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Efficiency Measurement of Multiple Components units in Data Envelopment Analysis Using Common Set of Weights

M. Fallah Jelodar [a]*, N. Shoja [a], M. Sanei [b], A. Gholam Abri [a]

(a) *Department of Mathematics, Islamic Azad University, Firoozkooh Branch,Firoozkooh,Iran.*

(b) *Department of Mathematics, Islamic Azad University, Tehran Center Branch,Tehran,Iran.*

Abstract

The current paper presents a mathematical programming model for use to the measure of efficiency where multiple performance measures are needed to examine the performance and productivity changes. In many applications of data envelopment analysis (DEA), the existing models are designed to obtain a single efficiency measure. However, in many real situations, the units under consideration may perform several different functions or can be separated into different components. In these cases, some inputs are often shared among those components and all components are involved in producing some outputs and all components have exclusive inputs and outputs. Therefore, measuring the efficiency of each component and measuring the aggregated efficiency of each unit are important. In this paper, car factories' efficiency is analyzed using data from 19 car factories in Iran. First, a DEA-efficiency analysis of multi-component DMUs, which proposed by Cook et al is presented and then by using a "common set of weights (csw)", a new model is proposed to measure the efficiency of each component and aggregated efficiency of units. One of the most important advantages of this model is that by solving only one linear programming problem all efficiency measures may be obtained. Secondly, by grouping the branches according to their organizational designation, their efficiency is measured.

Keywords: Data envelopment analysis, Efficiency, Multi-component efficiency, Aggregated efficiency, Common set of weights

*Corresponding author. Email Address: mehdi.fallah_jelodar@yahoo.com and mehdi.fallah@iaufb.ac.ir, Tel:+989123157650

1 Introduction

Data envelopment analysis is a method to determine the relative efficiency of a set of organizational units such as schools or bank branches when there are multiple incommensurate inputs and outputs. The applications of DEA present a range of issues relating to the homogeneity of the units under consideration. Efficiency measurement using tools such as DEA, as proposed by Charnes et al. [3], has tended to concentrate on achieving a single measure of efficiency for each member of a set of decision making units (DMUs). Note that a mathematical programming based method data envelopment analysis has proven an effective tool for evaluating the relative efficiency of peer decision making units when multiple performance measures are presented. In most real applications, a single measure of production efficiency, provided by DEA methodology, has been an adequate and useful means of comparing units and identifying best performance.
However, in most real situations, the DMU involved, can be separated into different components. In these situations, inputs, in particular resources, are often shared among those components. Also, all components are involved in producing some outputs. In such cases, we have to determine the performance of DMUs in each component. Therefore, measuring models that can deal with multiple performance measures and provide an integrated performance measure are needed. The idea of measuring efficiency relative to certain components of a DMU is not new. Fare and Grosskopf [5], for example, looked at a multi-stage process wherein intermediate outputs at one stage can be both final outputs and inputs to a later stage of production. This application of multi-component efficiency measurement does not involve shared resources. Cook et al. [4] proposed a method for measuring multi-component efficiency, which involved shared inputs.
This paper, first, addresses a model for deriving an "aggregate measure of efficiency" with component measurement based on a common set of weights, and derives a model for measuring the efficiency score of each component and aggregated efficiency of the units under consideration. Secondly, an application to Iranian car factories is presented. The Iranian car factories, like those of many countries, have undergone considerable change in operational policies in recent years. Because of these changes, the flexible evaluation of performance that tells managers whether or not they are "doing things right" has become an important topics to managing reality. This paper is structured as follows:
A summary of basic DEA models and multiple component efficiency is given in Section 2. Mathematical models for implementing the measures are presented in Section 3. The fourth section details the analytical results obtained from the application of the model to a real data set involving the data of 19 Iranian car factories. Conclusions appear in the final section.

