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Research Article

A Piecewise Approximate Method for Solving Second Order Fuzzy Differential Equations Under Generalized Differentiability

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Abstract

In this paper a numerical method for solving second order fuzzy differential equations under generalized differentiability is proposed. This method is based on the interpolating a solution by piecewise polynomial of degree 4 in the range of solution. Moreover we investigate the existence, uniqueness and convergence of approximate solutions. Finally the accuracy of piecewise approximate method by some examples are shown.

Keywords : Generalized differentiability; Numerical Solution; Fuzzy Differential Equations.

1 Introduction

Fuzzy differential equations (FDE) are a suitable tool to model problem in science and engineering in which uncertainties or vagueness pervade. There are many idea to define a fuzzy derivative and in consequence, to study FDE. The first and most popular approach is using the Hukuhara differentiability for fuzzy valued function. Kaleva in [19] proposed FDE using Hukuhara derivative and it was developed by some other authors [15, 23]. Hukuhara differentiability has the drawback that the solution of FDE need to have increasing length of its support, so in order to overcome this weakness, Bede and Gal [9], introduced the strongly generalized differentiability of fuzzy valued function. This concept allows us to solve the above-mentioned shortcoming, also the strongly generalized derivative is defined for a larger class of fuzzy valued functions than the Hukuhara derivatives.

Many researchers some numerical method for solving FDE under Hukuhara differentiability presented in [1, 2, 5], and under generalized differentiability investigated in [6, 7]. Higher-order fuzzy differential equations with Hukuhura differentiability were presented in [18, 13, 3, 4]. Khastan in [20], proposed a analytic method to solve higher-order fuzzy differential equations based on the selection different type of derivatives, they obtained several solution to fuzzy initial value problem. In this paper a numerical method for second order fuzzy differential equations is proposed. The idea of this method is based on interpolating the solution by polynomial of degree 4 in the range of solution, the step size used is of length $H = 3h$. Also existence, uniqueness and convergency of the approximate solutions are proved.

The paper is organized as follows: In section 2, some basic definitions are brought. A proposed method for solving second order fuzzy differential equations is introduced also the existence,

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uniqueness and convergency are proved in section 3. A numerical example are presented in section 4 and finally conclusion is drawn.

2 Notation and definitions

First notations which shall be used in this paper are introduced.

We denote by $\mathbb{R}_{\mathcal{F}}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which are defined over the real line.

For $0 < r \leq 1$, set $[u]^r = \Big\{t \in \mathbb{R}\Big| u(t) \geq r\Big\}$, and $[u]^0 = cl\Big\{t \in \mathbb{R}\Big| u(t) > 0\Big\}$. We represent $[u]^r = [u^-(r), u^+(r)]$, so if $u \in \mathbb{R}_{\mathcal{F}}$, the r-level set $[u]^r$ is a closed interval for all $r \in [0,\ 1]$. For arbitrary $u,\ v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u+v]^r = [u]^r + [v]^r$, $[ku]^r = k[u]^r$ respectively.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $u^-(r) = a + (b-a)r$ and $u^+(r) = c - (c-b)r$ are the endpoints of r-level sets for all $r \in [0, 1]$.

Definition 2.1 *[16]The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as*

$$D(u,\ v) = \sup_{r\in[0,\ 1]} \max\Big\{|u^-(r) - v^-(r)|,\ |u^+(r) - v^+(r)|\Big\}. \quad (2.1)$$

Consider $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, then the following properties are well-known for metric D,

1. *$D(u \oplus w,\ v \oplus w) = D(u,\ v)$, for all $u, v, w \in \mathbb{R}_{\mathcal{F}}$,*
2. *$D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v)$, for all $u, v \in \mathbb{R}_{\mathcal{F}}$, $\lambda \in \mathbb{R}$*
3. *$D(u \oplus v,\ w \oplus z) \leq D(u,\ w) + D(v,\ z)$, for all $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$,*
4. *$D(u \ominus v,\ w \ominus z) \leq D(u,\ w) + D(v,\ z)$, as long as $u \ominus v$ and $w \ominus z$ exist, where $u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}}$.*

where, $\ominus$ is the Hukuhara difference(H-difference), it means that $w \ominus v = u$ if and only if $u \oplus v = w$.

Definition 2.2 *[9] Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) & u = v + w, \\ & or \\ (ii) & v = u + (-1)w, \end{cases}$$

Then w is called the generalized Hukuhara difference of u and v.

Remark 2.1 *[9] Throughout the rest of this paper, we assume that $u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$.*

Note that a function $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$ is called fuzzy-valued function. The r-level representation of this function is given by $f(t; r) = [f^-(t; r)\ ,\ f^+(t; r)]$, for all $t \in [a, b]$ and $r \in [0, 1]$.

