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Research Article

Bernoulli operational matrix method for system of linear Volterra integral equations

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Abstract

In this paper, the numerical technique based on hybrid Bernoulli and Block-Pulse functions has been developed to approximate the solution of system of linear Volterra integral equations. System of Volterra integral equations arose in many physical problems such as elastodynamic, quasi-static visco-elasticity and magneto-electro-elastic dynamic problems. These functions are formed by the hybridization of Bernoulli polynomials and Block-Pulse functions which are orthonormal and have compact support on $[0,1]$. By these orthonormal bases we drove new operational matrix which was a sparse matrix. By use of this new operational matrix we reduces the system of integral equations to a system of linear algebraic equations that can be solved easily by any usual numerical method. The numerical results obtained by the presented method have been compared with some existed methods and they have been in good agreement with the analytical solutions and other methods that prove the profit and efficiency of the proposed method.

Keywords : System of Volterra integral equations; Bernoulli polynomials; Hybrid functions; Operational matrix.

1 Introduction

System of linear Volterra integral equations arises in many physical applications, e.g., linear quasi-static visco-elasticity problem [2], magneto-electro-elastic dynamic problems [3] and the elastodynamic problems of piezoelectric [4].

We consider the following system of linear Volterra integral equations

$$G(x)U(x) + \int_0^x K(x,s)U(s)ds = F(x),$$
$$i = 1,2,...,q, \tag{1.1}$$

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that

$$U(x) = [u_1(x), ..., u_q(x)]^T,$$
$$F(x) = [f_1(x), ..., f_q(x)]^T, \tag{1.2}$$

and

$$G(x) = \begin{pmatrix} g_{11}(x) & \cdots & g_{1q}(x) \\ \vdots & \ddots & \vdots \\ g_{q1}(x) & \cdots & g_{qq}(x) \end{pmatrix},$$

$$K(x,s) = \begin{pmatrix} k_{11}(x,s) & \cdots & g_{1q}(x) \\ \vdots & \ddots & \vdots \\ g_{q1}(x) & \cdots & g_{qq}(x) \end{pmatrix},$$

where the functions $g_{ij}(t), f_i(x) \in L^2[0,1)$ and the kernels $k_{ij}(x,s) \in L^2([0,1) \times [0,1))$ for $i,j = 1,2,...,q$ are known and $u_i(x)$ for $i = 1,2,...q$

are solutions to be determined. The theory on existence and uniqueness of a continuous solution for such equations was already established by Volterra and Brunner [2, 11].

Some existed numerical methods for approximating the solution of Eq.(1) are as follows. Maleknejad, Rabbani and Aghazadeh in [9] used expansion method to solve Volterra integral equations system of the second kind, in [8] Mirzaee obtained a numerical solution of these equations by using rationalized Haar functions, Saeed and Ahmed in [10] produced a method for numerical solution of the System of linear Volterra Integral equations of the second kind using Monte-Carlo method, in [1] Biazar and Pourabd solved these system of integral equations numerically based on Adomian decomposition method, Hashemizadeh and Basirat in [5] solved these equations by hybrid Block-Pulse and Legendre polynomials and in [6] Jiang and Chen used reproducing kernel method for solving this system of integral equations.

In this paper we construct orthonormal Bernoulli polynomial and hybrid them with Block-Pulse functions. By this new basis, we drive integral operational matrix and product matrix to discrete vollterra type of system of integral equation at collocation point.

The paper is organized as six sections including the introduction. In Section 1, we introduce Bernoulli polynomials and their properties. In Section 2, we would implement the hybrid of Block-Pulse functions and orthonormal Bernoulli polynomials operational matrix on the system of linear Volterra integral equations and convert them to a linear algebraic system of equations. In Section 3, we present error bound for solution of these systems. In Section 4, we present numerical examples that show the efficiency and accuracy of proposed method in analogy to some existed method. Finally Section 5, concludes the paper.

2 Hybrid of Bernoulli polynomials and Bock-pulse function

In this section, we introduce Bernoulli polynomials and their properties. We orthonormal these polynomial by Gram-Schmidt algorithm, then we hybrid these functions with Block-Pulse functions. Some properties of this set of functions are presented.