2 DEA Background

Consider n $DMUs$ with m inputs and s outputs. The input and output vectors of DMU_j, $(j = 1, \ldots, n)$, are $X_j = (x_{1j}, \ldots, x_{mj})^t$, $Y_j = (y_{1j}, \ldots, ysj)^t$, where $X_j \geq 0$, $X_j \neq 0$, $Y_j \geq 0$, $Y_j \neq 0$.
By using the non-empty, constant-returns-to-scale, convexity and possibility postulates,

the production possibility set (PPS) is made as follows:

$$T_c = \left\{ (X,Y) : \ X \geq \sum_{j=1}^{n} \lambda_j X_j,\ Y \leq \sum_{j=1}^{n} \lambda_j Y_j,\ \lambda_j \geq 0,\ j = 1 \ldots, n \right\}$$

Let DMU_o be the DMU under consideration. The envelopment form of the CCR model in the input-oriented case is as follows:

$$\begin{aligned} Min \quad & \theta - \varepsilon[\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+] \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}, \qquad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \qquad r = 1, \ldots, s \\ & \lambda_j \geq 0, \qquad j = 1, \ldots, n \end{aligned} \tag{2.1}$$

and its dual, the multiplier form of the CCR model in the input oriented case is:

$$\begin{aligned} Max \quad & \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \qquad j = 1, \ldots, n \\ & u_r \geq \varepsilon, \quad r = 1, \ldots, s \\ & v_i \geq \varepsilon, \quad i = 1 \ldots, m \end{aligned} \tag{2.2}$$

Clearly:

(1) DMU_o is efficient in model 1 if and only if θ^*=1, S^{-*}=0 and S^{+*}=0. (* means the optimal solution)

(2) DMU_o is efficient in model 2 if and only if there exists $(U^{*t}, V^{*t}) > 0$ such that $U^{*t} y_o$=1.

Component efficiency The DEA technique is based on mathematical programming and used for evaluating the relative efficiency of homogeneous unit. The relative efficiency of each DMU is a function of its inputs and outputs. A DMU is technically efficient if it uses inputs in the best way. IN other word, it does not have any waste in inputs and shortfall in outputs. When a multiple component system is evaluated as efficient, we can say that all of its components are efficient. But if it is inefficient by evaluation, using standard DEA models with regard to which its components are inefficient is not appropriate. Note that neither technical efficiency nor any other type of efficiency, including cost efficiency,

profit efficiency and total efficiency which are evaluated by standard DEA models, can be discussed in this field.
For the first time, Cook et al. [4] published a paper on component efficiency in 2001. Also, Beasley [1] conducted a research independently about instructional efficiency of London University whose results led to the publication of a paper in 2002. Then, Cook , himself completed his paper. Jahanshahloo et al. [7] considered the shared inputs and outputs as non-discretionary factors. The following subsection contains the organized and modified models that have been presented up to now.

2.1 Measuring the efficiency of DMUs with multiple components

In this subsection, we are going to extend the pervious method for the cases in which the evaluated units have multiple components. In this regard, consider the following figure:

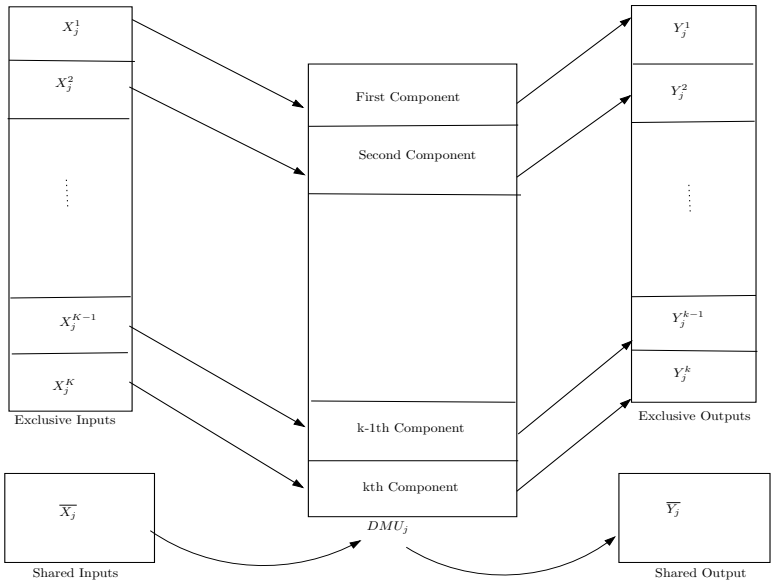

Fig. 1. A DMU with k components

Consider DMU_j, and note that $x_j^l = (x_{1j}^l, ..., x_{mj}^l), \quad l = 1, ..., k$ is the exclusive input of the lth component and $\overline{x_j} = (\overline{x_{1j}}, ..., \overline{x_{pj}})$ is the shared input for all components. Also, $y_j^l = (y_{1j}^l, ..., y_{sj}^l)$ is the exclusive output of the lth component and $\overline{y_j} = (\overline{y_{1j}}, ..., \overline{y_{tj}})$ is the shared output for all components. Let $\alpha^i = (\alpha_1^i, ..., \alpha_k^i) \;\; i = 1, ..., p$ be the contribution of shared inputs for all components. Now, consider the vector $(\alpha_l^1 \overline{x_{1j}}, ..., \alpha_l^p \overline{x_{pj}}, \;\; l = 1, ..., k)$, where $\alpha_l^i \overline{x_{ij}}$ is the contribution of the ith shared input for the lth component. Note that

