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Int. J. Industrial Mathematics Vol. 3, No. 3 (2011) 169-179

Numerical Method for Non-linear Fuzzy Volterra Integral Equations of the Second Kind

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Received 17 January 2011; revised 22 July 2011 ; accepted 6 August 2011.

Abstract

In this article a successive approximations method is used to solve nonlinear fuzzy Volterra integral equations along with representing its error. Then using quadrature rules we approximate the present integrals in the sequence of successive approximations and we obtain the numerical solution we have tended to.

Keywords : Fuzzy number; Fuzzy integral equation; Quadrature rules; numerical method

1 Introduction

In [4, 7, 8] Henstock integral is defined and its properties are investigated, and quadrature rules for this integral are introduced in [2]. Also in [2, 3], successive approximations method which can be obtained by Banach fixed point, is used to present a numerical method to solve nonlinear fuzzy Fredholm integral equations. In this article after introducing Henstock integral for fuzzy-number-valued functions and its quadrature rules, there presented theorems in the existence and uniqueness of the solution of the nonlinear fuzzy Volterra integral equations that has this form

$$x(t) = g(t) + (FH)\int_a^t k(t,s)f(s,x(s))ds \quad , \qquad t \in [a,b]$$

and then, we introduce successive approximations method for that. Later the error of the method is obtained and finally using the mentioned quadrature rules a numerical method is presented to solve this kind of equations.

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2 Preliminaries

Definition 2.1. *, [6]. A fuzzy number is a function* $u : \Re \longrightarrow [0,1]$ *having the properties:*

(i) *u is normal, that is* $\exists x_0 \in \Re$ *with* $u(x_0) = 1$*;*

(ii) *u is fuzzy convex set (that is* $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}\ \forall x, y \in \Re\ \lambda \in [0,1]$*);*

(iii) *u is upper semi-continuous on* $\Re$*;*

(iv) *the support* $\overline{\{x \in \Re : u(x) > 0\}}$ *is a compact set.*

The set of all fuzzy real numbers is denoted by ε^1. For $0 < \alpha \leq 1$, let us define $[u]^\alpha = \{x \in \Re : u(x) \geq \alpha\}$ and $[u]^0 = \overline{\{x \in \Re : u(x) > 0\}}$. Also, we define $u_-^\alpha = \inf [u]^\alpha$ and $u_+^\alpha = \sup [u]^\alpha$.

For $u, v \in \varepsilon^1$ and $\lambda \in \Re$, we have the sum $u + v$ and the product λu defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha\ \forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (as subsets of $\Re$), and $\lambda [u]^\alpha$ means the usual product between a scaler and a subset of $\Re$. We denote by $\sum$ the sum of real numbers and also the sum of fuzzy numbers with respect to + (if the terms are fuzzy numbers).

Also, we use the Hausdorff distance between fuzzy numbers given by $d_\infty : \varepsilon^1 \times \varepsilon^1 \longrightarrow \Re^+ \bigcup \{0\}$. as in [5]

$$d_\infty(u, v) = \sup_{\alpha \in [0,1]} \{d_H([u]^\alpha, [v]^\alpha)\} = \sup_{\alpha \in [0,1]} \max\{|u_-^\alpha - v_-^\alpha|, |u_+^\alpha - v_+^\alpha|\}$$

where $[u]^\alpha = [u_-^\alpha, u_+^\alpha], [v]^\alpha = [v_-^\alpha, v_+^\alpha] \subseteq \Re$ and d_H is the Hausdorff distance. We define $||.||_F = d_\infty(., \tilde{o})$.

Then we have the following theorem and it is known.

