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Research Article

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# Legendre polynomials for numerical solution of linear fuzzy Fredholm integral equations

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**Abstract**

In this paper, a numerical method is proposed for solving the fuzzy linear Fredholm integral equations of the second kind. For this result, we choose Legendre polynomials as basis functions and collocation method to estimate a solution for an unknown function in these equations. Finally, a numerical example will be stated to illustrate this method.

*Keywords* : Fuzzy number; Fuzzy Fredholm integral equation; Collocation method; Legendre polynomials.

## 1 Introduction

In recent years, many numerical methods have been proposed for solving fuzzy linear integral equations. For example, in [10], the authors used the divided differences and finite differences methods for solving a parametric of the fuzzy Fredholm integral equations of the second kind. Also, in [9], a numerical method is proposed for the approximate solution of fuzzy linear Fredholm functional integral equations of the second kind by using iterative interpolation. Moreover, in [2], a numerical procedure is proposed for solving the fuzzy linear Fredholm integral equations of the second kind by using Lagrange interpolation based on the extension principle. In [7], the classic Galerkin method for solving integral equations of the second kind was improved to fuzzy Galerkin method, and, the error analysis, namely, error estimate, stability and convergence of the extended method were discussed and some results were established. In [5], the homotopy analysis method (HAM) was applied for solving fuzzy linear Fredholm integral equations of the second kind. The results revealed the validity and the great potential of HAM in solving fuzzy integral equations.

In this paper, after converting the following fuzzy integral equation

$$F(t) = g(t)+(FH)\int_a^b k(t,s)F(s)ds \quad ; \quad t \in [a,b]$$

to the two crisp equations, we use the procedure proposed in [8] and approximate the unknown function.

The important point is that, we haven't used the technic of fuzzy linear system introduced in [3].

## 2 Preliminaries

**Definition 2.1** *[6] A fuzzy number is a function $u : \Re \longrightarrow [0,1]$ having the properties:*

**(i)** *$u$ is normal, that is $\exists x_0 \in \Re$ with $u(x_0) = 1$;*

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**(ii)** *$u$ is fuzzy convex set (that is $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ $\forall x, y \in \Re$ $\lambda \in [0,1]$);*

**(iii)** *$u$ is upper semi-continuous on $\Re$;*

**(iv)** *the support $\overline{\{x \in \Re : u(x) > 0\}}$ is a compact set.*

The set of all fuzzy real numbers is denoted by $\varepsilon^1$. For $0 < \alpha \leq 1$, let us define $[u]^\alpha = \{x \in \Re : u(x) \geq \alpha\}$ and $[u]^0 = \overline{\{x \in \Re : u(x) > 0\}}$. Also, we define $u_-^\alpha = \inf [u]^\alpha$ and $u_+^\alpha = \sup [u]^\alpha$.

For $u, v \in \varepsilon^1$ and $\lambda \in \Re$, we have the sum $u + v$ and the product $\lambda u$ defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$ $\forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (as subsets of $\Re$ ), and $\lambda[u]^\alpha$ means the usual product between a scaler and a subset of $\Re$. We denote by $\sum$ the sum of real numbers and also the sum of fuzzy numbers with respect to $+$ (if the terms are fuzzy numbers).

Also, we use the Hausdorff distance between fuzzy numbers given by $d_\infty : \varepsilon^1 \times \varepsilon^1 \longrightarrow \Re^+ \bigcup \{0\}$. as in [4]

$$d_\infty(u, v) = \sup_{\alpha \in [0,1]} \{d_H([u]^\alpha, [v]^\alpha)\}$$

$$= \sup_{\alpha \in [0,1]} \max\{|u_-^\alpha - v_-^\alpha|, |u_+^\alpha - v_+^\alpha|\}$$

where $[u]^\alpha = [u_-^\alpha, u_+^\alpha], [v]^\alpha = [v_-^\alpha, v_+^\alpha] \subseteq \Re$ and $d_H$ is the Hausdorff distance. We define $||.||_F = d_\infty(., \tilde{o})$.

Then we have the following theorem and it is known.