$$\sum_{l=1}^{k} \alpha_l^i = 1, \quad \alpha_l^i \geq 0, \quad i = 1, ..., p; \quad l = 1, ..., k$$

For more explanation, consider the following matrix:

$$\alpha = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & ... & \alpha_p^1 \\ \alpha_1^2 & \alpha_2^2 & ... & \alpha_p^2 \\ . & . & ... & . \\ \alpha_1^k & \alpha_2^k & ... & \alpha_p^k \end{pmatrix} = [\alpha^1, .., \alpha^p]$$

Note that the sum of each column is one.
Further, let $\beta^r = (\beta_1^r, ..., \beta_k^r), \quad r = 1, ..., t$ be the contribution of shared outputs for all components. Now, consider the vector $(\beta_l^1 \overline{y_r}, ..., \beta_k^r \overline{y_r}), \quad r = 1, ..., t$, where $\beta_l^r \overline{y_{rj}}$ is the contribution of the rth shared output for the lth component. For more convenience, consider the following matrix:

$$\beta = \begin{pmatrix} \beta_1^1 & \beta_2^1 & ... & \beta_t^1 \\ \beta_1^2 & \beta_2^2 & ... & \beta_t^2 \\ . & . & ... & . \\ \beta_1^k & \beta_2^k & ... & \beta_t^k \end{pmatrix} = [\beta^1, .., \beta^t]$$

Note that the sum of each column of this matrix is also equal to one.
According to the previous statements, the input vector of the lth component is as follows:

$$(x^l, \overline{x_j}\alpha^i) = (x_{1j}^l, ..., x_{mj}^l, \alpha_l^1 \overline{x_{1j}}, ..., \alpha_l^p \overline{x_{pj}}), \quad l = 1, ..., k$$

And the output vector of the lth component is:

$$(y^l, \overline{y}\beta^r) = (y_{1j}^l, ..., y_{sj}^l, \beta_l^1 \overline{y_{1j}}, ..., \beta_l^t \overline{y_{tj}}), \quad l = 1, ..., k$$

If we denote the efficiency of the lth component by e_j^l, then:

$$e_j^l = \frac{\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}}}{\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij}}, \quad l = 1, ..., k \tag{2.3}$$

and the aggregated efficiency of DMU_j is as follows:

$$e_j^a = \frac{\sum_{l=1}^{k} \sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k} \sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}}}{\sum_{l=1}^{k} \sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{l=1}^{k} \sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij}} \tag{2.4}$$

Clearly, the aggregated efficiency of each DMU is less than or equal to 1, i.e., $e_j^a \leq 1$. Also, the aggregated efficiency of each DMU is the convex combination of the efficiency of all its components, i.e.,

$$e_j^a = \sum_{l=1}^{k} \lambda_l e_j^l, \qquad \sum_{l=1}^{k} \lambda_l = 1, \quad \lambda_l \geq 0, \quad l = 1, ..., k$$

(See [4,7]). Therefore, the following model is obtained to evaluate DMU_o:

$$\begin{aligned} Max \quad & e_o^a \\ s.t. \quad & e_o^l \leq 1, \quad l = 1, ..., k \\ & e_j^a \leq 1, \quad j = 1, ..., n \\ & \sum_{l=1}^{k} \alpha_l^i = 1, \quad i = 1, ..., p \\ & \sum_{l=1}^{k} \beta_l^r = 1, \quad r = 1, ..., t \\ & all\ variables\ are\ nonnegative \end{aligned} \tag{2.5}$$