Theorem 2.1. *, [1].*

(i) $||.||_F$ *has the properties of a usual norm on* ε^1 *i.e* $||u||_F = o$ *iff* $u = \tilde{o}$, $||\lambda u||_F = |\lambda| ||u||_F$*, and* $||u + v||_F \leq ||u||_F + ||v||_F$*.*

(ii) $|\ ||u||_F - ||v||_F\ | \leq d_\infty(u, v)$ *and* $d_\infty(u, v) \leq ||u||_F + ||v||_F$ *for any* $u, v \in \varepsilon^1$*.*

The following theorem will be very helpful in what follows:

Theorem 2.2. *, [8].*

(i) $(\varepsilon^1, d_\infty)$ *is a complete metric space.*

(ii) $d_\infty(u + v, v + w) = d_\infty(u, w)\ \ \forall u, v, w \in \varepsilon^1$

(iii) $d_\infty(\lambda u, \lambda v) = |\lambda| d_\infty(u, v)\ \ \forall u, v \in \varepsilon^1,\ \ \forall \lambda \in \Re$

(iv) $d_\infty(u+v, w+e) \leq d_\infty(u,w) + d_\infty(v,e) \quad \forall u,v,w,e \in \varepsilon^1$

In [8] Congxin Wu and Zengtai Gong introduced the concept of the Henstock integral for a fuzzy-number-valued function.

Let $f : [a,b] \longrightarrow \varepsilon^1$. For $\Delta_n : a = x_0 < x_1 < ... < x_n = b$ a partition of the interval $[a,b]$, let us consider the intermadiate points $\zeta_i \in [x_{i-1}, x_i], i = 1, ..., n$, and $\delta : [a,b] \longrightarrow \Re^+$. The division $P = \{([x_{i-1}, x_i]; \zeta_i); i = 1, ..., n\}$ denoted shortly by $P = (\Delta_n, \zeta)$ is said to be $\delta - fine$ if:

$$[x_{i-1}, x_i] \subseteq (\zeta_i - \delta(\zeta_i), \zeta_i + \delta(\zeta_i)).$$

The function f is called Henstock integrable to $I \in \varepsilon^1$ if for every $\epsilon > 0$ there is a function $\delta : [a,b] \longrightarrow \Re^+$ such that for any $\delta - fine$ division P we have:

$$d_\infty(\Sigma_{i=1}^n (x_i - x_{i-1}) f(\zeta_i), I) < \epsilon.$$

Then I is called the Henstock integral of f and it is denoted by:

$$(FH) \int_a^b f(t)dt.$$

If the above $\delta : [a,b] \longrightarrow \Re^+$ is constant function, then one recaptures the concept of Riemann integral introduced by Goestchel and Voxman [6]. In this case $I \in \varepsilon^1$ will be called the Riemann integral of f on $[a,b]$ and will be denoted by:

$$(FR) \int_a^b f(t)dt.$$

Theorem 2.3. *, [8].*

(i) *If f and g are Henstock integrable mapping and if $d_\infty(f(t), g(t))$ is Lebesgue integrable, then:*

$$d_\infty(\ (FH) \int_a^b f(t)dt\ ,\ (FH) \int_a^b g(t)dt\ ) \leq (L) \int_a^b d_\infty(f(t), g(t))dt.$$

(ii) *Let $f : [a,b] \longrightarrow \varepsilon^1$ be a Henstock integrable bounded mapping. Then for any fixed $u \in [a,b]$, the function $\varphi_u : [a,b] \longrightarrow \Re$ defined by $\varphi_u(t) = d_\infty(f(u), f(t))$ is Lebesgue integrable on $[a,b]$.*

Definition 2.2. *, [5]. Let $f : [a,b] \to \varepsilon^1$ be a bounded mapping. Then the function $\omega_{[a,b]}(f, .) : \Re^+ \cup \{o\} \to \Re^+$*

$$\omega_{[a,b]}(f, \delta) = \sup\{\ d_\infty(f(x), f(y));\ x, y \in [a,b], |x-y| \leq \delta\}$$

is called the modulus of oscillation of f on $[a,b]$. If f is continuous on $[a,b]$, then $\omega_{[a,b]}(f, \delta)$ is called uniform modulus of continuity of f.

