarily). The tree arcs are either upward pointing (towards the root) or downward pointing (away from the root). Arc $(i, j) \in L$ can be considered a free arc if $x_{ij} > 0$, otherwise it is a restricted arc. It should be observed that all non basic arcs are restricted arcs, whereas basic arcs can be either free or restricted. We say that a basis solution is degenerate if it deals with restricted arc. The subgraph obtained from G where restricted arcs are deleted defines a forest (that is, a set of sub trees). We refer to each subtree in the forest as a component.

3 The upper bound of consecutive degenerate pivots

At first the two previously used methods are summarized.

In a basis solution like x, in which n is a number of arcs and m is the number of nodes in the network, there are k degenerate arcs. In this case it is clear that after at most $\binom{n-m+k}{k}$ consecutive degenerate pivots, this extreme point is deposited, or optimized. The aim is to reduce the upper bound of this consecutive degenerate pivots and reach to a polynomial bound according to n, m. Therefore it is assumed that, $B_0, B_1, ..., B_M$ are all the basis showing this extreme point, as it is known, in the network, each basis corresponds to a root spanning tree. Let $T_0, T_1, ..., T_M$ be the basis trees in a sequence of consecutive degenerate pivots, so optimality is recognized at T_M or the pivot executed on T_M is nondegenerate. It should be noticed that cycling prevention rule makes all of these trees strongly feasible, and according to the property of a strong tree, we can say that, in each strongly feasible basis tree, the orientation of each zero arc must be towards the root. On the other hand, because the pivots are degenerate, the non-degenerate basic arcs and the spanning graph has been formed by these non-degenerate basic arcs are common in all of the trees $(T_0, T_1, ..., T_M)$. Let these components be connected to each other via $p-1$ degenerate arcs, it is clear that $p \le m$. Let L be the maximum number of degenerate arcs in path from any node to root node in T_M , also the level of each node is the number of degenerate basic arcs that connected this node to the root. It is clear that all the nodes that are in the same component are of the same level. Therefore, the level of each component is equal to the level of nodes which it contains. Till now, the components can be achieved and the level of nodes and components can be defined. For each pivot, the arc that is true in $Maxz_{ij} - c_{ij} \geq 0 \mid (i,j) \in L$ will be an entering variable; with this rule of entering variable, we have at most $\binom{n-m+k}{k}$ consecutive degenerate pivots. Also to determine the leaving arc, adding entering arc to the basis forms the unique cycle W, then the orientation of W is defined along the entering arc, now the maximum possible flow is sent in the cycle along the orientation of W without violating any constraints of arcs, so some arcs in the cycle block further increase in the flow. Such arcs are called blocking arcs. If the blocking arc is unique, then it is selected as the leaving arc, and if there are several blocking arcs, according to cycling preventions rule, the algorithm would select the blocking arc farthest from the entering arc, when arcs are traversed along the orientation of the cycle. The above rule guarantees that the next basis is also strongly feasible. Researchers have changed the entering variable rule, and have founded 5 better upper bounds like mn and m^2 for a sequence of consecutive degenerate pivots. Now these rules will be summarized. Before that the two definitions are explained as followed: **Stage**: it is a subset of consecutive basis of $B_0, B_1, ..., B_M$ so that, each subset makes at least one level of T_M . For example, the first stage begins from B_0 that is corresponding to T_0 , and after making the first level of T_M, this stage will be finished, let this stage finish in T_S . So this stage contains basis $B_0, B_1, ..., B_S$, and second stage begins from B_{S+1} . **Stage length**: it is a number of bases in each stage. Now the aim is to present a new entering variable rule. To meet this aim, all the network arcs are stored in a one-dimensional array in a special order, and the arcs are entered according to their order. Two special orders in one V dimensional arrays which are corresponding to (LRC) and (LRS) rules should be explained, after that a new special order is represented and it will be shown that with this order, a smaller upper bound for a sequence of consecutive degenerate pivots can be achieved. Assume that $\bar{w}$ is dual values in T_M and arc (p,q) is a leaving arc in T_j . After omitting this arc in T_j , two subtrees will be there. It is clear that one of them does not contain the root; we call this sub tree as T_1. As it is known, after each pivot, the result tree must be a strong tree, therefore the direction of entering arc is towards the root, it means that its direction is from T_1 to $T - T_1$ (T_1 and $T - T_1$ shown in Figure **??**.