2.1 Hybrid Orthonormal Bernoulli Polynomials with Block-Pulse Functions and their properties

The Bernoulli polynomials of degree n are defined by [7]

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)\, x^n. \tag{2.3}$$

The first few Bernoulli polynomials for $n = 3$ are:

$$\begin{aligned}
&B_0(x) = 1,\\
&B_1(x) = x - \tfrac{1}{2},\\
&B_2(x) = x^2 - x + \tfrac{1}{6},\\
&B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,\\
&\vdots
\end{aligned}$$

An M-set of Block-Pulse function is defined over the interval [0,T) as

$$b_i(x) = \begin{cases} 1 & \frac{iT}{m} \le x < \frac{(i+1)T}{m} \\ 0 & \text{otherwise} \end{cases},$$

where $i = 0, 1, ..., m-1$ with m as a positive integer. Also, $h = \frac{T}{m}$ and b_i is the i-th BPF. In this paper it is assumed that $T = 1$, so BPFs are defined over $[0, 1)$ and $h = \frac{1}{m}$. There are some properties for BPFs; the most important properties are disjointness, orthogonality, and completeness.

By using Gram-Schmidt algorithm we obtain orthonormal Bernoulli polynomials to construct new basis, we call this new base as $OB_n(x)$. By orthonormal Bernoulli polynomials and Block-Pulse functions and their properties we can define hybrid of this function in the interval $[0, 1]$ as follow:

$$OBH_{i,j}(x) = \begin{cases} OB_n(mx - i + 1) & \frac{i-1}{m} \le x < \frac{i}{m} \\ 0 & \text{otherwise} \end{cases}, \tag{2.4}$$

where $i = 1, 2, ..., m$ and $j = 0, 1, ..., n$. These new functions have useful properties of orthonormal functions and Block-Pulse functions together. Thus, our new basis is $\{OBH_{1,0}, OBH_{1,1}, ..., OBH_{m,n}\}$ and we can approximate the functions with this base. In Fig. 1. The behavior of several Orthonormal Bernoulli Hybrid polynomials in the interval $[0, 1]$ is depicted. The property of $\int_0^1 OBH_n(x)dx = 0$ could be observed geometrically.

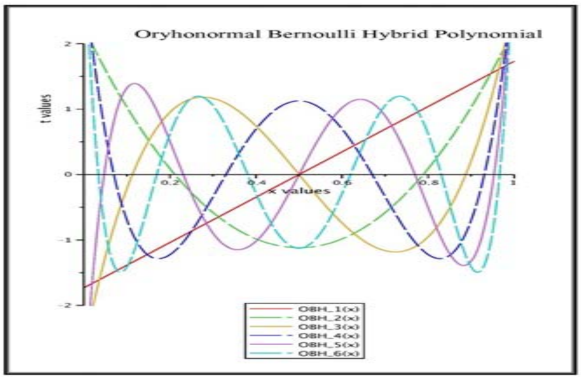

Figure 1: The graph of the first six of Orthonormal Bernoulli Hybrid polynomials.

2.2 Function approximation

Suppose that $H = L^2[0,1]$ and $\{OBH_{1,0}, OBH_{1,1}, ..., OBH_{m,n}\} \subset H$ be the set of hybrid of Block-Pulse and orthonormal Bernoulli polynomials and

$$Y = Span\{OBH_{1,0}, OBH_{1,1}, ..., OBH_{m,n}\} \quad (2.5)$$

and f be an arbitrary element in H. Since Y is a finite dimensional vector space, f has the unique best approximation out of Y such as $f_0 \in Y$, that is

$$\forall y \in Y, \| f - f_0 \| \leq \| f - y \| .$$

Since $f_0 \in Y$, there exists the unique coefficients $C_{10}, C_{11}, ..., C_{mn}$ such that

$$f \simeq f_0 = \sum_{j=1}^{m} \sum_{i=0}^{n} c_{ij} OBH_{ij}(t) = C^T OBH(t), \quad (2.6)$$

where

$$OBH(t) = Span\{OBH_{1,0}, OBH_{1,1}, ..., OBH_{m,n}\} \quad (2.7)$$

and

$$C = [c_{1,0}, c_{1,1}, ..., c_{m,n}]^T. \quad (2.8)$$

By using Eq.(2.6) we obtain

$$f_{ij} = \langle \sum_{l=0}^{n} \sum_{k=1}^{m} c_{kl} OBH_{kl}(t), OBH_{kl}(t) \rangle =$$

$$\sum_{l=0}^{n} \sum_{k=1}^{m} c_{kl} d_{kl}^{ij}, \quad i = 1, 2, ...n, j = 0, 1, ...m, \quad (2.9)$$

where $f_{ij} = \langle f, OBH_{ij}(t) \rangle$, $d_{kl}^{ij} = \langle OBH_{K!}(t), OBH_{ij}(T) \rangle$, and $\langle , \rangle$ denotes inner product.
Therefore

$$f_{ij} = C^T [d_{10}^{ij}, d_{11}^{ij}, ..., d_{1n}^{ij}, ..., d_{m0}^{ij}, d_{m1}^{ij}, ..., d_{mn}^{ij}],$$
$$i = 1, 2, ...n, j = 0, 1, ...m, \quad (2.10)$$