Clearly, the last problem is not linear. By the following transformation

$$\begin{array}{l} \overline{u_r}^l \beta_l^r = \widetilde{u_r}^l, \quad l = 1, ..., k; \quad r = 1, ..., t \\ \overline{v_i}^l \alpha_l^i = \widetilde{v_i}^l, ...l = 1, ..., k; \quad i = 1, ..., p \end{array} \tag{2.6}$$

and Charnes-Cooper transformation [2], the following linear programming problem is obtained:

$$\begin{array}{ll} Max & \sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{ro}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \widetilde{u_r}^l \overline{y_{ro}} \\ s.t. & \sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{io}^l + \sum_{l=1}^{k}\sum_{i=1}^{p} \widetilde{v_i}^l \overline{x_{io}} = 1 \\ & \sum_{r=1}^{s} u_r^l y_{ro}^l + \sum_{r=1}^{t} \widetilde{u_r}^l \overline{y_{ro}} - \sum_{i=1}^{m} v_i^l x_{io}^l - \sum_{i=1}^{p} \widetilde{v_i}^l \overline{x_{io}} \leq 0, \quad l = 1, ..., k \\ & \sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \widetilde{u_r}^l \overline{y_{rj}} - \sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l - \sum_{l=1}^{k}\sum_{i=1}^{p} \widetilde{v_i}^l \overline{x_{ij}} \leq 0, \quad j = 1, ..., n \\ & u_r^l \geq 0, \quad l = 1, ..., k; \quad r = 1, ..., s. \\ & \widetilde{u_r}^l \geq 0, \quad l = 1, ..., k; \quad r = 1, ..., t. \\ & v_i^l \geq 0, \quad l = 1, ..., k; \quad i = 1, ..., m. \\ & \widetilde{v_i}^l \geq 0, \quad l = 1, ..., k; \quad i = 1, ..., p. \end{array} \tag{2.7}$$

The efficiency of each component is obtained by solving this problem.

3 Proposed model to estimate the multi components efficiency

Recall the definitions of the previous section. The authors here proposed the following common set of weights model to obtain the efficiency scores of all components:

$$\begin{array}{lll} Max & \{e_1^a, ..., e_n^a\} & \\ s.t. & e_j^a \leq 1, & j = 1, ..., n \\ & e_j^l \leq 1, & j = 1, ..., n; \quad l = 1, ..., k \end{array} \tag{3.8}$$

or equivalently:

$$
\begin{aligned}
Max \ \ & \{e_j^a = \frac{\sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}}}{\sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{l=1}^{k}\sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij}} : \quad j = 1, ..., n\} \\
s.t. \ \ & \frac{\sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}}}{\sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{l=1}^{k}\sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij}} \leq 1, \qquad j = 1, ..., n \\
& \frac{\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}}}{\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij}} \leq 1, \quad j = 1, ..., n; \quad l = 1, ..., k \\
& \sum_{l=1}^{k} \alpha_l^i = 1, \quad i = 1, ..., p \\
& \sum_{l=1}^{k} \beta_l^r = 1, \quad r = 1, ..., t \\
& u_r \geq 0, \quad r = 1, ..., s. \\
& \overline{u_r} \geq 0, \quad r = 1, ..., t. \\
& v_i \geq 0, \quad i = 1, ..., m. \\
& \overline{v_i} \geq 0, \quad i = 1, ..., p. \\
& \alpha_l^i \geq 0, \quad l = 1, ..., k; \quad i = 1, ..., p \\
& \beta_l^r \geq 0, \quad l = 1, ..., k; \quad r = 1, ..., t
\end{aligned}
\tag{3.9}
$$

Clearly, this problem is a fractional goal programming model. By adding the proposal of Liu and Peng [6], it can be written as follows:

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{n} z_j \\
s.t. \quad & \frac{\sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}} + z_j}{\sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{l=1}^{k}\sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij}} = 1, \quad j = 1, ..., n \\
& \frac{\sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}}}{\sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{l=1}^{k}\sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij}} \leq 1, \quad j = 1, ..., n \\
& \frac{\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}}}{\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij}} \leq 1, \quad j = 1, ..., n; \quad l = 1, ..., k \\
& \sum_{l=1}^{k} \alpha_l^i = 1, \quad i = 1, ..., p \\
& \sum_{l=1}^{k} \beta_l^r = 1, \quad r = 1, ..., t \\
& u_r \geq 0, \quad r = 1, ..., s. \\
& \overline{u_r} \geq 0, \quad r = 1, ..., t. \\
& v_i \geq 0, \quad i = 1, ..., m. \\
& \overline{v_i} \geq 0, \quad i = 1, ..., p. \\
& z_j \geq 0, \quad j = 1, ..., n. \\
& \alpha_l^i \geq 0, \quad l = 1, ..., k; \quad i = 1, ..., p \\
& \beta_l^r \geq 0, \quad l = 1, ..., k; \quad r = 1, ..., t
\end{aligned}
\tag{3.10}
$$