It has been proved by Bazaraa [2] that after each pivot the dual values of the nodes in T_1 are strictly decreasing, and the nodes in $T - T_1$ remain the same so the sum of dual values is strictly decreasing. Now such information is classified in three properties as follows:

Property 1: In each pivot, since the leaving arc is degenerate, the sum of dual value is strictly decreasing.

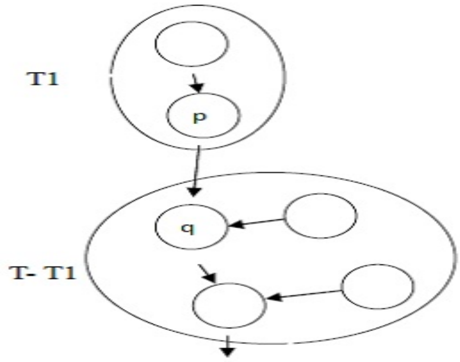

Figure 1

Property 2: In each tree of $T_0, T_1, ..., T_M$ for all nodes that are in root component we have $\bar{w}_i = w_i$

Property 3: T_M has smaller dual value. In other words, the dual values of nodes in any sub graph are not equal to their values in $\bar{w}$; they are strictly bigger than these values.

4 The special order in one dimensional array that resulted in LRC rule

Let arc (p,q) have entry criterion basis, and it be on the first level in T_0 . In other words, node p is an element of root component in T_0 . Notice that if $z_{pq} - c_{pq} \geq 0$, then arc (p,q) has entry criterion basis. Now set arc (p,q) as a first element of one-dimensional array, and after that without any special ordering, all of the network arcs are set as an element in this array.
According to this order, there is one-dimensiona array that exactly has n arcs and its first arc is (p,q). Now the entering arc is selected from this array. If the first arc should have the entry criterion basis, it would be selected as the entering arc, otherwise; this property will be checked for second arc and so on.
It is clear that in this array the entry criterion basis is checked for all of the network arcs, so after at most n consecutive degenerate pivots, this array will be finished. Assume that this array will finish in T_S . Bazaraa [2] proved that after finishing the first array, all of the components that are at first level in T_M are at the first level in T_S as well, and for all of the nodes for these components we have $\bar{w}_i = w_i$. It means that, with this array, the first level in T_M could be made. According to the definition of stage, it is clear that the first stage is finished in T_S and it contains $B_0, B_1, ..., B_S$ basses.

Bazaraa [2] proved by induction that after finishing each stage, it made one level in T_M. We had shown that the number of pivots in a stage is not more than n, and the number of stages in a sequence of degenerate pivots is bounded above by m (because T_M has at most m level). So the number of consecutive degenerate pivots with (LRC) entering variable rule bounded above by polynomial mn.

5 The special order in one dimensional array resulted through (LRS) rule

There is another special order for network arcs in one dimensional array which reduces the above bound to m^2. Assume arc (p, q) at first level in T_0 should have entry criterion basis. Now among all of the network arcs the arcs which have node p as their from node are selected, and these arcs are introduced as the first arcs in array, after that for each node in network it should be repeated.

5.1 Entering variable rule

For each node, for example node i, according to their order in array we select entering arc among enterable arcs, which they have node i as their from-node and it has to be maximum $z_{ij} - c_{ij}$ from these arcs.
Up to now, all of the network arcs are stored in one-dimensional array and represented an entering variable rule that corresponds to (LRS) rule. Now assume according to entering variable rule that arc (i, j) among all of the arcs, that node i is their from-node, must enter the basis. It has been proved in Bazaraa [2] that after entering arc (i, j) among the arcs set which have node i as their from-node, there will not be any enterable arc from this set. It means from any node, at most one arc can enter the basis. According to our assumption, this network includes m nodes, so in each array there are at most m pivots. It means, the length of each stage is at most m. As it is known, the number of stages in a sequence of consecutive degenerate pivot is bounded by m. Therefore, there are at most m stages with length m. So after at most m^2 consecutive degenerate pivots T_M is achieved.