So we get

$$\Phi = D^T C, \quad (2.11)$$

with

$$\Phi = [f_{10}, f_{11}, ..., f_{1n}, ..., f_{m0}, f_{m1}, ..., f_{mn}], \quad (2.12)$$

and

$$D = [d_{mn}^{ij}], \quad (2.13)$$

where D is a product matrix of integration of order $(nm) \times (nm)$ and is given by

$$D = \int_0^1 OBH(t) OBH^T(t) dt. \quad (2.14)$$

For example for $m = 2$ and $n = 3$, D is

$$\mathbf{D} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously, matrix D is a sparse matrix. We can also approximate the function $k(x,t) \in L^2([0,1] \times [0,1])$ as follows

$$k(x,t) \simeq OBH^T(x).K \; OBH(t), \quad (2.15)$$

where K is an $mn \times mn$ matrix that

$$\mathbf{k}_{ij} = \frac{\langle OBH_i(x), \langle k(x,t), OBH_j(t) \rangle \rangle}{\langle OBH_i(x), OBH_i(x) \rangle \langle OBH_j(t), OBH_j(t) \rangle}, \quad (2.16)$$

for $i, j = 1, 2, ..., mn$. So we have

$$k = D^{-1} \langle OBH(x), \langle k(x,t), OBH(t) \rangle \rangle D^{-1}. \quad (2.17)$$

2.3 Operational matrix of integration

The integration of the $OBH(t)$ defined in Eq.(2.7) is given by

$$\int_0^1 OBH(t') dt' \simeq \mathbf{P} \; OBH(t), \quad (2.18)$$

where $\mathbf{P}$ is the $(mn) \times (mn)$ operational matrix of integration and is given by

$$\mathbf{P} = \frac{1}{2m} \begin{pmatrix} P_0 & O \\ O & P_0 \end{pmatrix}, \quad (2.19)$$

where O is $N \times N$ zero matrix and P_0 is $N \times N$ matrix. Obviously, $\mathbf{P}$ is a sparse matrix too.

2.4 Product operational matrix

It is always necessary to evaluate the product of $OBH(x)$ and $OBH^T(x)$, that is called the product matrix of hybrid functions. Let

$$obh(x) = OBH(x)OBH^T(x), \tag{2.20}$$

where $obh(x)$ is $mn \times mn$ matrix. Multiplying the matrix $obh(x)$ by vector C that is defined in Eq. (2.8), we obtain

$$obh(x) \times C = \tilde{C} \times OBH^T(x) \tag{2.21}$$

where $\tilde{C}$ is $mn \times mn$ matrix which is called the coefficient matrix. To illustrate the calculation procedure in Eq. (2.21), we consider that $n = 2, m = 7$, so we have

$$\tilde{C} = \begin{pmatrix} \tilde{C}_1 & O \\ O & \tilde{C}_2 \end{pmatrix}, \tag{2.22}$$

where $\tilde{C}_1, i = 1,2$ are 7×7 matrices given by

$$\tilde{C}_i = \begin{pmatrix} c_{i0} & c_{i1} & \cdots & c_{i6} \\ c_{i2} & \begin{array}{c} c_{i1} \\ +\frac{2\sqrt{5}}{5}c_{i3} \end{array} & \cdots & \frac{6\sqrt{429}}{143}c_{i6} \\ c_{i3} & \begin{array}{c} \frac{2\sqrt{5}}{5}c_{i2} \\ +\frac{3\sqrt{105}}{35}c_{i4} \end{array} & \cdots & \begin{array}{c} \frac{15\sqrt{65}}{143}c_{i5} \\ +\frac{14\sqrt{5}}{55}c_{i7} \end{array} \\ c_{i4} & \begin{array}{c} \frac{3\sqrt{105}}{35}c_{i3} \\ +\frac{4\sqrt{21}}{21}c_{i5} \end{array} & \cdots & \begin{array}{c} \frac{100\sqrt{13}}{429}c_{i4} \\ +\frac{7\sqrt{1001}}{429}c_{i6} \end{array} \\ \vdots & \vdots & \vdots & \vdots \\ c_{i7} & \begin{array}{c} \frac{15\sqrt{65}}{143}c_{i5} \\ +\frac{14\sqrt{5}}{55}c_{i7} \end{array} & \cdots & \begin{array}{c} c_{i1}+\frac{14\sqrt{5}}{55}c_{i3} \\ +\frac{84}{187}c_{i5} \\ +\frac{400\sqrt{13}}{3553}c_{i7} \end{array} \end{pmatrix}.$$