Definition 2.3 *([16]) A fuzzy valued function $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $t_0 \in [a, b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(t), f(t_0)) < \epsilon$, whenever $t \in [a, b]$ and $|t - t_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $t_0 \in [a,\ b]$.*

Definition 2.4 *([12]) The generalized Hukuhara derivative of the fuzzy-valued function $f : (a, b) \rightarrow \mathbb{R}_{\mathcal{F}}$ at $t_0 \in (a, b)$ is defined as*

$$f'_{gH}(t_0) = \lim_{h\to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h}. \quad (2.2)$$

If $f'_{gH}(t_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (2.2) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at t_0.

Definition 2.5 *([12]) Let $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ and $t_0 \in (a, b)$, with $f^-(t; r)$ and $f^+(t; r)$ both differentiable at t_0 for all $r \in [0, 1]$. We say that*

- *f is $[(i) - gH]$-differentiable at t_0 if*

$$f'_{i.gH}(t_0; r) = [(f^-)'(t_0; r)\ ,\ (f^+)'(t_0; r)], \quad (2.3)$$

- *f is $[(ii) - gH]$-differentiable at t_0 if*

$$f'_{ii.gH}(t_0; r) = [(f^+)'(t_0; r)\ ,\ (f^-)'(t_0; r)]. \quad (2.4)$$

Definition 2.6 *([12]) We say that a point $t_0 \in (a, b)$, is a switching point for the differentiability of f, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that*

type(I) *at t_1 (2.3) holds while (2.4) does not hold and at t_2 (2.4) holds and (2.3) does not hold, or*

type(II) *at t_1 (2.4) holds while (2.3) does not hold and at t_2 (2.3) holds and (2.4) does not hold.*

Theorem 2.1 *[6] Let $T = [a, a+\beta] \subset \mathbb{R}$, with $\beta > 0$ and $f \in \mathcal{C}^n_{gH}([a,b],\ \mathbb{R}$ F). For $s \in T$*

(i) *If $f^{(i)}$, $i = 0, 1, \dots, n-1$ are $[(i)-gH]$-differentiable, provided that type of gH-differentiability has no change. Then*

$$f(s) = f(a) \oplus f'_{i.gH}(a) \odot (s-a) \oplus f''_{i.gH}(a) \odot \frac{(s-a)^2}{2!} \oplus \dots \oplus f^{(n-1)}_{i.gH}(a) \odot \frac{(s-a)^{n-1}}{(n-1)!} \oplus R_n(a,s),$$

where

$$R_n(a,s) := \int_a^s \Big(\int_a^{s_1} \dots \Big(\int_a^{s_{n-1}} f^{(n)}_{i.gH}(s_n) ds_n \Big) ds_{n-1} \dots \Big) ds_1.$$

(ii) *If $f^{(i)}$, $i = 0, 1, \dots, n-1$ is $[(ii)-gH]$-differentiable, provided that type of gH-differentiability has no change. Then*

$$f(s) = f(a) \ominus (-1) f'_{ii.gH}(a) \odot (s-a) \ominus (-1) f''_{ii.gH}(a) \odot \frac{(a-s)^2}{2!} \ominus (-1) \dots \ominus (-1) f^{(n-1)}_{ii.gH}(a) \odot \frac{(a-s)^{n-1}}{(n-1)!} \ominus (-1) R_n(a,s),$$

where

$$R_n(a,s) := \int_a^s \Big(\int_a^{s_1} \dots \Big(\int_a^{s_{n-1}} f^{(n)}_{ii.gH}(s_n) ds_n \Big) ds_{n-1} \dots \Big) ds_1.$$

(iii) *If $f^{(i)}$ are $[(i)-gH]$-differentiable for $i = 2k-1$, $k \in \mathbb{N}$, and $f^{(i)}$ are $[(ii)-gH]$-differentiable for $i = 2k$, $k \in \mathbb{N} \cup \{0\}$. Then*

$$f(s) = f(a) \ominus (-1) f'_{ii.gH}(a) \odot (s-a) \oplus f''_{i.gH}(a) \odot \frac{(a-s)^2}{2!} \ominus (-1) \dots \ominus (-1) f^{(\frac{i-1}{2})}_{ii.gH}(a) \odot \frac{(a-s)^{\frac{i}{2}-1}}{(\frac{i}{2}-1)!} \oplus f^{(\frac{i}{2})}_{i.gH}(a) \odot \frac{(a-s)^{\frac{i}{2}}}{(\frac{i}{2})!} \ominus (-1) \dots \ominus (-1) R_n(a,s),$$

where

$$R_n(a,s) := \int_a^s \Big(\int_a^{s_1} \dots \Big(\int_a^{s_{n-1}} f^{(n)}_{i.gH}(s_n) ds_n \Big) ds_{n-1} \dots \Big) ds_1.$$