6 The new method for unit-stalling

In previous section two entering variable rules similar to (LRC) and (LRS) were introduced, these rules could reduce the exponential upper bounds corresponding to a sequence of consecutive degenerate pivots to polynomial upper bounds like mn and m^2. Now in this section, a new special order for arcs in one dimensional array will be proposed, and a new entering variable rule, after that it will be shown that with these rules the upper bound corresponding to a sequence of consecutive degenerate pivot is reduced to k. In other words, after at most k consecutive degenerate pivots we can achieve T_M. At first we explain some of the properties that we need in theorem formed as follow:In previous section two entering variable rules similar to (LRC) and (LRS) were introduced, these rules could reduce the exponential upper bounds corresponding to a sequence of consecutive degenerate pivots to polynomial upper bounds like mn and m^2. Now in this section, a new special order for arcs in one dimensional array will be proposed, and a new entering variable rule, after that it will be shown that with these rules the upper bound corresponding to a sequence of consecutive degenerate pivot is reduced to k. In other words, after at most k consecutive degenerate pivots we can achieve T_M. At first we explain some of the properties that we need in theorem formed as follow:

Theorem 6.1 *The number of degenerate arcs is equal to the number of components.*

Proof. In the first section, it was illustrated how to achieve the components and it was assumed that p is a number of components. Now if we consider each component as a node, then the spanning tree corresponding to these nodes will have exactly $p - 1$ arcs. Now after rooting this tree, a root spanning tree with p nodes and p arcs is obtained which according to the previous information all of these arcs are degenerate (notice that the degenerate arc is a zero basis arc). Therefore, it will be clear that the number of degenerate arcs is equal to the number of components. Now before representing the theorem number 2, this point should be agreed upon: Considering the arc (i, j), and assume that node i is a node in component k and j is a node in G_s, so $k \neq s$. So according to this assumption we agree that arc (i, j) exits from G_k and enters G_s.

Theorem 6.2 *In each basis tree from each component, there exits exactly one degenerate arc.*

Proof. By contradiction, suppose that there is one component like G_K which has two exiting degenerate arcs. There are two cases to consider:

Case1: these two degenerate arcs enter one component like G_s. As we know, each tree has p components and $p - 1$ degenerate arcs (except root arc). Now if there is a component with two exiting degenerate arcs which they enter one component, then there could be a component that is not connected with other components. This is in contrast with being a spanning tree.

Case2: These two degenerate arcs enter the two different components like G_S and G_t. we illustrate an example of this case in Figure 2. We assume without loss of generality that G_s is in a unique path which connects G_K to root, therefor the path that connects G_t to root has one degenerate arc from G_k to G_t which is away from the root and is in contrast with being a strong tree.

Now considering the above theorems, a new special order in one-dimensional array will be introduced and a new entering variable rule. Suppose that we are in To. At first considering all of the nodes which are in root component, now by selecting all of the zero arcs flow in this feasible basis solution (it is clear that, $n - m + k$ is the number of these arcs), and choosing from these $n - m + k$ arcs, the arcs which have their end-node in root component, then are stored in one set like F.

6.1 The new special order in one dimensional array

Suppose that arc (p, q) is at first level in T_o and should have entry criterion basis, it means that $z_{pq} - c_{pq} \geq 0$ or the dual value of node p is strictly bigger than $\bar{w}$ (according to property 3). Now suppose that, node p is in G_k and at first level in T_o, considering all of the nodes which are in G_k, so introduce all of the arcs which would have these nodes as their from-node as first arcs in array. After that, this process should be continued for other components. Now we have one-dimensional array with at most n arcs.

6.2 The entering variable rule

It will enter arc, when it satisfies these two following restrictions:

(i) The arc must have maximum $z_{ij} - c_{ij} \geq 0$ among all of the arcs which have the same component for their from-node.

(ii) The arc must be in set F.

If an arc exists that satisfies in the two above conditions then it enters, otherwise; we do not have an entering arc from this component and next component arcs must be checked. Assume that the first array finishes in T_S, the method to write the second array, is exactly like the first array. Also it will be shown, after finishing the first array, the first level of T_M is made. So it is clear that for all of the nodes in common components at first level in T_S and T_M there is $\bar{w}_i = w_i$ in T_S. But set F will change in each stage, for the clarification of this case, set F is represented which corresponds to the second stage. Considering the components in T_S, which are at first level in T_S, and for all of the nodes that are in these components we have $\bar{w}_i = w_i$. Now selecting from zero arcs, the arcs that their end-node is in these components, introduce these arcs as elements of set F. Therefore, for each stage, like above the set F has to be obtained.

Theorem 6.3 *Suppose that there is a sequence of consecutive degenerate pivots by considering new array and new entering variable rule, we will have the entering arc in each pivot which is the basic arc in T_M.*

In other words, it will be shown that by considering (i) and (ii) restrictions for entering variable, the entering arc from each component is the basic arc in T_M.

Proof.