Some properties of the modulus of oscillation are presented below:

Theorem 2.4. , *[5]. The following properties hold:*

(i) $d_\infty(f(x), f(y)) \leq \omega_{[a,b]}(f, |x-y|)$ *for any* $x, y \in [a,b]$;

(ii) $\omega_{[a,b]}(f, \delta)$ *is increasing function on* δ ;

(iii) $\omega_{[a,b]}(f, 0) = 0$;

(iv) $\omega_{[a,b]}(f, \delta_1 + \delta_2) \leq \omega_{[a,b]}(f, \delta_1) + \omega_{[a,b]}(f, \delta_2)$ *for any* $\delta_1, \delta_2 \geq 0$;

(v) $\omega_{[a,b]}(f, n\delta) \leq n\omega_{[a,b]}(f, \delta)$ *for any* $\delta \geq 0$ *and* $n \in \mathbf{N}$;

(vi) $\omega_{[a,b]}(f, \lambda\delta) \leq (\lambda + 1)\omega_{[a,b]}(f, \delta)$ *for any* $\delta, \lambda \geq 0$;

(vii) *If* $[c,d] \subseteq [a,b]$ *then* $\omega_{[c,d]}(f, \delta) \leq \omega_{[a,b]}(f, \delta)$.

Definition 2.3. , *[2]. For* $L \geq 0$, *a function* $f : [a,b] \to \varepsilon^1$ *is L-Lipschitz if*

$$d_\infty(f(x), f(y)) \leq L|x-y|$$

for any $x, y \in [a,b]$.

3 Quadrature rules for the Henstock integral

Here we present the quadrature rules obtained in [2], which contain as particular cases the trapezoidal, middle point and three point rules.

Theorem 3.1. , *[2]. Let* $f : [a,b] \to \varepsilon^1$ *be a bounded and Henstock integrable function. Then for any partition* $\Delta : a = t_0 < t_1 < ... < t_n = b$ *and* $\xi_i \in [t_{i-1}, t_i]$ *we have:*

$d_\infty(\ (FH)\int_a^b f(t)dt\ ,\ \sum_{i=1}^n (t_i - t_{i-1}) f(\xi_i)\ ) \leq$

$\sum_{i=1}^n (t_i - t_{i-1})\omega_{[t_{i-1}, t_i]}(f, t_i - t_{i-1}) \leq$

$(b-a)\omega_{[a,b]}(f, \nu(\Delta))$.

where $\nu(\Delta) = \max_{i=1,...,n}\{t_i - t_{i-1}\}$ *is the norm of the partition* Δ.

Particular election of the points ξ_i leads to the following result:

Corollary 3.1. , *[2]. Let* $f : [a,b] \to \varepsilon^1$ *be a bounded and Henstock integrable function. Then:*

$$d_\infty(\ (FH)\int_a^b f(t)dt\ ,\ \frac{b-a}{2}(f(a) + f(b))\ ) \leq \frac{b-a}{2}\omega_{[a,b]}(f, \frac{b-a}{2}).$$

For Lipschitzian functions the following result holds:

Theorem 3.2. *, [2] . Let $f : [a, b] \to \varepsilon^1$ be a L-Lipschitz function. Then for any partition $\Delta : a = t_0 < t_1 < ... < t_n = b$ and $\xi_i \in [t_{i-1}, t_i], i = 1, ..., n$ we have:*

$$d_\infty(\ (FH)\int_a^b f(t)dt\ ,\ \sum_{i=1}^{n}(t_i - t_{i-1})f(\xi_i)) \leq \frac{L}{2}\sum_{i=1}^{n}[(t_i - \xi_i)^2 + (\xi_i - t_{i-1})^2] \leq \frac{L}{2}\sum_{i=1}^{n}(t_i - t_{i-1})^2.$$

Particular election of the points ξ_i leads to the following result:

Corollary 3.2. *, [3]. Let $f : [a, b] \to \varepsilon^1$ be a L-Lipschitz function. Then we have:*

$$d_\infty(\ (FH)\int_a^b f(t)dt\ ,\ \frac{b-a}{2}(f(a) + f(b))\ ) \leq L\frac{(b-a)^2}{4}\ .$$

