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On Humbert Matrix Function $\Psi_1(A,B;C,C';z,w)$ of Two Complex Variables under Differential Operator

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Abstract

This paper deals with the study of the Humbert function with matrix arguments $\Psi_1(A,B;C,C';z,w)$.
The convergent properties, an integral representation of $\Psi_1(A,B;C,C';z,w)$ and contiguous function relations are presented. Some results are obtained from operating the differential operator D on Humbert matrix function. Moreover a solution of certain partial differential equation is given.

Keywords : Humbert's matrix functions; Integral form; Contiguous function relations; Hypergeometric matrix differential equation; Differential operator.

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1 Introduction

Humbert's functions of scalar coefficients and variables appear in many fields such as statistical distribution theory, heat flow, astrophysics and related areas (see for instance [4, 6, 19]) and Srivastava and Karlsson [17]. Humbert's functions of real variables were generalized to these functions with matrix argument in [7] and [13], see also the books of Mathai [11, 12]. Recently, Upadhyaya and Dhami have presented some properties of the Humbert's functions of matrix arguments in [20, 21]. Jdar and Corts introduced and studied the hypergeometric matrix functions in [9, 10]. Some properties of gamma and beta matrix functions were given in [8].

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Our main purpose in this paper is to obtain an extension of the hypergeometric matrix function to functions of more than one variable. Humbert's matrix functions will be introduced as functions of two complex variables with matrix coefficients. The structure of this paper is as follows:
Section 2 is organized to establish the seven Humbert's matrix functions and calculate their corresponding radius of regularity. In section 3 some integral representation for the Humbert matrix function $\Psi_1(A, B; C, C'; z, w)$ is given. The functions contiguous to the Humbert matrix function Ψ_1 and its relations are given in section 4. In Section 5, a solution of certain partial differential equation is proposed.
A matrix A in $\mathbb{C}^{N\times N}$ is a positive stable matrix if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A and its two-norm denoted by

$$||A|| = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2},$$

where for a vector y in $\mathbb{C}^N$, $||y||_2 = (y^T y)^{\frac{1}{2}}$ is Euclidean norm of y.

Let $\alpha(A)$ and $\gamma(A)$ be the real numbers which were defined in [9] by

$$\alpha(A) = \max\{Re(z) : z \in \sigma(A)\}, \quad \gamma(A) = \min\{Re(z) : z \in \sigma(A)\}. \tag{1.1}$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus (see [3]), it follows that

$$f(A)g(A) = g(A)f(A). \tag{1.2}$$

Hence, if B in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{1.3}$$

The reciprocal Gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is a well-defined matrix.
Furthermore, if

$$A + nI \quad \text{is invertible for all integer } n \geq 0, \tag{1.4}$$

see [9]. The pochhammer symbol or shifted factorial defined by

$$\begin{aligned}(A)_n &= A(A+I)...(A+(n-1)I) \\ &= \Gamma(A+nI)\Gamma^{-1}(A); \quad n \geq 1; (A)_0 = I.\end{aligned} \tag{1.5}$$

Jódar and Cortés have proved in [9] that

$$\Gamma(A) = \lim_{n\longrightarrow\infty} (n-1)![(A)_n]^{-1} n^A. \tag{1.6}$$

The Schur decomposition of A, was given by [5] in the form:

$$\|e^{tA}\| \leq e^{t\alpha(A)} \sum_{k=0}^{r-1} \frac{(\|A\| r^{\frac{1}{2}} t)^k}{k!}; \quad t \geq 0,$$

and

$$\|n^A\| \leq n^{\alpha(A)} \sum_{k=0}^{r-1} \frac{(\|A\| r^{\frac{1}{2}} \ln n)^k}{k!}; \quad n \geq 1. \tag{1.7}$$

The hypergeometric matrix function is defined by the matrix power series in the form

$${}_2F_1(A, B; C; z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n [(C)_n]^{-1}}{n!} z^n. \tag{1.8}$$