With the powerful properties of Eq. (2.21), we can convert the Volterra part of Integral equations to an algebraic equation.

3 Implementation of hybrid Bernoulli function method on system of linear volterra integral equations

Consider the system of linear Volterra integral equation Eq. (1.1), we can show this system in the following form

$$\sum_{j=1}^{q} g_{ij}(x)u_i(x) + \sum_{j=1}^{q} \int_0^x k_{ij}(x,s)u_j(s)ds = f_i(x); \;\; i = 1,2,...,q. \tag{3.23}$$

We put

$$u_i(x) \simeq U_i^T OBH(x), i = 1,...,q, \tag{3.24}$$

where U_i for $i = 1,...,q$ are unknown nm-vectors and $OBH(x)$ is given by Eq. (2.7). Likewise, $g_{ij}(x), k_{ij}(x,s)$ and $f_i(x)$ for $i,j = 1,...,q$ are expanded into the hybrid functions as follows

$$k_{ij}(x,s) \simeq OBH^T(x)K_{ij}OBH(s), g_{ij}(x) \simeq G_{ij}^T OBH(x), i,j = 1,...,q, \tag{3.25}$$

$$f_i(x) \simeq F_i^T OBH(x), \quad i = 1,...,q, \tag{3.26}$$

where K_{ij} for $i,j = 1,...,q$ are known $mn \times mn$-matrices and G_{ij}, F_i for $i,j = 1,...,q$, are known nm-vectors. After substituting the approximate Eqs. (3.24), (3.25), (3.26) in (3.23) we get

$$\sum_{j=1}^{q}(G_{ij}^T OBH(x)OBH^T(x)U_j) + \sum_{j=1}^{q}\int_0^x OBH^T(x)K_{ij}OBH(s)OBH^T(s)U_j\, ds = OBH^T(x)F_i, \quad i = 1,...,q, \tag{3.27}$$

by using Eqs. (2.18) and (2.21) we can convert Eq. (3.27) to the following equations

$$\sum_{j=1}^{q} OBH^T(x)\tilde{G_{ij}}U_j + \sum_{j=1}^{q} OBH^T(x)K_{ij}\tilde{U}_j POBH(x) = OBH^T(x)F_i, \quad i = 1,...,q, \tag{3.28}$$

now we have q equations with $q \times n \times m$ unknowns $U_1, U_2, ..., U_q$ (each of these vectors have nm unknowns). In order to find U_i, for $i = 1,...,q$, we collocate Eq. (3.28) in nm points $x_p, p = 1,...,nm$, by Gauss-Legendre points in the interval $[0,1]$. So we have the following system of linear equations

$$\sum_{j=1}^{q} OBH^T(x_p)\tilde{G_{ij}}U_j + \sum_{j=1}^{q} OBH^T(x_p)K_{ij}\tilde{U}_j POBH(x_p) = OBH^T(x_p)F_i, \;\; i = 1,...,q, p = 1,...,mn. \tag{3.29}$$

By solving linear system Eq. (3.29) we can drive U_i, for $i = 1,...,q$, so we can approximate $U_i(x)$ as $U_i^T OBH(x)$ for $i = 1,...,q$, that they are the approximate solution for system of linear Volterra integral equations Eq. (3.23).

Table 1: Absolute error for Example 4.1.