(iv) *Suppose that $f \in \mathcal{C}^n_{gH}([a,b],\ \mathbb{R}$ F) ,$n \geq 3$.*

Furthermore let f in $[a,\xi]$ is $[(i)-gH]$-differentiable and in $[\xi, b]$ is $[(ii)-gH]$-differentiable, in fact ξ is switching point type I for first order derivative of f and $t_0 \in [a,\xi]$ in a neighborhood of ξ. Moreover suppose that second order derivative of f in ζ_1 of $[t_0, \xi]$ have switching point type II. Moreover type of differentiability for $f^{(i)}$, $i \leq n$ on $[\xi,\ b]$ don't change. So

$$\begin{aligned} f(s) &= f(t_0) \oplus f'_{i.gH}(t_0) \odot (\xi - t_0) \ominus f''_{ii.gH}(t_0) \odot (t_0 - \zeta_1) \odot (\xi - t_0) \oplus f''_{i.gH}(\zeta_1) \Big(\frac{(\xi-\zeta_1)^2}{2} - \frac{(t_0-\zeta_1)^2}{2} \Big) \\ &\odot \ominus (-1) f'_{ii.gH}(\xi) \odot (s-\xi) \ominus (-1) f''_{ii.gH}(\xi) \odot \frac{(s-\xi)^2}{2!} \\ &\ominus (-1) \int_{t_0}^{\xi} \Big(\int_{t_0}^{\zeta_1} \Big(\int_{t_0}^{s_2} f'''_{ii.gH}(s_4) ds_4 \Big) ds_2 \Big) ds_1 \\ &\oplus \int_{t_0}^{\xi} \Big(\int_{\zeta_1}^{s_1} \Big(\int_{\zeta_1}^{s_3} f'''_{i.gH}(s_5) ds_5 \Big) ds_3 \Big) ds_1 \\ &\ominus (-1) \int_{\xi}^{s} \Big(\int_{\xi}^{t_1} \Big(\int_{t_0}^{t_2} f'''_{ii.gH}(t_3) dt_3 \Big) dt_2 \Big) dt_1. \end{aligned}$$

3 Piecewise Approximate Method (PWA Method)

Consider the following second order fuzzy differential equation

$$\begin{cases} y''(t) = f(t, y(t)), & t \in I = [0, T], \\ y(0) = y_0, y'(0) = y'_0, \end{cases} \tag{3.5}$$

where the derivative $y^{(i)}$, $i = 1, 2$, is considered in the sense of gH-differentiable, where at the end points of I we consider only the one-sided derivatives, and the fuzzy function $f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is sufficiently smooth function. The initial data y_0, y'_0 are assumed in $\mathbb{R}_{\mathcal{F}}$. The interval I may be $[0, T]$ for some $T > 0$ or $I = [0, \infty)$. We assume that $f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy function, such that there exists $k > 0$ such that

$$D(f(t, x), f(t, z)) \leq kD(x, z), \quad \forall t \in I, \ x, z \in \mathbb{R}_{\mathcal{F}}. \tag{3.6}$$

Our construction of the fuzzy approximate solution $s(t)$ is as follows:
let $y(t)$ be the fuzzy solution of (3.5) determined by the fuzzy initial value problem y_0 and y'_0 . We divided the range of solution $[0, T]$ into subintervals of equal length $H = 3h = \frac{T}{n}$, and let $I_k = [kH, (k + 1)H]$, where $k = 0, \cdots, n - 1$. Let $s(t)$, $0 \leq t \leq T$ be a fuzzy approximate function of degree m.
In this paper we assume that $m = 4$, and we approximate fuzzy solution of (3.5) by fuzzy piecewise polynomial of order 4. Piecewise approximate solution $s(t)$ on $I_k = [kH, (k + 1)H]$, is construct step by step as follows:

Step 1: We define the first component of $s(t)$ by $s_0(t)$, in three cases:

Case(i): Let us suppose that the unique solution of problem (3.5), $y(t)$ is $[(i) - gH]$-differentiable, therefore

$$s_0(t) = y(0) \oplus t \odot y'_{i.gH}(0) \oplus \sum_{i=2}^{4} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.7}$$

for $0 \leq t \leq H$,

Case(ii): Now, consider $y(t)$ is $[(ii) - gH]$-differentiable, then $s_0(t)$ is obtained as follows:

$$s_0(t) = y(0) \ominus (-1)t \odot y'_{ii.gH}(0) \oplus \sum_{i=2}^{4} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.8}$$

for $\quad 0 \leq t \leq H$,

In Eqs (3.7) and (3.8), the coefficients $\alpha_{i,0}$ for $i = 2, 3, 4$ as yet undetermined and to be obtained where $s_0(t)$ satisfy the relations:

$$s''_0(jh) = f(jh, s_0(jh)), \tag{3.9}$$

for $j = 1, 2, 3$. By using Hausdorff distance(2.1), for $j = 1, 2, 3$ we obtain:

$$(s_0^-)''(jh, r) = f^-(jh, s_0(jh, r)), \tag{3.10}$$

$$(s_0^+)''(jh, r) = f^+(jh, s_0(jh, r)), \tag{3.11}$$

by solving (3.10) and (3.11), the value of $\alpha_{i,0}$ for $i = 2, 3, 4$ are obtained and $s_0(t)$ is constructed.

Step 2: The approximate solution $s(t)$ in interval $[H, 2H]$ is obtained as follows:

$$s(t) = \sum_{i=0}^{1} s_0^{(i)}(t) \odot \frac{(t - H)^i}{i!} \oplus \sum_{i=2}^{4} \alpha_{i,k} \odot \frac{(t - H)^i}{i!}, \tag{3.12}$$

where $s_0(t)$ is obtained by step 1. The value of $\alpha_{i,k}$ are to be determined where $s(t)$ satisfy the following relations:

$$s''(jh) = f(jh, s(jh)). \tag{3.13}$$

This means for $j = 4, 5, 6$,

$$(s^-)''(jh, r) = f^-(jh, s(jh, r)), \tag{3.14}$$

$$(s^+)''(jh, r) = f^+(jh, s(jh, r)), \tag{3.15}$$

by solving (3.14) and (3.15), the values of $\alpha_{i,k}$ are obtained.

Step 3: The approximate solution $s(t)$ in interval $[kH, (k + 1)H]$ for $k = 2, \cdots, n - 1$ is obtained as follows:

$$s(t) = \sum_{i=0}^{1} s_{3k}^{(i)}(t) \odot \frac{(t - kH)^i}{i!} \oplus \sum_{i=2}^{4} \alpha_{i,k} \odot \frac{(t - kH)^i}{i!}, \tag{3.16}$$

The value of $\alpha_{i,k}$ are to be determined where $s(t)$ satisfy the following relations:

$$s''(jh) = f(jh, s(jh)). \tag{3.17}$$

This means for $j = 3k+1, 3k+2, 3k+3$; $k = 2, \cdots, n-1$,

$$(s^-)''(jh, r) = f^-(jh, s(jh, r)), \tag{3.18}$$

$$(s^+)''(jh, r) = f^+(jh, s(jh, r)), \tag{3.19}$$

by solving (3.18) and (3.19), the values of $\alpha_{i,k}$ are obtained.

Finally the PWA method is obtained as follows

$$s(t) = \sum_{i=0}^{1} s_{3k}^{(i)}(t) \odot \frac{(t-kH)^i}{i!} \oplus \sum_{i=2}^{4} \alpha_{i,k} \odot \frac{(t-kH)^i}{i!}, \tag{3.20}$$

where

$$s_0(t) = y(0) \oplus t \odot y'_{i.gH}(0) \oplus \sum_{i=2}^{4} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.21}$$

if $y(t)$ is $[(i)-gH]-differentiable$.

$$s_0(t) = y(0) \ominus (-1)t \odot y'_{ii.gH}(0) \oplus \sum_{i=2}^{4} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.22}$$

if $y(t)$ is $[(ii)-gH]-differentiable$.

3.1 Existence and uniqueness

In this section we prove that there exist a unique fuzzy function $s(t)$ where approximate the solution of second order fuzzy differential equation (3.5), provided that the size of the subinterval h satisfies some constraints.

Theorem 3.1 *If $h = \min\{h_1, h_2, h_3\}$, where*

$$h_1 < \sqrt{\frac{2}{L}}, h_2 < \sqrt{\frac{6}{L}}, h_3 < \sqrt{\frac{24}{L}} \tag{3.23}$$

then the approximate solution defined by (3.20), exists and unique.