If n is large enough, then

$$\|(C + nI)^{-1}\| \leq \frac{1}{n - \|C\|}; \quad n > \|C\|, \tag{1.9}$$

where C in $\mathbb{C}^{N \times N}$ such that $C + nI$ is invertible for all integers $n \geq 0$.
Let us denote

$$\gamma(n) = \|(C)^{-1}\| \|(C + I)^{-1}\| ... \|(C + (n-1)I)^{-1}\|; \quad n > 0, \tag{1.10}$$

2 Definition of Humbert's matrix functions.

Let A, A', B, B', C and C' be matrices in $\mathbb{C}^{N \times N}$ such that $C + nI$ and $C' + nI$ are invertible for all integer $n \geq 0$.
We define Humbert's matrix functions as follows:

$$\Phi_1(A, B, C; z, w) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n} (B)_n [(C)_{m+n}]^{-1}}{m! n!} z^m w^n,$$

$$\Phi_2(A, A', C; z, w) = \sum_{m,n=0}^{\infty} \frac{(A)_m (A')_n [(C)_{m+n}]^{-1}}{m! n!} z^m w^n,$$

$$\Phi_3(A, C; z, w) = \sum_{m,n=0}^{\infty} \frac{(A)_m [(C)_{m+n}]^{-1}}{m! n!} z^m w^n,$$

$$\Psi_1(A, B; C, C'; z, w) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n} (B)_m [(C)_m]^{-1} [(C')_n]^{-1}}{m! n!} z^m w^n,$$

$$\Psi_2(A; C, C'; z, w) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n} [(C)_m]^{-1} [(C')_n]^{-1}}{m! n!} z^m w^n,$$

$$\Xi_1(A, A', B, C; z, w) = \sum_{m,n=0}^{\infty} \frac{(A)_m (A')_n (B)_m [(C)_{m+n}]^{-1}}{m! n!} z^m w^n,$$

$$\Xi_2(A, B, C; z, w) = \sum_{m,n=0}^{\infty} \frac{(A)_m (B)_m [(C)_{m+n}]^{-1}}{m! n!} z^m w^n.$$

Now, let The Humbert matrix function $\Psi_1(A, B; C, C'; z, w)$ of two complex variables by

$$\begin{aligned} \Psi_1(A, B; C, C'; z, w)) &= \sum_{m,n=0}^{\infty} \frac{(A)_{m+n} (B)_m [(C)_m]^{-1} (C')_n]^{-1}}{m! n!} z^m w^n \\ &= \sum_{m,n=0}^{\infty} U_{m,n}(z, w), \end{aligned} \tag{2.11}$$

where

$$U_{m,n}(z, w) = \frac{(A)_{m+n} (B)_m [(C)_m (C')_n]^{-1}}{m! n!} z^m w^n.$$

We study this function by calculating its radius of convergence R. For this purpose, we recall the relation of [16] and keep in mind that $\sigma_{m,n} \geq 1$. Hence

$$\begin{aligned}
\frac{1}{R} &= \lim_{m+n\to\infty} \sup \left(\frac{\|U_{m,n}\|}{\sigma_{m,n}} \right)^{\frac{1}{m+n}} \\
&= \lim_{m+n\to\infty} \sup \left(\left\| \frac{(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!\sigma_{m,n}} \right\| \right)^{\frac{1}{m+n}} \\
&= \lim_{m+n\to\infty} \sup \Bigg[\Bigg\| \frac{(m+n)^{-A}(A)_{m+n}}{(m+n-1)!}(m+n-1)!(m+n)^A \frac{m^{-B}(B)_m}{(m-1)!}(m-1)!m^B \\
&\qquad \frac{m^{-C}[(C)_m]^{-1}}{(m-1)!}(m-1)!m^C . \frac{m^{-C'}[(C')_n]^{-1}}{(n-1)!}(n-1)!n^{C'} \frac{1}{m!n!\sigma_{m,n}} \Bigg\| \Bigg]^{\frac{1}{m+n}}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{R} &= \lim\sup_{m+n\to\infty} \Bigg[\Bigg\| \Big([\Gamma(A)]^{-1}[\Gamma(B)]^{-1}\Gamma(C)\Gamma(C') \Big) \\
&\qquad (m+n)^A m^{B-C} n^{-C'} \frac{(m+n-1)!}{m!(n-1)!\sigma_{m,n}} \Bigg\| \Bigg]^{\frac{1}{m+n}} \\
&\leq \lim\sup_{m+n\to\infty} \left[\frac{\|(m+n)^A\| \|m^B\| \|m^{-C}\| \|n^{-C'}\| (m+n-1)!}{m!(n-1)!} \right]^{\frac{1}{m+n}}
\end{aligned}$$