**Theorem 2.1** *See [1]*

**(i)** *$||.||_F$ has the properties of a usual norm on $\varepsilon^1$ i.e $||u||_F = o$ iff $u = \tilde{o}$, $||\lambda u||_F = |\lambda| ||u||_F$, and $||u + v||_F \leq ||u||_F + ||v||_F$.*

**(ii)** *$|\ ||u||_F - ||v||_F\ | \leq d_\infty(u, v)$ and $d_\infty(u, v) \leq ||u||_F + ||v||_F$ for any $u, v \in \varepsilon^1$.*

The following theorem is very important in concept of distance between fuzzy numbers:

**Theorem 2.2** *See [11]*

**(i)** *$(\varepsilon^1, d_\infty)$ is a complete metric space.*

**(ii)** $d_\infty(u + v, v + w) = d_\infty(u, w)$ $\forall u, v, w \in \varepsilon^1$

**(iii)** $d_\infty(\lambda u, \lambda v) = |\lambda| d_\infty(u, v)$ $\forall u, v \in \varepsilon^1$, $\forall \lambda \in \Re$

**(iv)** $d_\infty(u + v, w + e) \leq d_\infty(u, w) + d_\infty(v, e)$ $\forall u, v, w, e \in \varepsilon^1$

In [11] Congxin Wu and Zengtai Gong introduced the concept of the Henstock integral for a fuzzy-number-valued function.

Let $f : [a, b] \longrightarrow \varepsilon^1$. For $\Delta_n : a = x_0 < x_1 < ... < x_n = b$ a partition of the interval $[a, b]$, let us consider the intermadiate points $\zeta_i \in [x_{i-1}, x_i], i = 1, ..., n$ , and $\delta : [a, b] \longrightarrow \Re^+$. The division $P = \{([x_{i-1}, x_i]; \zeta_i); i = 1, ..., n\}$ denoted shortly by $P = (\Delta_n, \zeta)$ is said to be $\delta - fine$ if:

$$[x_{i-1}, x_i] \subseteq (\zeta_i - \delta(\zeta_i), \zeta_i + \delta(\zeta_i)).$$

The function $f$ is called Henstock integrable to $I \in \varepsilon^1$ if for every $\epsilon > 0$ there is a function $\delta : [a, b] \longrightarrow \Re^+$ such that for any $\delta - fine$ division $P$ we have:

$$d_\infty(\Sigma_{i=1}^n (x_i - x_{i-1}) f(\zeta_i), I) < \epsilon.$$

Then $I$ is called the Henstock integral of $f$ and it is denoted by:

$$(FH) \int_a^b f(t)dt.$$

If the above $\delta : [a, b] \longrightarrow \Re^+$ is constant function, then one recaptures the concept of Riemann integral introduced by Goestchel and Voxman [6]. In this case $I \in \varepsilon^1$ will be called the Riemann integral of $f$ on $[a, b]$ and will be denoted by:

$$(FR) \int_a^b f(t)dt.$$

**Theorem 2.3** *See [11]*

**(i)** *If $f$ and $g$ are Henstock integrable mapping and if $d_\infty(f(t), g(t))$ is Lebesgue integrable, then:*

$$d_\infty(\ (FH) \int_a^b f(t)dt\ ,\ (FH) \int_a^b g(t)dt\ ) \leq (L)$$

**Table 1:** The values of $(s_j, P_6(\ \underline{F}(s_j, \alpha_r)\ ))$ for the membership degrees of $\alpha_r = 0, 1, ..., 5$.

| $s_j$ | $r_0 = 0$ | $r_1 = 0.2$ | $r_2 = 0.4$ | $r_3 = 0.6$ | $r_4 = 0.8$ | $r_5 = 1$ |
|---|---|---|---|---|---|---|
| -1 | 0 | 0.2819 | 0.5639 | 0.8458 | 1.1277 | 1.4096 |
| -0.8333 | 0 | 0.2968 | 0.5936 | 0.8904 | 1.1872 | 1.4839 |
| -0.6667 | 0 | 0.3143 | 0.6287 | 0.9430 | 1.2574 | 1.5717 |
| -0.5 | 0 | 0.3351 | 0.6702 | 1.0052 | 1.3403 | 1.6754 |
| -0.3333 | 0 | 0.3596 | 0.7192 | 1.0787 | 1.4383 | 1.7979 |
| -0.1667 | 0 | 0.3885 | 0.7770 | 1.1656 | 1.5541 | 1.9426 |
| 0 | 0 | 0.4227 | 0.8454 | 1.2681 | 1.6908 | 2.1135 |

**Table 2:** the values of $(s_j, P_6(\ \overline{F}(s_j, \alpha_r)\ ))$ for the membership degrees of $\alpha_r = 0, 1, ..., 5$.