Finally, we generalize the quadrature formula from the last corollary for a partition $\Delta : a = t_0 < t_1 < ... < t_n = b$. According to [8], the Henstock integral has the property

$$(FH)\int_a^b f(t)dt = \sum_{i=0}^{n-1}(FH)\int_{t_i}^{t_{i+1}} f(t)dt$$

and consequently,

$$d_\infty(\ (FH)\int_{t_i}^{t_{i+1}} f(t)dt\ ,\ \frac{(t_{i+1} - t_i)}{2}(f(t_i) + f(t_{i+1})\ ) \leq L\frac{(t_{i+1} - t_i)^2}{4}\ .$$

where $t_i,\ i = \overline{0, n}$ realize a uniform partition of the interval $[a, b]$, and L is the Lipschitz constant of f. Then, $t_{i+1} - t_i = \frac{b-a}{n}$ and $t_i = a + i\frac{b-a}{n}\ , \forall i = \overline{0, n}$. Using the properties of the distance between fuzzy numbers presented in Theorem 2.2 and the above inequality we obtain the generalization of the trapezoidal inequality for Lipschitzian fuzzy-number-valued functions:

$$d_\infty(\ (FH)\int_a^b f(t)dt\ ,\ \sum_{i=0}^{n-1}\frac{(t_{i+1} - t_i)}{2}(f(t_i) + f(t_{i+1}))\ ) \leq L\frac{(b-a)^2}{4n}\ .$$

4 numerical method

In this section, we consider the nonlinear fuzzy Volterra integral equation

$$x(t) = g(t) + (FH)\int_a^t k(t, s)f(s, x(s))ds\ \ ,\quad t \in [a, b]$$

such that the functions

$$g : [a, b] \to \varepsilon^1 \quad and \quad f : [a, b] \times \varepsilon^1 \to \varepsilon^1$$

are continuous. The following theorems state the existence and uniqueness of the solution of above equation.

Theorem 4.1. *Consider the nonlinear fuzzy Volterra integral equation:*

$$x(t) = Tx = g(t) + (FH)\int_a^t k(t,s)f(s,x(s))ds \quad , \quad t \in [a,b]$$

assume that:

(i) $g : [a,b] \longrightarrow \varepsilon^1$ *satisfies the following conditions:*

(1) $g \in C[a,b]$

(2) $\exists c_g \ ; \ \forall t \in [a,b] \quad ||g(t)||_F \leq c_g$

(ii) $f : [a,b] \times \varepsilon^1 \longrightarrow \varepsilon^1$ *satisfies the following conditions:*

(1) $f \in C([a,b] \times \varepsilon^1)$

(2) $\exists c_1, c_2 > 0 \ ; \forall u \in \varepsilon^1 \quad ||f(s,u)||_F \leq c_1||u||_F + c_2$

(iii) $k : [a,b] \times [a,b] \longrightarrow \Re$ *satisfies the following conditions:*

(1) $\forall t, s \in [a,b] \quad k(t,s) \geq 0 \ , \quad \forall t \leq t_0 \quad k(t,s) \geq k(t_0,s)$

(2) $\sup_{a \leq t \leq b} \int_a^t k(t,s)ds < \frac{1}{c_1}$

(3) *the function* $t \mapsto \int_a^t k(t,s)ds$ *is continuous on* $[a,b]$

(4) $\forall t_0 \in [a,b] \qquad s \mapsto k(t_0,s) \in L^1[a,b]$

then integral equation has a solution $x \in C[a,b]$.

Theorem 4.2. *Assume that the function f(.,.) given by the previous theorem, satisfies the following condition:*

$$\forall u, v \in \varepsilon^1 \qquad d_\infty(\ f(s,u), f(s,v)\ ) \leq L(d_\infty(u,v))^r$$

for some constants $L > 0$ *and* $0 < r \leq 1$.
Then, under the conditions of the previous theorem, integral equation has a unique continuous solution on $[a,b]$.

Now, we apply the methods developed in the previous section to give numerical solutions for the integral equations.

Theorem 4.3. *Under the hypotheses of the above theorems, if* $L(b-a)M_K < 1$, *where* $M_K = \sup_{a<t,s<b} k(t,s)$, *then the iterative procedure*

$$x_0(t) = g(t),$$

$$x_m(t) = g(t) + (FH)\int_a^t k(t,s)f(s,x_{m-1}(s))ds$$

converges to the unique solution x^* of the integral equation and the error estimation is

$$d^*_\infty(x^*, x_m) \leq \frac{(L(b-a)M_K)^m}{1-L(b-a)M_K}(b-a)M_K(c_1c_g+c_2)$$

where $d^*_\infty(f,g) = \sup_{a\leq t\leq b} d_\infty(f(t),g(t))$ denotes the uniform distance between fuzzy-number-valued functions.