Proof: Let $t = jh$ and $j = 3k+\eta$ for $\eta = 1,2,3$, therefore

$$s''((3k+\eta)h) = s''_{3k+\eta} = \sum_{i=2}^{4} \alpha_{i,k} \frac{(\eta h)^{i-2}}{(i-2)!} \tag{3.24}$$

By solving system (3.24) we obtain:

$$\alpha^+_{2,k} = 3(s^+_{3k+1})'' - 3(s^+_{3k+2})'' + (s^+_{3k+3})'', \tag{3.25}$$

$$\alpha^+_{3,k} = \frac{1}{h}[-\frac{5}{2}(s^+_{3k+1})'' + 4(s^+_{3k+2})'' - \frac{3}{2}(s^+_{3k+3})''], \tag{3.26}$$

$$\alpha^+_{4,k} = \frac{1}{h^2}[(s^+_{3k+1})'' - 2(s^+_{3k+2})'' + (s^+_{3k+3})''], \tag{3.27}$$

and

$$\alpha^-_{2,k} = 3(s^-_{3k+1})'' - 3(s^-_{3k+2})'' + (s^-_{3k+3})'', \tag{3.28}$$

$$\alpha^-_{3,k} = \frac{1}{h}[-\frac{5}{2}(s^-_{3k+1})'' + 4(s^-_{3k+2})'' - \frac{3}{2}(s^-_{3k+3})''], \tag{3.29}$$

$$\alpha^-_{4,k} = \frac{1}{h^2}[(s^-_{3k+1})'' - 2(s^-_{3k+2})'' + (s^-_{3k+3})''], \tag{3.30}$$

To prove the existence and uniqueness of $s(t)$, let us define the operator $G_\nu : \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ by $\alpha_{j,k} = G_\nu(\alpha_{j,k})$ for $j = 2,3,4$ and $v = 1,2,3$. According to condition (3.6) and equations (3.25), (3.26), (3.27) and (3.28), (3.29), (3.30) we conclude that

$$D(G_1(\alpha_{2,k}), G_1(\alpha^*_{2,k}) \leq L\frac{h^2}{2} D(\alpha_{2,k}, \alpha^*_{2,k})|3-3+1|, \tag{3.31}$$

$$D(G_2(\alpha_{3,k}), G_2(\alpha^*_{3,k}) \leq L\frac{h^3}{6} D(\alpha_{3,k}, \alpha^*_{3,k})|\frac{1}{h}(-\frac{5}{2}+8-\frac{9}{2})|, \tag{3.32}$$

$$D(G_3(\alpha_{4,k}), G_3(\alpha^*_{4,k}) \leq L\frac{h^4}{24} D(\alpha_{4,k}, \alpha^*_{4,k})|\frac{1}{h^2}(\frac{1}{2}-4+\frac{9}{2})|, \tag{3.33}$$

From Equations (3.31), (3.32), (3.33), and

$$h_1 < \sqrt{\frac{2}{L}}, \quad h_2 < \sqrt{\frac{6}{L}}, \quad h_3 < \sqrt{\frac{24}{L}}$$

it follows that $G_\nu, \quad \nu = 1, 2, 3$ are contraction operators. This implies the existence and uniqueness of approximate solution under the stated conditions of theorem.

3.2 Consistency relations and convergence

It is well-known that a linear method will be convergent if and only if, It is both consistent and stable.

Theorem 3.2 *The piecewise approximate functions (3.20), are consistent.*

proof: In the case of [(i)-gH]-differentiability, $s(t)$ is defined on I_k as:

$$s(t) = \sum_{i=0}^{1} s_{3k}^{(i)}(t) \odot \frac{(t-3kh)^i}{i!} \oplus \sum_{i=2}^{4} \alpha_{i,k} \odot \frac{(t-3kh)^i}{i!}, \tag{3.34}$$

and the parametric form of $s(t) = (s^-(t,r), s^+(t,r))$ is as following:

$$s^-(t,r) = \sum_{i=0}^{1} \frac{(s_{3k}^-)^{(i)}(t)}{i!}(t-3kh)^i + \sum_{i=2}^{4} \frac{\alpha_{i,k}^-}{i!}(t-3kh)^i, \tag{3.35}$$

$$s^+(t,r) = \sum_{i=0}^{1} \frac{(s_{3k}^+)^{(i)}(t)}{i!}(t-3kh)^i + \sum_{i=2}^{4} \frac{\alpha_{i,k}^+}{i!}(t-3kh)^i, \tag{3.36}$$

without lose generality, we just proof consistency for s^+, and for s^- is similar.