| $s_j$ | $r_0 = 0$ | $r_1 = 0.2$ | $r_2 = 0.4$ | $r_3 = 0.6$ | $r_4 = 0.8$ | $r_5 = 1$ |
|---|---|---|---|---|---|---|
| -1 | 2.8193 | 2.5374 | 2.2554 | 1.9735 | 1.6916 | 1.4096 |
| -0.8333 | 2.9679 | 2.6711 | 2.3743 | 2.0775 | 1.7807 | 1.4839 |
| -0.6667 | 3.1434 | 2.8291 | 2.5147 | 2.2004 | 1.8861 | 1.5717 |
| -0.5 | 3.3508 | 3.0157 | 2.6806 | 2.3456 | 2.0105 | 1.6754 |
| -0.3333 | 3.5958 | 3.2362 | 2.8766 | 2.5170 | 2.1575 | 1.7979 |
| -0.1667 | 3.8852 | 3.4967 | 3.1081 | 2.7196 | 2.3311 | 1.9426 |
| 0 | 4.2271 | 3.8044 | 3.3817 | 2.9590 | 2.5363 | 2.1135 |

$$\int_a^b d_\infty(f(t), g(t))dt.$$

**(ii)** *Let $f : [a, b] \longrightarrow \varepsilon^1$ be a Henstock integrable bounded mapping. Then for any fixed $u \in [a, b]$, the function $\varphi_u : [a, b] \longrightarrow \Re$ defined by $\varphi_u(t) = d_\infty(f(u), f(t))$ is Lebesgue integrable on $[a, b]$.*

# 3 Numerical method

Consider the fuzzy linear Fredholm integral equation of the second kind:

$$F(t) = g(t) + (FH)\int_a^b k(t, s)F(s)ds \quad ; \quad t \in [a, b] \tag{3.1}$$

where $k : [a, b] \times [a, b] \longrightarrow \Re$ and $g : [a, b] \longrightarrow \varepsilon^1$ are known functions, but $F : [a, b] \longrightarrow \varepsilon^1$ is an unknown function. Considering the nonnegative kernel $k$ , the above fuzzy equation replaced by the two following crisp equations

$$\underline{F}(t, \alpha) = \underline{g}(t, \alpha) + \int_a^b k(t, s)\underline{F}(s, \alpha)ds; \ \alpha \in [0, 1] \tag{3.2}$$

and

$$\overline{F}(t, \alpha) = \overline{g}(t, \alpha) + \int_a^b k(t, s)\overline{F}(s, \alpha)ds; \alpha \in [0, 1]. \tag{3.3}$$

The Legendre polynomials $L_m(t)$ on the interval $[-1, 1]$ are given by the following recursive formula.

$$\begin{aligned} L_0(t) &= 1 \\ L_1(t) &= t \\ L_{m+1}(t) &= \frac{2m+1}{m+1}\, tL_m(t) \\ &- \frac{m}{m+1}\, L_{m-1}(t) \quad m = 1, 2, 3, ... \end{aligned}$$

We estimate the two unknown functions $\underline{F}(t, \alpha), \overline{F}(t, \alpha)$ through the following Legendre polynomials.

$$\underline{F}(t, \alpha) \approx P_m(\ \underline{F}(t, \alpha)\ ) = \sum_{i=0}^{m} \underline{a}_{i\alpha} L_i(t) \tag{3.4}$$

and

$$\overline{F}(t, \alpha) \approx P_m(\ \overline{F}(t, \alpha)\ ) = \sum_{i=0}^{m} \overline{a}_{i\alpha} L_i(t). \tag{3.5}$$

A=

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.7675 | -0.9028 | 0.6851 | -0.4164 | 0.2533 | -0.1365 | 0.0604 |
| 0.7253 | -0.7185 | 0.3829 | -0.0981 | -0.0043 | 0.0721 | -0.0876 |
| 0.6755 | -0.5310 | 0.1368 | 0.0808 | -0.0768 | 0.0164 | 0.0593 |
| 0.6166 | -0.3398 | -0.0529 | 0.1479 | -0.0476 | -0.0839 | 0.1264 |
| 0.5471 | -0.1440 | -0.1863 | 0.1310 | 0.0185 | -0.1202 | 0.0597 |
| 0.4649 | 0.0570 | -0.2631 | 0.0576 | 0.0750 | -0.0670 | -0.0517 |
| 0.3679 | 0.2642 | -0.2832 | -0.0446 | 0.0940 | 0.0453 | -0.1037 |