proof: Using the first part of the Theorem 2.2 for $m \in \mathbf{N}$ we have

$$d^*_\infty(x^*, x_m) \leq d^*_\infty(x^*, x_{m+1}) + d^*_\infty(x_{m+1}, x_m)$$

moreover

$$d^*_\infty(x^*, x_{m+1}) = d^*_\infty(T(x^*), T(x_m)) \leq L(b-a)M_K d^*_\infty(x^*, x_m)$$

and

$$\begin{aligned} d^*_\infty(x_{m+1}, x_m) &= d^*_\infty(T(x_m), T(x_{m-1})) \\ &\leq L(b-a)M_K d^*_\infty(x_m, x_{m-1}) \\ &\leq (L(b-a)M_K)^2 d^*_\infty(x_{m-1}, x_{m-2}) \\ &\vdots \\ &\leq (L(b-a)M_K)^m d^*_\infty(x_1, x_0) \end{aligned}$$

so

$$d^*_\infty(x^*, x_m) \leq \frac{(L(b-a)M_K)^m}{1-L(b-a)M_K} d^*_\infty(x_1, x_0)$$

finally

$$\begin{aligned} d^*_\infty(x_1, x_0) &= \sup_{a<t<b} d_\infty(x_1(t), x_0(t)) \\ &= \sup_{a<t<b} d_\infty(\ x_0(t) + (FH)\int_a^t k(t,s)f(s, x_0(s))\ ds\ ,\ x_0(t)\ ) \\ &= \sup_{a<t<b} d_\infty(\ (FH)\int_a^t k(t,s)f(s, x_0(s))\ ds\ ,\ \tilde{0}\ ) \\ &\leq \sup_{a<t<b}(L)\int_a^t\ d_\infty(\ k(t,s)f(s, x_0(s))\ ,\ \tilde{0}\ )\ ds \\ &= \sup_{a<t<b}(L)\int_a^t\ k(t,s) d_\infty(\ f(s, x_0(s))\ ,\ \tilde{0}\ )\ ds \\ &= \sup_{a<t<b}(L)\int_a^t\ k(t,s)||f(s, g(s))||_F\ ds \\ &\leq (b-a)M_K(c_1c_g+c_2) \end{aligned}$$

In this way we obtain the inequality of the error estimation. □

The above theorem states that the sequence of successive approximations $(x_m)_{m\in\mathbf{N}}$, converges to the solution x^* of integral equation. To approximate this solution by the terms of the sequence of successive approximations, the integrals must be computed. In this

aim, we use the quadrature formula obtained in previous section.
Consider the uniform partition of the interval $[a, b]$:

$$\Delta = \{a = t_0 < t_1 < ... < t_n = b\}$$

with $t_i = a + i\frac{(b-a)}{n}$, $\forall i = \overline{0, n}$. Then, on the knots of the partition Δ, the successive approximations are:

$$x_0(t_i) = g(t_i),$$

$$x_m(t_i) = g(t_i) + (FH)\int_a^{t_i} k(t_i, s)f(s, x_{m-1}(s))ds.$$

Computing the integrals, we apply the quadrature formula in the Corollary 3.2 and obtain the following algorithm.

$$y_0(t_i) = g(t_i),$$

$$y_1(t_i) = g(t_i) + \sum_{j=0}^{i-1} \frac{(b-a)}{2n}[k(t_i, t_j)f(t_j, g(t_j)) + k(t_i, t_{j+1})f(t_{j+1}, g(t_{j+1}))],$$

$$y_2(t_i) = g(t_i) + \sum_{j=0}^{i-1} \frac{(b-a)}{2n}[k(t_i, t_j)f(t_j, y_1(t_j)) + k(t_i, t_{j+1})f(t_{j+1}, y_1(t_{j+1}))]$$

and by induction, we have

$$y_m(t_i) = g(t_i) + \sum_{j=0}^{i-1} \frac{(b-a)}{2n}[k(t_i, t_j)f(t_j, y_{m-1}(t_j)) + k(t_i, t_{j+1})f(t_{j+1}, y_{m-1}(t_{j+1}))]$$

for $m \geq 3$, $m \in \mathbf{N}$, and $\forall i = \overline{0, n}$.