	Absolute errors for	$u_1(x)$	Absolute errors for	$u_2(x)$
x	OBH function by $m=2, n=4$	Method in [6] with $N=25, M=30$	OBH function by $m=2, n=4$	Method in [6] with $N=25, M=30$
0.0	1.0×10^{-7}	$1,12071E-7$	2.0×10^{-7}	$7,43647E-8$
0.1	1.0×10^{-9}	$1,93035E-4$	5.2×10^{-7}	$2,13517E-6$
0.2	1.0×10^{-9}	$1,93351E-4$	7.0×10^{-7}	$1,10363E-4$
0.3	0.0×10^{-9}	$1,60621E-4$	1.0×10^{-7}	$1,36687E-4$
0.4	1.0×10^{-9}	$2,85454E-4$	3.0×10^{-7}	$3,40355E-4$
0.5	2.0×10^{-9}	$4,46399E-4$	4.0×10^{-7}	$6,70169E-4$
0.6	2.0×10^{-9}	$2,85732E-4$	2.0×10^{-7}	$5,15149E-4$
0.7	3.0×10^{-9}	$1,60049E-4$	6.0×10^{-7}	$3,45906E-4$
0.8	5.0×10^{-9}	$1,91258E-4$	1.0×10^{-7}	$4,59884E-4$
0.9	1.0×10^{-5}	$1,80286E-5$	2.0×10^{-7}	$5,67914E-5$
1.0	2.0×10^{-5}	$1,79989E-6$	2.0×10^{-7}	$1,15123E-6$

Table 2: Absolute error for Example 4.2.

	Absolute errors for	$u_1(x)$	Absolute errors for	$u_2(x)$
x	OBH function by $m=2, n=3$	Method in [8] with $k=32$	OBH function by $m=2, n=3$	Method in [8] with $k=32$
0.0	4.7×10^{-10}	0.0×10^{-4}	1.4×10^{-11}	0.0×10^{-4}
0.1	3.0×10^{-11}	0.9×10^{-4}	5.0×10^{-11}	0.1×10^{-4}
0.2	1.0×10^{-10}	0.6×10^{-4}	3.0×10^{-11}	0.7×10^{-4}
0.3	4.0×10^{-10}	0.2×10^{-4}	3.0×10^{-10}	0.4×10^{-4}
0.4	2.2×10^{-10}	0.3×10^{-4}	0.0×10^{-11}	0.5×10^{-4}
0.5	5.1×10^{-10}	0.2×10^{-4}	6.7×10^{-11}	0.1×10^{-4}
0.6	3.1×10^{-10}	0.6×10^{-4}	5.0×10^{-10}	0.2×10^{-4}
0.7	4.1×10^{-10}	0.9×10^{-4}	4.2×10^{-10}	0.8×10^{-4}
0.8	5.3×10^{-10}	0.1×10^{-4}	3.7×10^{-10}	0.8×10^{-4}
0.9	6.0×10^{-10}	0.9×10^{-4}	6.4×10^{-10}	0.2×10^{-4}
1.0	7.5×10^{-10}	0.1×10^{-4}	6.7×10^{-10}	0.8×10^{-4}

Table 3: Absolute error for Example 4.3.

	Absolute errors for	$u_1(x)$	Absolute errors for	$u_2(x)$
x	OBH function by $m=2, n=5$	Method in [6] with $N=25, M=30$	OBH function by $m=2, n=5$	Method in [6] with $N=25, M=30$
0.0	2.0×10^{-6}	$3,17983E-6$	1.1×10^{-7}	$8,41599E-7$
0.1	2.0×10^{-6}	$2,78782E-6$	2.5×10^{-7}	$2,72792E-7$
0.2	2.7×10^{-7}	$3,80744E-7$	1.6×10^{-6}	$7,24135E-8$
0.3	2.1×10^{-6}	$1,50503E-6$	3.6×10^{-7}	$4,63035E-7$
0.4	2.2×10^{-6}	$2,49236E-6$	2.0×10^{-7}	$1,05532E-6$
0.5	1.2×10^{-6}	$2,74089E-6$	1.0×10^{-6}	$1,49415E-6$
0.6	1.3×10^{-7}	$2,23777E-6$	1.2×10^{-6}	$1,52353E-6$
0.7	1.6×10^{-7}	$1,20489E-7$	1.2×10^{-6}	$1,01451E-6$
0.8	2.5×10^{-6}	$2,65504E-6$	3.6×10^{-6}	$2,79125E-7$
0.9	1.0×10^{-6}	$1,73647E-6$	4.0×10^{-6}	$2,19568E-6$
1.0	1.0×10^{-6}	$1,72341E-6$	3.0×10^{-6}	$2,67258E-6$

4 Numerical examples

To show the efficiency of the proposed numerical method, we implement it on three system of Volterra integral equation as test problem. We note that

$$\|e_n\|_\infty = max(e_n),$$

that

$$e_n(x_i) = f(x_i) - OBH(f_n(x_i)),$$

where $OBH(f_n(x_i))$ and $f(x_i)$ are the approximate solution of order n with hybrid of Bernstein and Block-Pulse approximation and exact solutions of the integral equations, respectively.
For implementation of proposed method we used "Maple" 15.