On differentiating equation (3.36) and setting $t = jh$ with $j = 3k+1,\ 3k+2,\ 3k+3$, we obtain

$$(s^+)''((3k+\eta)h) = (s^+)''_{3k+\eta} = \sum_{i=2}^{4} \alpha_{i,k}^+ \frac{(\eta h)^{i-2}}{(i-2)!}, for \quad \eta = 1(1)3, \tag{3.37}$$

on eliminating $\alpha_{i,k}^+$, we obtain:

$$s_{3(k-1)}^+ - 2s_{3k}^+ + s_{3(k+1)}^+ = h^2\{\frac{405}{12}(s_{3k+1}^+)'' - \frac{486}{12}(s_{3k+2}^+)'' + \frac{189}{12}(s_{3k+3}^+)''\} \tag{3.38}$$

Hence, the associative polynomials $\rho(\xi)$ and $\sigma(\xi)$ are

$$\rho(\xi) = \xi^6 - 2\xi^3 + 1, \quad \sigma(\xi) = \frac{405}{12}\xi^4 - \frac{486}{12}\xi^5 + \frac{189}{12}\xi^6, \tag{3.39}$$

clearly $\rho(1) = 0, \rho'(1) = 0$ and $\rho''(1) = 2\sigma(1)$, and the method is consistent. Also the condition of stability is fulfilled since the zeros of $\rho(\xi)$ do not exceed unity in modulus, multiple zeros of multiplicity 2 and thus the method is convergent.

Table 1: Error of PWA method by Hausdorff distance in example 4.1

	Error of PWA method	
t	Case (i)	Case(ii)
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0

Table 2: Error of PWA method by Hausdorff distance in example 4.2

t	Case (i)	Case(ii)
0	0	0
0.1	0.000003073	0.0000030737
0.2	0.000007067	0.0000070678
0.3	0.000010994	0.0000109946
0.4	0.000018675	0.0000186745
0.5	0.000027282	0.0000272813
0.6	0.000035617	0.0000356173
0.7	0.000047701	0.0000477022
0.8	0.000060486	0.00006048718
0.9	0.000072667	0.0000726680

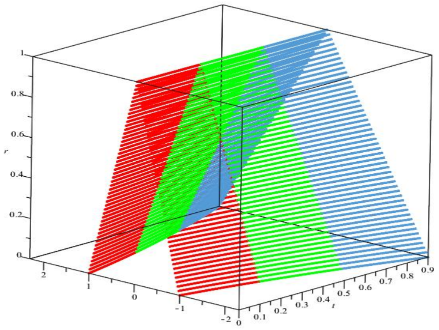

Figure 1: Approximate solution for case(i) in example 4.1. Red points: $s_0(t)$; Green points: $s_3(t)$; Blue. points: $s_6(t)$

4 Numerical Example

Example 4.1 *[20] Let us consider the following second-order fuzzy initial value problem*

$$y''(t) = \sigma_0, \quad y_0 = \gamma_0, \quad y'(0) = \gamma_1, \qquad (4.40)$$

where $\sigma_0 = \gamma_0 = \gamma_1$ are the triangular fuzzy number having r-level sets $[r-1, 1-r]$.

Case(i) If $y(t)$ is $[(i)-gH]$-differentiable, the real solution is:

$$y^-(t,r) = (r-1)\{\frac{t^2}{2} + t + 1\},$$
$$y^+(t,r) = (1-r)\{\frac{t^2}{2} + t + 1\},$$

Now we use PWA method to obtain piecewise approximate solution $s(t)$. Let $I_k = [kH, (k+1)H]$, for $k = 0, 1, 2$, $H = 3h$ and $h = 0.1$. $s_0(t)$, $s_3(t)$ and $s_6(t)$ are obtained as follows:

$$\begin{aligned}
s_0^-(t) &= (r-1) + t(r-1) + \frac{t^2}{2}(r-1), \\
s_0^+(t) &= (1-r) + t(1-r)t + \frac{t^2}{2}(1-r), \\
s_3^-(t) &= 1.345r - 1.345 \\
&+ (t-0.3)(1.3r-1.3) \\
&+ \frac{(t-0.3)^2}{2}(r-1), \\
s_3^+(t) &= 1.345 - 1.345r \\
&+ (t-0.3)(1.3-1.3r) \\
&+ \frac{(t-0.3)^2}{2}(1-r), \\
s_6^-(t) &= 1.78r - 1.78 \\
&+ (t-0.6)(1.6r-1.6) \\
&+ \frac{(t-0.6)^2}{2}(r-1), \\
s_6^+(t) &= 1.78 - 1.78r \\
&+ (t-0.6)(1.6-1.6r)) \\
&+ \frac{(t-0.6)^2}{2}(1-r),
\end{aligned}$$

The approximate solution $s_i(t)$ in Case(i), for $i = 0, 1, 2$, is plotted in Fig 1.