Example 4.1. *Consider the triangular fuzzy number $A = (0, 1, 2)$ having the level sets $[A]^\alpha = [\alpha, 2 - \alpha]$, $\alpha \in [0, 1]$ and the functions $g : [0, 1] \to \varepsilon^1$, $K : [0, 1] \times [0, 1] \to \Re$ given by $g(t) = A = (0, 1, 2)$, $k(t, s) = \frac{1}{\sqrt{1+t+s}}$ for all $t, s \in [0, 1]$.*
The function $arctan : \Re \to \Re$ is continuous and strictly increasing, so for all $u \in \varepsilon^1$ we have $arctan(u) \in \varepsilon^1$ and we can define

$$[arctan(u)]^\alpha = [arctan(u_-^\alpha), arctan(u_+^\alpha)] \quad , \quad \alpha \in [0, 1]$$

The following integral equation is made using the above functions.

$$x(t) = A + (FH)\int_0^t \frac{1}{\sqrt{1+t+s}} arctan(x(s))\ ds \quad , \quad t \in [0, 1]$$

First, we partition the interval $[0, 1]$ by $t_i = i\frac{1}{5}$, $i = 0, 1, ..., 5$ and continue the algorithm to $m = 10$ to make the following sequence of successive approximations in the t_i points.

$$y_0(t_i) = A,$$

$$y_m(t_i) = A + \sum_{j=0}^{i-1} \frac{1}{10}\left(\frac{1}{\sqrt{1+t_i+t_j}} arctan(y_{m-1}(t_j)) + \frac{1}{\sqrt{1+t_i+t_{j+1}}} arctan(y_{m-1}(t_{j+1}))\right)$$

Using parametric representation of fuzzy numbers this sequence is converted to the two following sequences,

$$(y_0(t_i))_-^{\alpha_k} = \alpha_k \ ,$$

$$(y_m(t_i))_-^{\alpha_k} = \alpha_k + \sum_{j=0}^{i-1} \frac{1}{10} \ (\ \frac{arctan(\ (y_{m-1}(t_j))_-^{\alpha_k} \ )}{\sqrt{1+t_i+t_j}} + \frac{arctan(\ (y_{m-1}(t_{j+1}))_-^{\alpha_k} \ )}{\sqrt{1+t_i+t_{j+1}}} \ )$$

and

$$(y_0(t_i))_+^{\alpha_k} = 2 - \alpha_k \ ,$$

$$(y_m(t_i))_+^{\alpha_k} = 2 - \alpha_k + \sum_{j=0}^{i-1} \frac{1}{10} \ (\ \frac{arctan(\ (y_{m-1}(t_j))_+^{\alpha_k} \ )}{\sqrt{1+t_i+t_j}} + \frac{arctan(\ (y_{m-1}(t_{j+1}))_+^{\alpha_k} \ )}{\sqrt{1+t_i+t_{j+1}}} \ )$$

for $\alpha_k = \frac{k}{10}, \ k = 0, 1, ..., 10$ *and* $m = 1, 2, ..., 10$.
Table 1, shows the values of $(y_{10}(t_i))_-^{\alpha_k}$ *for the membership degrees of* $\alpha_k = \frac{k}{10}, \ k = 0, 1, ..., 10$.

Table 1

	$t=0$	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1$
α						
0	*0*	*0*	*0*	*0*	*0*	*0*
0.1	*0.1000*	*0.1175*	*0.1343*	*0.1507*	*0.1672*	*0.1838*
0.2	*0.2000*	*0.2347*	*0.2677*	*0.2999*	*0.3319*	*0.3639*
0.3	*0.3000*	*0.3512*	*0.3996*	*0.4463*	*0.4923*	*0.5378*
0.4	*0.4000*	*0.4669*	*0.5293*	*0.5891*	*0.6472*	*0.7042*
0.5	*0.5000*	*0.5815*	*0.6567*	*0.7280*	*0.7964*	*0.8628*
0.6	*0.6000*	*0.6950*	*0.7817*	*0.8629*	*0.9401*	*1.0141*
0.7	*0.7000*	*0.8074*	*0.9042*	*0.9940*	*1.0786*	*1.1590*
0.8	*0.8000*	*0.9186*	*1.0245*	*1.1217*	*1.2125*	*1.2984*
0.9	*0.9000*	*1.0288*	*1.1427*	*1.2464*	*1.3425*	*1.4329*
1.0	*1.0000*	*1.1381*	*1.2590*	*1.3683*	*1.4691*	*1.5633*

Table 2, shows the values of $(y_{10}(t_i))_+^{\alpha_k}$ *for the membership degrees of* $\alpha_k = \frac{k}{10}, \ k = 0, 1, ..., 10$.