For positive numbers μ and positive integers n, we can write

$$m = \mu n \tag{2.12}$$

Using the relation (1.8) and the Stirling formula,we get

$$\begin{aligned}
\frac{1}{R} \quad &\leq \lim_{n(\mu+1)\to\infty} \sup \Bigg[\big(n(\mu+1)\big)^{\alpha(A)} \sum_{k=0}^{N-1} \frac{(\|A\| N^{\frac{1}{2}} \ln((\mu+1)n)^k)}{k!} \big(\mu n\big)^{\alpha(B)} \\
&\sum_{k=0}^{N-1} \frac{(\|B\| N^{\frac{1}{2}} \ln \mu n)^k}{k!} \big(\mu\, n\big)^{-\gamma(C)} \sum_{k=0}^{N-1} \frac{(\|C\| N^{\frac{1}{2}} \ln \mu n)^k}{k!} \big(n\big)^{-\gamma(C')} \\
&\sum_{k=0}^{N-1} \frac{(\|C'\| N^{\frac{1}{2}} \ln n)^k}{k!} \cdot \frac{\sqrt{2\pi\{n(\mu+1)-1\}}\{\frac{n(\mu+1)-1}{e}\}^{n(\mu+1)-1}}{\sqrt{2\pi\mu n}(\frac{\mu n}{e})^{\mu n}\sqrt{2\pi(n-1)}(\frac{(n-1)}{e})^{(n-1)}} \Bigg]^{\frac{1}{n(\mu+1)}} .
\end{aligned}$$

So,

$$\begin{aligned}
\sum_{k=0}^{N-1} \frac{(\|A\| N^{\frac{1}{2}} \ln(n(\mu+1))^k}{k!} &\leq (N \ln(n(\mu+1))^{N-1} \sum_{k=0}^{N-1} \frac{(\|A\|)^k}{k!} \\
&= (N \ln(n(\mu+1))^{N-1} e^{\|A\|}
\end{aligned}$$

and

$$\frac{1}{R} \leq \lim_{n(\mu+1)\to\infty} \sup \Bigg[\left(n^{\alpha(A)+\alpha(B)-\gamma(C)-\gamma(C')}\right) \left((N \ln((\mu+1)n)^{N-1} e^{\|A\|}\right)$$

$$\left((N \ln(\mu\, n)^{N-1} e^{\|B\|}\right) \left((N \ln(\mu\, n)^{N-1} e^{\|C\|}\right) \left((N \ln(n)^{N-1} e^{\|C'\|}\right) \Bigg]^{\frac{1}{n(\mu+1)}}.$$

Therefore

$$\lim_{n(\mu+1)\to\infty} \sup \left[\frac{\sqrt{2\pi\{n(\mu+1)-1\}}\{\frac{n(\mu+1)-1}{e}\}^{n(\mu+1)-1}}{\sqrt{2\pi\mu n}(\frac{\mu n}{e})^{\mu n} \sqrt{2\pi(n-1)}(\frac{(n-1)}{e})^{(n-1)}} \right]^{\frac{1}{n(\mu+1)}} = 1,$$

i.e. the radius of convergence of the Humbert matrix function Ψ_1 is one.
By analogous way, one can prove that the other Humbert's matrix functions are regular in the hypersphere S_R; $R = 1$.