As we know there is at least one component at first level in T_M. Assume, while the generality is preserved, that G_k is at first level in T_M and restrictions (i) and (ii) in G_k happens on arc (p, q) in T_0, so q is into root component and p is into component G_k. it means that the maximum $z_{ij} - c_{ij} \geq 0$ among the arcs, which have G_k nodes as their from-node, happens on arc (p, q). According to theorem 6.2, in each basis tree, from each component, exactly one degenerate arc exits. So, we should proof that arc (p, q) is basis arc in T_M. If arc (p, q) be a basis arc in T_M the proof is complete. Otherwise, suppose that one other zero arc like (f, t) which f is into component G_k and t is into root component will be a basis arc in T_M.

As we know, $z_{pq} - c_{pq} = w_p - w_q - c_{pq}$ and because q is in root component then $w_q = \bar{w}_q$. Beside this, according to our assumption arc (p, q) has maximum $z_{ij} - cij \geq 0$ among the arcs, which have G_k nodes as their from-node. After entering arc (p, q) we will have

$$(z_{pq} - c_{pq})_{NEW} = (w_p - \bar{w}_q - c - pq) - \delta_{pq} = 0$$

and for arc (f, t) we will have

$$(z_{ft} - c_{ft})_{NEW} = (w_f - w_t - c_{ft}) - \delta_{pq} =$$

$$(w_f - \delta_{pq}) - \bar{w}_t - c_{ft} = w_f^* - \bar{w}_t - c_{ft} < 0 \qquad (a)$$

. By the way, we have assumed arc (f, t) as basis arc in T_M, so in T_M,

$$z_{ft} - c_{ft} = \bar{w}_f - \bar{w}_t - c_{ft} = 0 \qquad (b)$$

With compering equations (a) and (b) we can conclude that $w_f^* < w_f$ and it is in contrast with property 3. After proving the above theorem, it can be inferred that after finishing the first array the first level of T_M will make. This follows because in first array just basic arcs at first level in T_M can enter the basis. It means that the zero basic arcs, as they are at first level in T_M, also are at first level in T_S (on the supposition that first array finishes in T_S). So by induction, simply we can show that after finishing each array, one level of T_M will make.

6.3 The upper bound of consecutive degenerate pivots

As it is known, there is a sequence of consecutive degenerate pivots; therefore just the arcs corresponding to zero flow in a feasible basis solution can enter the basis. Supposing that k is the number of degenerate arcs (it means there are exactly k zero arc in basis), so $n-m+k$ will be the number of zero arc in this solution. It is clear that in each pivot, one of these zero arcs enters the basis. Now it can be claimed that if one zero arc enters the basis then it could not be leaving arc in all the next degenerate pivots in this sequence. To prove this claim assume that arc (p,q) in stage C enters the basis. Now it will be shown that this arc cannot leave the basis in all next stages or all the next pivots. According to the entering variable rule, the arc that satisfies in (i) and (ii) restrictions enters the basis, also according to theorem 6.3, after entering arc (p,q) there will be $w_p = \bar{W}_p$ and $w_q = \bar{W}_q$. Now suppose that arc (p,q) in one of next degenerate pivots, for example in T_j leaving the basis, so node p will be in subtree T_1 (T_1 is a subtree from T_j which does not contain root node), also according to property 1, for node p we have $w_p < \bar{W}_p$ and it is in contrast with T_M dual values. Till now it was shown that in each degenerate pivot from this sequence the entering arc is basis arc in T_M and the entering arc cannot be a leaving arc in all of the next degenerate pivots. On the other hand, k is the number of degenerate arcs in the basis, it means there are exactly k zero arcs in basis so after at most k consecutive degenerate pivots we can achieve T_M, or after at most k consecutive degenerate pivots there will be a non-degenerate pivot.

7 Conclusion

This paper advances a new algorithm featuring a new special order in one-dimensional array and a new entering variable rule for anti-stalling. This article has proved that in each sequence of consecutive degenerate pivots, the entering arc in each pivot is basic arc in final tree or T_M and it cannot be a leaving arc in all of the next degenerate pivots in this sequence. In addition, with this new algorithm we will have at most k consecutive degenerate pivots or after at most k consecutive degenerate pivots we can achieve non-degenerate pivots.

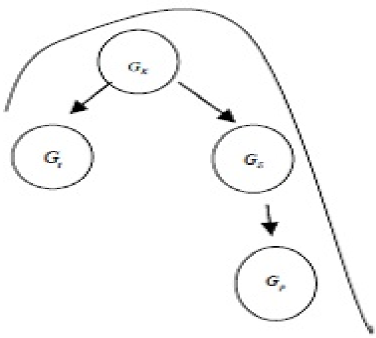

Figure 2: The unique path from G_t to root.

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