Case(ii) If $y(t)$ is $[(ii) - gH]$-differentiable, the real solution is:

$$y^-(t,r) = (r-1)\{\frac{t^2}{2} - t + 1\},$$
$$y^+(t,r) = (1-r)\{\frac{t^2}{2} - t + 1\},$$

in this case $s_0(t)$, $s_3(t)$ and $s_6(t)$ are obtained as follows:

$$\begin{aligned} s_0^-(t) &= (r-1)+t(1-r)+\frac{t^2}{2}(r-1),\\ s_0^+(t) &= (1-r)+t(r-1)t+\frac{t^2}{2}(1-r),\\ s_3^-(t) &= .745r-.745+(t-0.3)(0.7-.7r)\\ &+ \frac{(t-0.3)^2}{2}(r-1),\\ s_3^+(t) &= .745-.745r+(t-0.3)(0.7r-.7)\\ &+ \frac{(t-0.3)^2}{2}(1-r),\\ s_6^-(t) &= .58r-.58+(t-0.6)(.4-.4r)\\ &+ \frac{(t-0.6)^2}{2}(r-1),\\ s_6^+(t) &= .58-.58r+(t-0.6)(.4r-.4r)\\ &+ \frac{(t-0.6)^2}{2}(1-r), \end{aligned}$$

The approximate solution $s_i(t)$ in Case(ii), for $i = 0, 1, 2$, is plotted in Fig 2.

Example 4.2 *[20] Consider the fuzzy initial value problem*

$$y''(t)+y(t)=\sigma_0, \quad y(0)=\gamma_0, \quad y'(0)=\gamma_1,$$

where σ_0 is the fuzzy number having r-level sets $[r, 2-r]$. $[\gamma_0]^r = [\gamma_1]^r = [r-1, 1-r]$.

Case(i) If $y(t)$ is $[(i)-gH]$-differentiable, the real solution is:

$$\begin{aligned} y^-(t,r) &= r(1+\sin(t))-\sin(t)-\cos(t),\\ y^+(t,r) &= (2-r)(1+\sin(t))\\ &- \sin(t)-\cos(t), \end{aligned}$$

Let $I_k = [kH, (k+1)H]$, for $k = 0, 1, 2$, $H = 3h$ and $h = 0.1$. $s_0(t)$, $s_3(t)$ and $s_6(t)$ are obtained as follows:

$$\begin{aligned} s_0^-(t) &= (r-1)+t(r-1)\\ &+ \frac{t^2}{2}(.9992+0.00099r)\\ &+ \frac{t^3}{3!}(1.016-1.01817r)\\ &+ \frac{t^4}{4!}(-1.1778+.1986r),\\ s_0^+(t) &= (1-r)+t(1-r)\\ &+ \frac{t^2}{2}(1.001-0.00099r)\\ &+ \frac{t^3}{3!}(-1.021+1.0182r)\\ &+ \frac{t^4}{4!}(-.7807-.1985r),\\ s_3^-(t) &= (1.295r-1.2509)\\ &+ (t-0.3)(.9554r-.6599)\\ &+ \frac{(t-0.3)^2}{2}(1.251-.2947r)\\ &+ \frac{(t-0.3)^3}{3!}(.6688-.972r)\\ &+ \frac{(t-0.3)^4}{4!}(-1.356+.4791),\\ s_3^+(t) &= (1.3402-1.296r)\\ &+ (t-0.3)(1.2509-.9554r)\\ &+ \frac{(t-0.3)^2}{2}(.6612+.2946r)\\ &+ \frac{(t-0.3)^3}{3!}(-1.275+.972r)\\ &+ \frac{(t-0.3)^4}{4!}(-.3978-.4791r),\\ s_6^-(t) &= (1.565r-1.39)\\ &+ (t-0.6)(.8254r-.2608)\\ &+ \frac{(t-0.6)^2}{2}(1.39-.564r)\\ &+ \frac{(t-0.6)^3}{3!}(.26201-.839r)\\ &+ \frac{(t-0.3)^4}{4!}(-1.413+.7169r),\\ s_6^+(t) &= (1.74-1.565r)\\ &+ (t-0.6)(1.3901-.8254r)\\ &+ \frac{(t-0.6)^2}{2}(.26208+.56394r)\\ &+ \frac{(t-0.6)^3}{3!}(-1.416+.839r)\\ &+ \frac{(t-0.3)^4}{4!}(0.0206-.7168r), \end{aligned}$$

The approximate solution $s_i(t)$ in Case(i), for $i = 0, 1, 2$, is plotted in Fig 3.