3 An integral representation.

We begin this section by considering the binomial function of a matrix exponent. If u and b are complex numbers with $|u| < 1$, then $(1-u)^b = \exp(b \log(1-u))$, where Log is the principal branch of the logarithm function, (see[18]). The Taylor expansion of $(1-u)^{-a}$ about $u = 0$ is given by

$$(1-u)^{-a} = \sum_{n\geq 0} \frac{(a)_n}{n!} u^n, \quad |u| < 1, \quad a \in C. \tag{3.13}$$

Now, we consider the function of the complex variable a defined by (3.13). Let $f_n(a)$ be the function defined by

$$f_n(a) = \frac{(a)_n}{n!} u^n = \frac{a(a+1)...(a+n-1)}{n!} u^n, \quad a \in C. \tag{3.14}$$

For a fixed complex number u with $|u| < 1$, it is clear that f_n is a holomorphic function of variable a defined in the complex plane. Given a closed bounded disc $D_R = \{a \in C : |a| \leq R\}$, one gets

$$|f_n(a)| \leq \frac{(|a|)_n}{n!} |u|^n \leq \frac{(R)_n}{n!} |u|^n, \quad n \geq 0,\ |a| \leq R.$$

Since $\sum_{n\geq 0} \frac{(R)_n}{n!} |u|^n < +\infty$, by the Weierstrass theorem for the convergence of holomorphic function (see[18]) it follows that

$$g(a) = \sum_{n\geq 0} \frac{(a)_n}{n!} u^n = (1-u)^{-a}$$

is holomorphic in C. By application of the holomorphic functional calculus ([3]) for any matrix A in $\mathbb{C}^{N\times N}$, the image of g by this functional calculus acting on A yields

$$(1-u)^{-A} = \sum_{n\geq 0} \frac{(A)_n}{n!} u^n, \quad |u| < 1, \tag{3.15}$$

where $(A)_n$ is given by (1.6).

Let B and C be matrices in $\mathbb{C}^{N\times N}$ such that

$$BC = CB. \tag{3.16}$$

where

$$C, B \text{ and } C-B \quad \text{are positive stable.} \tag{3.17}$$

By (3.13) and (3.17), one gets

$$\begin{aligned}(B)_m[(C)_m]^{-1} &= \Gamma(B+mI)\Gamma^{-1}(B)[\Gamma(C+mI)\Gamma^{-1}(C)]^{-1}\\ &= \Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C-B)\Gamma(B+mI)\Gamma^{-1}(C+mI)\Gamma(C).\end{aligned} \tag{3.18}$$

By lemma 2 in [8] and (3.16) and (3.17), we find that

$$\begin{aligned}\int_0^1 t^{B+(m-1)I}(1-t)^{C-B-I}dt &= B(B+mI, C-B)\\ &= \Gamma(C-B)\Gamma(B+mI)\Gamma^{-1}(C+mI).\end{aligned} \tag{3.19}$$

Also, by (3.18) and (3.19), we get

$$(B)_m[(C)_m]^{-1} = \Gamma^{-1}(B)\Gamma^{-1}(C-B)\Big(\int_0^1 t^{B+(m-1)I}(1-t)^{C-B-I}dt\Big)\Gamma(C) \tag{3.20}$$

Hence, formally one can write

$$\begin{aligned}\Psi_1(A,B;C,C';z,w) &= \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n\\ &= \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}\Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C)}{(C')_n m!n!}\\ &\quad \Big(\int_0^1 t^{B+(m-1)I}(1-t)^{C-B-I}dt\Big) z^m w^n.\end{aligned}$$