then we have

$$
\begin{aligned}
Min \quad & \sum_{l=1}^{k} z_l \\
s.t. \quad & \sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}} - \sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l - \sum_{l=1}^{k}\sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij} + z_j = 0, \quad j = 1, ..., n \\
& \sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}} - \sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l - \sum_{l=1}^{k}\sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij} \le 0, \quad j = 1, ..., n \\
& \sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{r=1}^{t} \overline{u_r}^l \beta_l^r \overline{y_{rj}} - \sum_{i=1}^{m} v_i^l x_{ij}^l - \sum_{i=1}^{p} \overline{v_i}^l \alpha_l^i \overline{x}_{ij} \le 0, \quad j = 1, ..., n; \quad l = 1, ..., k \\
& \sum_{l=1}^{k} \alpha_l^i = 1, \quad i = 1, ..., p \\
& \sum_{l=1}^{k} \beta_l^r = 1, \quad r = 1, ..., t \\
& u_r \ge 0, \quad r = 1, ..., s. \\
& \overline{u_r} \ge 0, \quad r = 1, ..., t. \\
& v_i \ge 0, \quad i = 1, ..., m. \\
& \overline{v_i} \ge 0, \quad i = 1, ..., p. \\
& z_j \ge 0, \quad j = 1, ..., n. \\
& \alpha_l^i \ge 0, \quad l = 1, ..., k; \quad i = 1, ..., p \\
& \beta_l^r \ge 0, \quad l = 1, ..., k; \quad r = 1, ..., t
\end{aligned}
\tag{3.11}
$$

Now, consider

$$
\begin{aligned}
\overline{u}_r^l \beta_l^r &= \widetilde{u}_r^l, \quad l = 1, ..., k; \quad r = 1, ..., t \\
\overline{v}_i^l \alpha_l^i &= \widetilde{v}_i^l, \quad l = 1, ..., k; \quad i = 1, ..., p
\end{aligned}
\tag{3.12}
$$

Therefore, the following linear programming model is obtained:

$$
\begin{aligned}
Min \quad & \sum_{j=1}^{n} z_j \\
s.t. \quad & \sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \widetilde{u_r}^l \overline{y_{rj}} - \sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l - \sum_{l=1}^{k}\sum_{i=1}^{p} \widetilde{v_i}^l \overline{x}_{ij} + z_j = 0, \quad j = 1, ..., n \\
& \sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \widetilde{u_r}^l \overline{y_{rj}} - \sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l - \sum_{l=1}^{k}\sum_{i=1}^{p} \widetilde{v_i}^l \overline{x}_{ij} \le 0, \quad j = 1, ..., n \\
& \sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{r=1}^{t} \widetilde{u_r}^l \overline{y_{rj}} - \sum_{i=1}^{m} v_i^l x_{ij}^l - \sum_{i=1}^{p} \widetilde{v_i}^l \overline{x}_{ij} \le 0, \quad j = 1, ..., n; \quad l = 1, ..., k \\
& u_r \ge 0, \quad r = 1, ..., s. \\
& \widetilde{u_r}^l \ge 0, \quad r = 1, ..., t. \\
& v_i \ge 0, \quad i = 1, ..., m. \\
& \widetilde{v_i}^l \ge 0, \quad i = 1, ..., p. \\
& z_j \ge 0, \quad j = 1, ..., n.
\end{aligned}
\tag{3.13}
$$

Clearly, DMU_j is efficient if and only if $z_j = 0$. The aggregated efficiency of DMU_j is as follows:

$$e_j^a = \frac{\sum_{l=1}^{k}\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{l=1}^{k}\sum_{r=1}^{t} \widetilde{u}_r^{\,l}\overline{y_{rj}}}{\sum_{l=1}^{k}\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{l=1}^{k}\sum_{i=1}^{p} \widetilde{v}_i^{\,l}\overline{x}_{ij}} \quad j = 1, ..., n; \quad l = 1, ..., k \tag{3.14}$$

and the efficiency score of each component is as follows:

$$e_j^l = \frac{\sum_{r=1}^{s} u_r^l y_{rj}^l + \sum_{r=1}^{t} \widetilde{u}_r^{\,l}\overline{y_{rj}}}{\sum_{i=1}^{m} v_i^l x_{ij}^l + \sum_{i=1}^{p} \widetilde{v}_i^{\,l}\overline{x}_{ij}} \quad j = 1, ..., n; \quad l = 1, ..., k \tag{3.15}$$

It is obvious that the aggregated efficiency of each DMU is the convex combination of all its components' efficiency scores.