By using (3.4) and (3.5) in the equations (3.2) and (3.3)

$$\sum_{i=0}^{m} \underline{a}_{i\alpha} L_i(t) = \underline{g}(t,\alpha) + \int_a^b k(t,s) \sum_{i=0}^{m} \underline{a}_{i\alpha} L_i(s) ds, \tag{3.6}$$

$$\alpha \in [0,1].$$

and

$$\sum_{i=0}^{m} \overline{a}_{i\alpha} L_i(t) = \overline{g}(t,\alpha) + \int_a^b k(t,s) \sum_{i=0}^{m} \overline{a}_{i\alpha} L_i(s) ds, \tag{3.7}$$

$$\alpha \in [0,1].$$

So, we define the residual equations the given form

$$\underline{R}_m(t) = \sum_{i=0}^{m} \underline{a}_{i\alpha} L_i(t) - \underline{g}(t,\alpha) - \int_a^b k(t,s) \sum_{i=0}^{m} \underline{a}_{i\alpha} L_i(s) ds \tag{3.8}$$

and

$$\overline{R}_m(t) = \sum_{i=0}^{m} \overline{a}_{i\alpha} L_i(t) - \overline{g}(t,\alpha) - \int_a^b k(t,s) \sum_{i=0}^{m} \overline{a}_{i\alpha} L_i(s) ds. \tag{3.9}$$

To determine the unknown coefficients $\underline{a}_{i\alpha}, \overline{a}_{i\alpha}$, we choose some points of collocation as

$$\underline{R}_m(s_j) = \overline{R}_m(s_j) = 0 \quad ; \quad j = 0,1,...,m.$$

Such as [8] collocation points are

$$s_j = a + \frac{(b-a)}{m} j \quad ; \quad j = 0,1,...,m.$$

Therefore, we have two linear systems of equations

$$A_m \underline{X_\alpha} = \underline{b}_{m\alpha} \quad , \quad A_m \overline{X_\alpha} = \overline{b}_{m\alpha}$$

in which

$A_m = [\ L_i(s_j) - \int_a^b k(s_j,s) L_i(s) ds\ ]_{j=0}^{m}$ ; $i = 0,1,...,m$

$\underline{X_\alpha}^T = [\ \underline{a}_{i\alpha}\ ]_{i=0}^{m}$ , $\underline{b_m} = [\underline{g}(s_j,\alpha)]$ ; $j = 0,1,...,m$

$\overline{X_\alpha}^T = [\ \overline{a}_{i\alpha}\ ]_{i=0}^{m}$ , $\overline{b_m} = [\overline{g}(s_j,\alpha)]$ ; $j = 0,1,...,m$.

For investigating convergence of procedure and its speed you can refer [8].

**Example 3.1** *Consider the following fuzzy integral equation*

$$F(t) = (0,1,2) + (FH) \int_{-1}^{0} e^{t+s} F(s) ds \ ; \ t \in [-1,0]$$

*where, $(0,1,2)$ is a triangular fuzzy number with level sets as:*

$$[(0,1,2)]^\alpha = [\alpha, 2-\alpha].$$

*Therefore, using parametric representation of fuzzy numbers, we have*

$$\underline{F}(t,\alpha) = \alpha + \int_{-1}^{0} e^{t+s} \underline{F}(s,\alpha) ds \quad ; \quad \alpha \in [0,1]$$

*and*

$$\overline{F}(t,\alpha) = 2 - \alpha + \int_{-1}^{0} e^{t+s} \overline{F}(s,\alpha) ds \quad ; \quad \alpha \in [0,1].$$

*Choosing $\alpha_r = \frac{1}{5} r$ ; $r = 0,1,...,5$ and collocation points as*

$$s_j = -1 + \frac{1}{6} j \ ; \ j = 0,1,...,6$$

*we can use Legendre polynomials $L_0(t), L_1(t), ..., L_6(t)$ to approximate unknown functions $\underline{F}(t,\alpha)$ and $\overline{F}(t,\alpha)$ and get:*

## 4 Conclusion

The aim of this paper, is to propose a simple numerical method to solve linear fuzzy Fredholm integral equations with nonegative kernels. In this method, we use Legendre polynomials to approximate the unknown functions of crisp linear integral equations obtained from the fuzzy integral equation. Such a method for crisp equations has been used in [8] and the convergence of its was investigated there.

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