Since $(A)_{m+n} = (A)_n(A+n)_m$, for $|z|<1$, by the relation (3.15), we get

$$\begin{aligned}\Psi_1(A,B;C,C';z,w) &= \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n\\ &= \Big(\int_0^1 t^{B-I}(1-t)^{C-B-I}(1-tz)^{-A}\ {}_1\Psi_1(A,-;-,C';\frac{w}{1-tz}dt\Big)\\ &\quad \times \Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C).\end{aligned}$$

We have proved the following theorem:

Theorem 3.1. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ such that $CB = BC$ where C, B, C-B are positive stable. Then for $|z|<1$ we have*

$$\begin{aligned}\Psi_1(A,B;C,C';z,w) &= \Big(\int_0^1 t^{B-I}(1-t)^{C-B-I}(1-tz)^{-A}\ {}_1\Psi_1(A,-;-,C';\frac{w}{1-tz}dt\Big)\\ &\quad \times \Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C).\end{aligned} \tag{3.21}$$

4 The contiguous function relations.

In this section some recurrence relations are carried out on the Horn matrix function. In this connection the following contiguous functions relations follow directly by increasing or decreasing one in original relation, we use the notations

$$\begin{aligned} \Psi_1 =& \Psi_1(A,B;C,C';z,w), \\ \Psi_1(A+) =& \Psi_1(A+I,B;C,C';z,w), \\ \Psi_1(A-) =& \Psi_1(A-I,B;C,C';z,w), \end{aligned} \tag{4.22}$$

together with notations of $\Psi_1(B+)$, $\Psi_1(B-)$, $\Psi_1(C+)$, $\Psi_1(C-)$, $\Psi_1(C'+)$, $\Psi_1(C'-)$, for the other functions contiguous to Ψ_1. Now, we may write the functions contiguous to Ψ_1 in the form

$$\begin{aligned} \Psi_1(A+) =& \sum_{m,n=0}^{\infty} \frac{(A+I)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ =& A^{-1} \sum_{m,n=0}^{\infty} \frac{(A+(m+n)I)(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ =& A^{-1} \sum_{m,n=0}^{\infty} (A+(m+n)I)U_{m,n}(z,w), \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} \Psi_1(A-) =& \sum_{m,n=0}^{\infty} \frac{(A-I)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ =& \sum_{m,n=0}^{\infty} \frac{(A+(m+n-1)I)^{-1}(A-I)(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ =& \sum_{m,n=0}^{\infty} (A+(m+n-1)I)^{-1}(A-I)U_{m,n}(z,w). \end{aligned} \tag{4.24}$$

Similarly

$$\begin{aligned} \Psi_1(B+) =& B^{-1} \sum_{m,n=0}^{\infty} (B+mI)U_{m,n}(z,w), \\ \Psi_1(B-) =& \sum_{m,n=0}^{\infty} (B+(m-1)I)^{-1}(B-I)U_{m,n}(z,w), \\ \Psi_1(C+) =& \sum_{m,n=0}^{\infty} (C+mI)^{-1}CU_{m,n}(z,w), \\ \Psi_1(C-) =& \sum_{m,n=0}^{\infty} (C-I)^{-1}(C+(m-1)I)U_{m,n}(z,w), \\ \Psi_1(C'+) =& \sum_{m,n=0}^{\infty} (C'+n\,I)^{-1}C'U_{m,n}(z,w), \end{aligned}$$

$$\Psi_1(C'-) = \sum_{m,n=0}^{\infty} (C'-I)^{-1}(C'+(n-1)I)U_{m,n}(z,w).$$

For any integer $k \geq 1$, we deduce that

$$\Psi_1(A+kI) = \prod_{r=1}^{k}(A+(r-1)I)^{-1} \sum_{m,n=0}^{\infty} \prod_{r=1}^{k}(A+(m+n+(r-1)I)U_{m,n}(z,w), \tag{4.25}$$

$$\Psi_1(A-kI) = \prod_{r=1}^{k}(A-rI) \sum_{m,n=0}^{\infty} \prod_{r=1}^{k}(A+(m+n-r)I)^{-1}U_{m,n}(z,w), \tag{4.26}$$