4 Example

In this section we are going to apply our proposed method to a real data set. In order to do this, consider 19 Iranian car factories with the following data:

Table 1
Inputs and Outputs of Car Factories

DMUs	Stocks	Total Assets	Sales	Capital	Total Equity	Net Profit
Company1	100,000,000	492,726	844851	100,000	167,683	43,000
Company2	40,000,000	321,486	298007	40,000	115,972	48,949
Company3	972,000,000	10,660,537	11,462,010.0	972,000	1,909,307	840,107
Company4	3,200,000,000	10,186,199	2,414,483.0	3,200,000	5,636,671	1,259,910
Company5	1,200,000,000	6,043,419	4796789	1,200,000	2,244,373	912,000
Company6	142,000,000	611,790	579128	142,000	293,385	65,024
Company7	120,000,000	359,500	2,147,552	120,000	293,076	162,858
Company8	72,000,000	829,380	687994	72,000	129,203	5,040
Company9	10,000,000	193,570	135061	10,000	36,841	1,300
Company10	600,000,000	5,369,319	5167457	600,000	1,568,362	720,000
Company11	40,000,000	272,434	189,781.0	40,000	52,121	6,159
Company12	7,000,000,000	29,649,891	32,886,940.0	7,000,000	15,588,292	12,263,233
Company13	60,000,000	204,703	129664	60,000	66,156	2,288
Company14	34,000,000	167,676	172,681.0	34,000	55,498	35,747
Company15	600,000,000	12,355,705	6,034,301.0	600,000	879,072	684,072
Company16	13,000,000	148,175	218298	13,000	21,059	5,200
Company17	170,100,000	1,257,490	1679620	170,100	414,424	93,555
Company18	5,000,000,000	64,766,600	43,633,916.0	5,000,000	6,802,398	4,164,647
Company19	150,000,000	820,490	1,600,077.0	150,000	163,426	366

where Stocks (the number of), Total Assets (1000 million Rials) and Capital (1000 million Rials) are inputs; Sales (1000 million Rials), Total Equity (Rials) and Net Profit(1000 million Rials) are suggested outputs. All of these car factories have two components: 1- Department of Production, 2- Department of Administration. Now, consider the following figure:

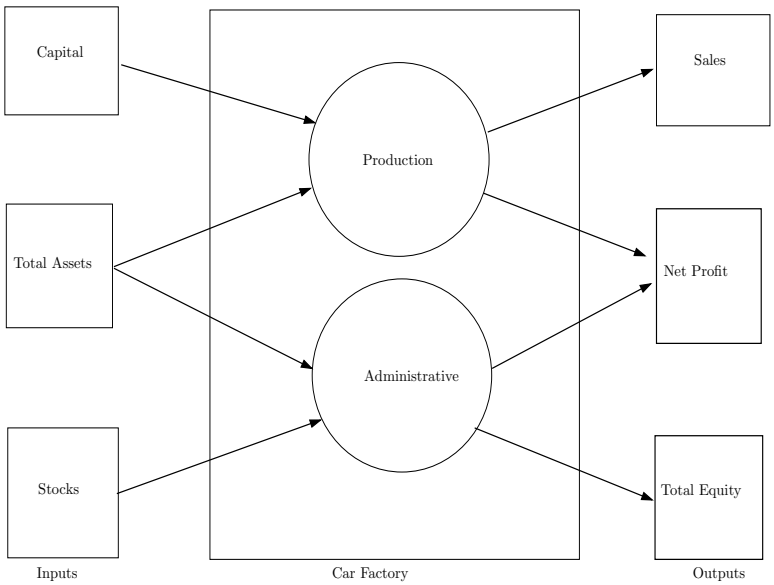

Fig. 2. A Car Factory with two components

Capital and stocks are exclusive inputs of the first and second components and total assets is the shared input. Also, sales and total equity are exclusive outputs of the components and net profit is the shared output.
By using model (2.7) the following efficiency scores are obtained:

Table 2
Efficiency Scores

DMUs	Aggregated Efficiency	First Component Efficiency	Second Component Efficiency
Company1	0.640574	0.000635	1
Company2	1	1	1
Company3	0.735246	1	0.709122
Company4	0.704606	0.014023	0.792339
Company5	0.709981	0.010820	0.853510
Company6	0.806255	0.001220	0.950728
Company7	1	1	1
Company8	0.599226	1	0.374097
Company9	1	1	1
Company10	0.939651	1	0.900295
Company11	0.467532	0.077381	0.550223
Company12	1	1	1
Company13	0.444467	0.000253	0.507700
Company14	0.673767	1	0.639217
Company15	0.758871	1	0.713396
Company16	0.938249	1	0.000287
Company17	0.864048	0.170090	1
Company18	0.569334	1	0.501034
Company19	0.595755	1	0.000260

Now, consider company 4. Clearly it is inefficient because all of its components have efficiency scores less than one. Also, unit 1 is inefficient because the first component has an efficiency score less than unity. Of course company 2 is efficient because all of its components are efficient.
Again consider model (3.13). By using the data, the following results are obtained:

Table 3
Efficiency Scores in new model

DMUs	z_j	Aggregated Efficiency	First Component Efficiency	Second Component Efficiency
1	0.0005	0.43798	0.41531	0.47998
2	0.0002	0.58942	0.45190	0.82791
3	0.0048	0.51939	0.52917	0.50327
4	0.0192	0.24498	0.09432	0.53696
5	0.0062	0.39421	0.27807	0.60886
6	0.0007	0.36285	0.24175	0.59067
7	0	1	1	1
8	0.0005	0.32865	0.34441	0.30286
9	0.0001	0.43844	0.41391	0.47492
10	0.0025	0.57693	0.48183	0.73913
11	0.0003	0.24001	0.20922	0.29470
12	0.0203	0.64886	0.46398	0.99730
13	0.0004	0.15790	0.09984	0.26973
14	0.0002	0.46298	0.36704	0.64074
15	0.0049	0.38333	0.40815	0.34684
16	0.0001	0.52869	0.64159	0.34375
17	0.0008	0.51118	0.46059	0.60002
18	0.0336	0.38618	0.39718	0.36856
19	0.0008	0.36407	0.43768	0.22940

Consider company 7; $z_7 = 0$ so it is efficient, since all of its components have efficiency scores of one. Other units are inefficient because their derivation variables are positive. Also, their components have efficiency score of less than unity. The number of efficient units is decreased by using the common set of weights.

5 Conclusions

The DEA model presented here, can be used for the analysis of any real situation where a significant number of inputs and outputs are included and management views the production process as a multi-stage process. The provision of component efficiency facilitates managerial actions only on those components where the DMU is underperforming. The particular application area investigated is that involving the economical functions within Iranian car factories. Using the data from Iranian car factories, this study has measured the multi-component efficiency of the economical functions in 19 Iranian car factories. We used a "common set of weights" to propose a new model for measuring the efficiency score of multiple component units. One of the most important advantages of this model is that by solving only one linear programming problem, we can obtain the efficiency scores of all components and the aggregated efficiency of all DMUs. It can be observed that the number of efficient units is decreased by using the common set of weights.

References

[1] J.E. Beasley, Allocating fixed costs and resources via data envelopment analysis, Eur. J. Oper. Res. 147 (2003) 197-216.

[2] A. Charnes, W.W. Cooper, Programming with linear fractional functions, Naval Research Logistics Quarterly 9 (1962) 181-186

[3] A. Charnes, W.W. Cooper, E. Rhodes, Measuring the efficiency of decision making units, European Journal of Operational Research 2 (6) (1995) 429-444.

[4] W.D. Cook, M. Hababou, H.J.H. Tuenter, Multicomponent efficiency measurement and shared inputs in DEA: an application to sales and service performance in bank branches, Journal of Productivity Analysis 14 (2000) 209-224.

[5] R. Fare, S. Grosskopf, Productivity and intermediate products: a frontier approach, Economic Letters 50 (1) (1996) 65-70.

[6] Fuh-Hwa Franklin Liu, Hao Hsuan Peng. Ranking of units on the DEA frontier with common weights. Computers and Operations Research. In press.

[7] G.R. Jahanshahloo, A.R. Amirteimoori, S. Kordrostami. Measuring the multi-component efficiency with shared inputs and outputs in data envelopment analysis, Applied Mathematics and Computation 155 (2004) 283-293.