Example 4.1 *Consider the following system of Volterra integral equations of the second kind [6]:*

$$\begin{cases} u_1(x) = f_1(x) + \int_0^x (x-s)^3 u_1(s)ds + \\ \qquad \int_0^x (x-s)^2 u_2(s)ds, \\ \\ u_2(x) = f_2(x) + \int_0^x (x-s)^4 u_1(s)ds + \\ \qquad \int_0^x (x-s)^3 u_2(s)ds, \end{cases}$$

$u_1(x)$ and $u_2(x)$ are chosen such that the exact solution is $u_1(x) = 1 + x^2, u_2(x) = 1 + x - x^3$. For this example we take $n = 4$ and $m = 2$, then we compared our results with [6]. The answers are tabulated in Table 1. Also Fig. 1 shows $u_2(x)$ and its approximation.

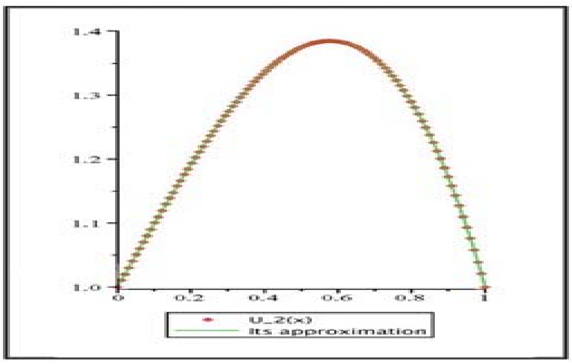

Figure 2: Numerical and exact graph of the solution for example 4.1 by $n = 4$ for $u_2(x)$.

Example 4.2 *Consider the following system of Volterra integral equations:*

$$\begin{cases} u_1(x) - \int_0^x u_2(s)ds = 1 - x^2, \\ \\ u_2(x) - \int_0^x u_1(s)ds = x, \end{cases}$$

the exact solution is $u_1(x) = 1$, $u_2(x) = 2x$. Table 2 gives the absolute errors for $u_1(x)$ and $u_2(x)$ by OBH functions method with comparison by absolute errors of rationalized Haar function method [8].

Example 4.3 *Consider the following system of linear Volterra integral equations of the second kind [6]:*

$$\begin{cases} u_1(x) = f_1(x) + \int_0^x (sin(x-s) - 1)u_1(s)ds \\ \qquad + \int_0^x (1 - s\ cosx)u_2(s)ds, \\ \\ u_2(x) = f_2(x) + \int_0^x u_1(s)ds + \\ \qquad \int_0^x (x-s)u_2(s)ds, \end{cases}$$

$u_1(x)$ and $u_2(x)$ are chosen such that the exact solution is $u_1(x) = cosx, u_2(x) = sinx$. For this example we take $n = 5$ and $m = 2$, then we compared our results by the results of paper [6]. The answers are tabulated in Table 3. Also Fig. 2 shows $u_1(x)$ and its approximation.

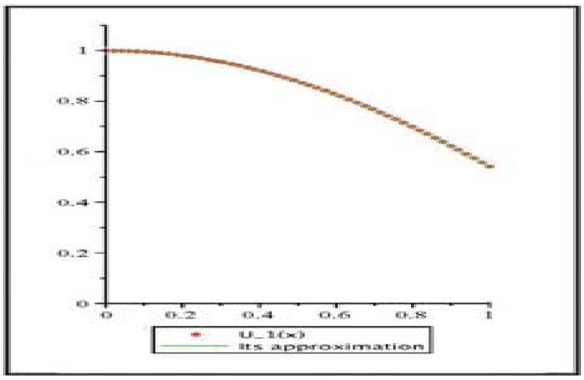

Figure 3: Numerical and exact graph of the solution for example 4.3 by $n = 5$ for $u_1(x)$.

5 Conclusion

A new method based on Bernoulli polynomial and Block-Pulse function has been formulated and employed for the numerical solution of system of Volterra integral equations which arose in many physical problems such as elastodynamic. By these orthonormal bases we drove new operational matrix which was a sparse matrix. By use of this new operational matrix we reduced the system of linear Volterra integral equations to a system of linear algebraic equations that can be solved easily by known methods. Using the proposed method in solving system of linear Volterra integral equation shows the high capability of this method compared to other methods and numerical results given in tables and figures.

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