$$\Psi_1(B+kI) = \prod_{r=1}^{k}(B+(r-1)I)^{-1} \sum_{m,n=0}^{\infty} \prod_{r=1}^{k}(B+(m+(r-1)I))U_{m,n}(z,w), \tag{4.27}$$

$$\Psi_1(B-kI) = \prod_{r=1}^{k}(B-rI) \sum_{m,n=0}^{\infty} \prod_{r=1}^{k}(B+(m-r)I)^{-1}U_{m,n}(z,w), \tag{4.28}$$

$$\Psi_1(C+kI) = \prod_{r=1}^{k}(C+(r-1)I) \sum_{m,n=0}^{\infty} \prod_{r=1}^{k}(C+(m+(r-1)I))^{-1}U_{m,n}(z,w), \tag{4.29}$$

$$\Psi_1(C-kI) = \prod_{r=1}^{k}(C-rI)^{-1} \sum_{m,n=0}^{\infty} \prod_{r=1}^{k}(C+(m-r)I)U_{m,n}(z,w), \tag{4.30}$$

$$\Psi_1(C'+kI) = \prod_{r=1}^{k}(C'+(r-1)I) \sum_{m,n=0}^{\infty} \prod_{r=1}^{k}(C'+(m+(r-1)I))^{-1}U_{m,n}(z,w) \tag{4.31}$$

and

$$\Psi_1(C'-kI) = \prod_{r=1}^{k}(C'-rI)^{-1} \sum_{m,n=0}^{\infty} \prod_{r=1}^{k}(C'+(m-r)I)U_{m,n}(z,w). \tag{4.32}$$

By similar arguments, we can get some examples of contiguous function relations directly as follows:

$$\begin{aligned} \Psi_1(A+,B+) =& A^{-1}B^{-1} \sum_{m,n=0}^{\infty} (A+(m+n)I)(B+mI)U_{m,n}(z,w) \\ \Psi_1(A-,B-) =& (A-I)(B-I) \sum_{m,n=0}^{\infty} [(A+(m+n-1)I)]^{-1} \\ & (B+(m-1)I)^{-1}U_{m,n}(z,w), \end{aligned} \tag{4.33}$$

$$\begin{aligned}
\Psi_1(B+;C+) &= \sum_{m,n=0}^{\infty} CB^{-1}(B+mI)(C+(m+n)I)^{-1}U_{m,n}(z,w),\\
\Psi_1(B-;C-) &= \sum_{m,n=0}^{\infty} (B-I)(B+(m-1)I)^{-1}(C-I)^{-1}\\
&\qquad (C+(m+n-1)I)U_{m,n}(z,w),\\
\Psi_1(B++;C'+) &= \sum_{m,n=0}^{\infty} C'B^{-1}(B+I)^{-1}(B+mI)(B+(m+1)I)\\
&\qquad (C'+(m+n)I)^{-1}U_{m,n}(z,w),\\
\Psi_1(B++;(C+I)+) &= \sum_{m,n=0}^{\infty} (C+I)B^{-1}(B+I)^{-1}(B+mI)(B+(m+1)I)\\
&\qquad (C+(m+n+1)I)^{-1}U_{m,n}(z,w).
\end{aligned} \tag{4.34}$$

Consider the differential operator D, as given in [14, 15, 16], takes the form

$$D = \begin{cases} d_1 + d_2, & m,n \geq 1;\\ 1, & \text{otherwise}, \end{cases} \tag{4.35}$$

where $d_1 = z\frac{\partial}{\partial z}$ and $d_2 = w\frac{\partial}{\partial w}$. This operator has the particularly pleasant property

$$Dz^m w^n = (m+n)z^m w^n.$$

Now, the following contiguous function relations for the Humbert matrix function can be deduced

$$\begin{aligned}
(D\ I + A)\ \Psi_1 &= \sum_{m,n=0} \frac{(A+(m+n)I)(A)_{m+n}(B)_m[(C)_{m+n}]^{-1}}{m!n!} z^m w^n\\
&= \sum_{m,n=0} (A+(m+n)I)U_{m,n}(z,w) = A\ \Psi_1(A+).
\end{aligned} \tag{4.36}$$