Case(ii) If $y(t)$ is $[(ii) - gH]$-differentiable, the real solution is:

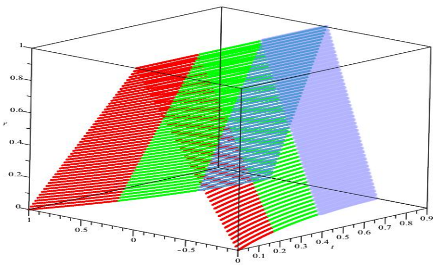

Figure 2: Approximate solution for case(ii) in example 4.1. Red points: $s_0(t)$; Green points: $s_3(t)$; Blue points: $s_6(t)$

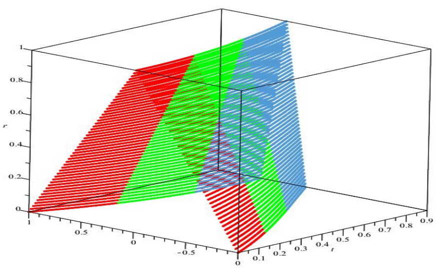

Figure 3: Approximate solution for case(ii) in example 4.2. Red points: $s_0(t)$; Green points:$s_3(t)$; Blue points: $s_6(t)$

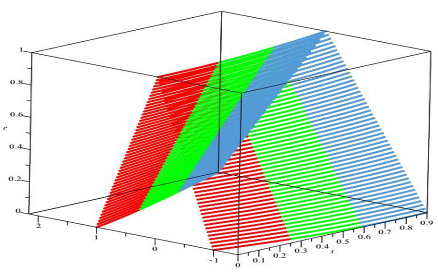

Figure 4: Approximate solution for case(i) in example 4.2. Red points: $s_0(t)$; Green points: $s_3(t)$; Blue points: $s_6(t)$

$$y^-(t,r) = r(1-\sin(t)) + \sin(t) - \cos(t),$$

$$\begin{aligned} y^+(t,r) &= (2-r)(1-\sin(t)) \\ &\quad + \sin(t) - \cos(t), \end{aligned}$$

$s_0(t)$, $s_3(t)$ and $s_6(t)$ are obtained as follows:

$$\begin{aligned}
s_0^-(t) &= (r-1) + t(1-r) \\
&+ \frac{t^2}{2}(1.0011 - 0.00099r) \\
&+ \frac{t^3}{3!}(-1.021 + 1.0182r) \\
&+ \frac{t^4}{4!}(-.78074 - .19851r), \\
s_0^+(t) &= (1-r) + t(r-1) \\
&+ \frac{t^2}{2}(.9992 + 0.00099r) \\
&+ \frac{t^3}{3!}(1.0157 - 1.01818r) \\
&+ \frac{t^4}{4!}(-1.1778 + .1985r), \\
s_3^-(t) &= (.7045r - .6599) \\
&+ (t-0.3)(1.251 - .95537r) \\
&+ \frac{(t-0.3)^2}{2}(.66114 + .2946r) \\
&+ \frac{(t-0.3)^3}{3!}(-1.7527 + .97199r) \\
&+ \frac{(t-0.3)^4}{4!}(-.3978 - .4791r), \\
s_3^+(t) &= (.7492 - .70447r) \\
&+ (t-0.3)(.9554r - .65985) \\
&+ \frac{(t-0.3)^2}{2}(1.2504 - .29463r) \\
&+ \frac{(t-0.3)^3}{3!}(.66872 - .97199r) \\
&+ \frac{(t-0.3)^4}{4!}(-1.3559 + .47905r), \\
s_6^-(t) &= (.43533r - .26066) \\
&+ (t-0.6)(1.3901 - .825399r) \\
&+ \frac{(t-0.6)^2}{2}(.26207 + .56394r) \\
&+ \frac{(t-0.6)^3}{3!}(-1.41597 + .83899r) \\
&+ \frac{(t-0.3)^4}{4!}(0.0207 - .71680r), \\
s_6^+(t) &= (.610 - .43533r) \\
&+ (t-0.6)(.8254r - .26074) \\
&+ \frac{(t-0.6)^2}{2}(1.3899 - .56394r) \\
&+ \frac{(t-0.6)^3}{3!}(.26201 - .838989r) \\
&+ \frac{(t-0.3)^4}{4!}(-1.413 + .7168r),
\end{aligned}$$

The approximate solution $s_i(t)$ in Case(ii), for $i = 0, 1, 2$, is plotted in Fig 4.

5 Conclusion

In this paper a new approach for solving second order fuzzy differential equations under generalized differentiability was proposed. We used piecewise fuzzy polynomial of degree 4 based on the Taylor expansion for approximating solutions of second order fuzzy differential equations. Also, we can develop this method for Nth-order fuzzy differential equations under generalized derivatives.

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