Table 2

	$t=0$	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1$
α						
0	*2.0000*	*2.1946*	*2.3567*	*2.4976*	*2.6237*	*2.7385*
0.1	*1.9000*	*2.0910*	*2.2504*	*2.3894*	*2.5139*	*2.6275*
0.2	*1.8000*	*1.9870*	*2.1436*	*2.2805*	*2.4033*	*2.5155*
0.3	*1.7000*	*1.8827*	*2.0362*	*2.1707*	*2.2917*	*2.4024*
0.4	*1.6000*	*1.7779*	*1.9281*	*2.0601*	*2.1790*	*2.2880*
0.5	*1.5000*	*1.6728*	*1.8193*	*1.9484*	*2.0651*	*2.1723*
0.6	*1.4000*	*1.5671*	*1.7095*	*1.8355*	*1.9497*	*2.0549*
0.7	*1.3000*	*1.4609*	*1.5988*	*1.7213*	*1.8327*	*1.9356*
0.8	*1.2000*	*1.3540*	*1.4869*	*1.6055*	*1.7138*	*1.8142*
0.9	*1.1000*	*1.2464*	*1.3737*	*1.4879*	*1.5927*	*1.6902*
1.0	*1.0000*	*1.1381*	*1.2590*	*1.3683*	*1.4691*	*1.5633*

If we plot $(t_i,(y_{10}(t_i))^{\alpha_k}_{-}),(t_i,(y_{10}(t_i))^{\alpha_k}_{+})$ *points, the following figurate will be resulted that shows the approximate solution for the integral equation is a fuzzy number.*

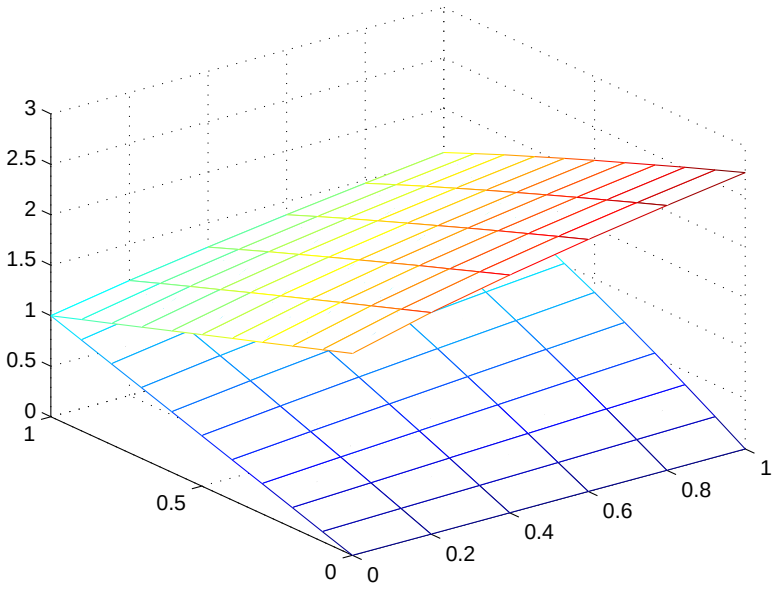

Fig. 1. A fuzzy number (numerical solution of fuzzy integral equation.)

5 Conclusion

The aim of this article has been to present a numerical method for solving nonlinear fuzzy Volterra integral equations. The method approximates the solution for equation in uniform partition points of the interval $[a,b]$. Also, theorems have been presented in the existence and uniqueness of the solution of the Volterra integral equations that have been proved before and have been presented later. The error of this method has been proved, which can be used to show the convergence of the method. A method similar to this

method was previously used by B. Bede and A. M. Bica for Fredholm equations, but our hypothesis for using the method in Volterra equations is different and a little more here.

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