By the same way, we get

$$\begin{aligned}
&(d_1\ I + B)\ \Psi_1 = B\ \Psi_1(B+),\\
&(d_1 I + C)\ \Psi_1 = (C-I)\Psi_1(C-) + \Psi_1\\
&(d_2 I + C')\ \Psi_1 = (C'-I)\ \Psi_1(C'-) + \Psi_1.
\end{aligned} \tag{4.37}$$

From (4.33) and (4.34), it follows at once that

$$\begin{aligned}
&(A-(C+C')+2I)\ \Psi_1 = A\ \Psi_1(A+) - (C-I)\ \Psi_1(C-) - (C'-I)\ \Psi_1(C'-),\\
&(A-B)\ \Psi_1 = A\ \Phi_1(A+) - B\ \Phi_1(B+) - d_2\ \Psi_1(B),\\
&(B-C+I)\ \Psi_1 = B\ \Psi_1(B+) - (C-I)\ \Psi_1(C-),\\
&(A-(B+C'-I))\Psi_1 = A\ \Psi_1(A+) - B\ \Psi_1(B+) - (C'-I)\ \Psi_1(C'-).
\end{aligned} \tag{4.38}$$

From these relations the other contiguous function relations can be deduced. Now, Acting the operating with D on the Humbert matrix function yields

$$\begin{aligned} D\ \Psi_1 &= \sum_{m,n=1}^{\infty} \frac{(m+n)(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ &= \sum_{m1,n=0}^{\infty} \frac{m(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ &+ \sum_{m=0,n=1}^{\infty} \frac{n(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ &= z \sum_{m,n=0}^{\infty} (A+(m+n)I)(B+mI)[(C+mI)]^{-1} U_{m,n}(z,w) \\ &= w \sum_{m,n=0}^{\infty} (A+(m+n)I)[(C'+n\ I)]^{-1} U_{m,n}(z,w). \end{aligned} \tag{4.39}$$

Alternatively, equation (4.36) can be written in the form

$$\begin{aligned} D\ \Psi_1 &= zABC^{-1}\ \Psi_1(A+,B+;C+,C';z,w) \\ &+ wAC'^{-1}\ \Psi_1(A+,B;C,C'+;z,w). \end{aligned} \tag{4.40}$$

Now, let us operate with D on the series defining $\Phi_1(A-)$, we thus obtain from the relation (4.24) that

$$\begin{aligned} \Psi_1(A-) &= \sum_{m,n=0}^{\infty} (A-I)[A+(m+n-1)I]^{-1} U_{m,n}(z,w) \\ &= \sum_{m,n=0}^{\infty} \left(I-(m+n)[A+(m+n-1)I]^{-1}\right) U_{m,n}(z,w) \\ &= \Psi_1 - (A-I)^{-1} D\ \Psi_1(A-), \end{aligned}$$

i.e. the $\Psi_1(A,B;C;z,w)$ is a solution of the partial differential equation

$$D\ \Psi_1(A-) - (A-I)\ \Psi_1 + (A-I)\ \Psi_1(A-) = 0. \tag{4.41}$$

Now, for $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, we have

$$\begin{aligned} \Psi_1(2I,B;C,C';z,w) &= \sum_{m,n=0} \frac{(2I)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ &= \sum_{m,n=0} \frac{((m+n+1)I)(I)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ &= \sum_{m,n=0} \frac{((m+n)I)(I)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \quad . \\ &+ \sum_{m,n=0} \frac{(I)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n \\ &= D\ \Psi_1(I,B;C,C';z,w) + \Psi_1(I,B;C,C';z,w) \end{aligned}$$

It turns out that

$$D\ \Psi_1(I,B;C,C';z,w) = \Psi_1(2I,B;C,C';z,w) - \Psi_1(I,B;C,C';z,w). \tag{4.42}$$

On the other hand, we can use $\Psi_1(C+)$ to get

$$\begin{aligned}\Psi_1(C+) = &\sum_{m,n=0}^{\infty} (C+mI)^{-1}CU_{m,n}(z,w)\\ =&\Psi_1 - \ C^{-1}\ d_1\ \Psi_1(C+),\end{aligned}$$

$$d_1\ \Psi_1(C+) = C\ \Psi_1 - C\ \Psi_1(C+). \tag{4.43}$$

From (4.42), we obtain

$$d_1\ \Psi_1(C+) + d_2\ \Psi_1(C+) = C\ \Psi_1 - C\ \Psi_1(C+) + d_2\ \Psi_1(C+),$$

i.e. Operate with D on the function $\Psi_1(C+)$, we obtain

$$D\ \Psi_1(C+) = C\ \Psi_1 - C\ \Psi_1(C+) + d_2\ \Psi_1(C+), \tag{4.44}$$

and

$$D\ \Psi_1(C'+) = C'\ \Psi_1 - C'\ \Psi_1(C'+) + d_1\ \Psi_1(C'+). \tag{4.45}$$

5 The Humbert matrix: differential equation.

The operators D, d_1 and d_2 which have been already used in the derivation of the contiguous function relations, are helpful in deriving differential equation satisfied by (2.11).

$$\begin{aligned}&\Big[d_1(d_1\ I + C - I) + d_2(d_2\ I + C' - I)\Big]\ \Psi_1\\ &= \sum_{m=1,n=0}^{\infty} \frac{m(C+(m-1)I)(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n\\ &+ \sum_{m=0,n=1}^{\infty} \frac{m(C'+(n-1)I)(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^n\\ &= \sum_{m=1,n=0}^{\infty} \frac{(A)_{m+n}(B)_m[(C)_{m-1}]^{-1}[(C')_n]^{-1}}{(m-1)!n!} z^m w^n\\ &+ \sum_{m=0,n=1}^{\infty} \frac{(A)_{m+n}(B)_m[(C)_m]^{-1}[(C')_{n-1}]^{-1}}{m!(n-1)!} z^m w^n.\end{aligned}$$

A shift of index yields

$$
\begin{aligned}
&\Big(d_1(d_1\ I + C - I) + d_2(d_2\ I + C' - I)\Big)\ \Psi_1 \\
&= \sum_{m=1,n=0}^{\infty} \frac{(A)_{m+1+n}(B)_{m+1}[(C)_{m+1}]^{-1}[(C')_n]^{-1}}{m!n!} z^{m+1} w^n \\
&+ \sum_{m=0,n=1}^{\infty} \frac{(A)_{m+n+1}(B)_m[(C)_m]^{-1}[(C')_n]^{-1}}{m!n!} z^m w^{n+1} \\
&= z \sum_{m,n=0}^{\infty} ((A + (m+n)I)((B + mI))U_{m,n}(z,w) \\
&+ w \sum_{m,n=1}^{\infty} ((A + (m+n)I)U_{m,n}(z,w) \\
&= z(D\ I + A)(d_1\ I + B)\ \Psi_1 + w(D\ I + A)\ \Psi_1.
\end{aligned}
$$

It is easy to see that the Humbert matrix function $\Psi_1(A, B; C, C'; z, w)$ should be a solution of a partial differential equation given by

$$
\begin{aligned}
&\Bigg[\Big(z(D\ I + A)(d_1\ I + B) + w(D\ I + A)\Big) \\
&- \Big(d_1(d_1\ I + C - I) + d_2(d_2\ I + C' - I)\Big)\Bigg]\Psi_1(A, B; C, C'; z, w) = 0.
\end{aligned}
\tag{5.46}
$$

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