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Numerical Solutions of Two-dimensional Linear and Nonlinear Volterra Integral Equations: Homotopy Perturbation Method and Differential Transform Method

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Abstract

In this paper, He's homotopy perturbation method is applied to solve two-dimensional linear and nonlinear Volterra integral equations. Then homotopy perturbation method (HPM) is compared with the differential transform method (DTM) for solving two dimensional integral equations. We also give some examples to demonstrate the accuracy of the method. From the computational view point, the homotopy perturbation method is more efficient and easy to use.

Keywords : Two-dimensional Volterra integral equations; Homotopy perturbation method; Differential transform method.

1 Introduction

As we know, much work has been done on developing and analyzing numerical methods for solving one-dimensional integral equations of the second kind [3, 7, 27], but in two-dimensional cases a small amount of work has been done [3, 4, 13, 14].

On the other hand, in recent years, He's homotopy perturbation method has been developed for solving differential and integral equations. The homotopy perturbation method (HPM) was proposed by Ji-Huan He (see [15]-[24]) in 1998. In this method, the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. Using the homotopy technique from topology, a homotopy is constructed

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with an embedding parameter $p \in [0, 1]$, which is considered as a (small parameter). Considerable research has been recently conducted in applying this method to a wide class of linear and nonlinear equations (see [1], [2], [5]-[11]) and also in [26] this method is used for solving system of linear Fredholm integral and integro-differential equations. In [29] this method is applied to one-dimensional system of Fredholm-Volterra type integral equations and in [9, 12] to nth-order integro-differential and nonlinear Volterra-Fredholm integral equations. In this paper, we propose HPM for solving a class of two-dimensional linear and nonlinear Volterra integral equations and compare the homotopy perturbation method with the differential transform method.

2 Description of HPM and DTM

This section is devoted to reviewing HPM and DTM for solving the two-dimensional Volterra integral equations of the from:

$$u(x,t) - \int_0^t \int_0^x k(x,t,y,z,u(y,z))dydz = f(x,y) \tag{2.1}$$

where k and f are continuous functions and k has the following degenerate form:

$$k(x,t,y,z,u(y,z)) = \sum_{i=0}^{p} v_i(x,t)w_i(y,z,u(y,z)). \tag{2.2}$$

To explain HPM, we reconstitute (2.1) as:

$$L(u) = u(x,t) - f(x,t) - \int_0^t \int_0^x k(x,t,y,z,u(y,z))dydz = 0 \tag{2.3}$$

with solution $u(x,t)$, and we define the homotopy $H(u,p)$ by

$$H(u,p) = (1-p)N(u) + pL(u) = 0, \tag{2.4}$$

where $N(u)$ is a functional operator with a known solution, say, u_0, which can be obtained easily. Obviously from (2.4), we have:

$$H(u,0) = N(u), \qquad H(u,1) = L(u)$$

and changing p from 0 to 1 we continuously trace an implicitly defined curve from a starting point $H(u_0,0)$ to a solution function $H(u,1)$.

The solutions to problem (2.3) can be written as [15]:

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2u_2 + \cdots . \tag{2.5}$$

When $p \to 1$ Eq. (2.4) corresponds to Eq. (2.3) and series (2.5) gives the approximate or exact solution of Eq. (2.3), i.e.,

$$u(x,t) = \lim_{p\to 1} u = \sum_{n=0}^{\infty} u_n. \tag{2.6}$$

The series (2.6) is convergent in most of the cases, and also the rate of convergence depends on $N(u)$ and $L(u)$, see [15].

Taking $N(u) = u(x,t) - f(x,t)$, we substitute (2.5) into (2.4) and equate the terms with identical powers of p, obtaining

$$u_0(x,t) = f(x,t),$$
$$u_{N+1}(x,t) = \int_0^t \int_0^x k(x,t,y,z) H_N(y,z) dydz, \quad N = 0,1,2,\cdots.$$

Where the H_N's are so-called He's polynomials [10] which can be calculated by using the formula:

$$H_N(u_0, u_1, \cdots, u_N) = \frac{1}{N!}\frac{\partial^N}{\partial p^N}((\sum_{k=0}^{N} p^k u_k)^r)_{p=0}, \quad N = 0,1,2,\cdots.$$

In the DTM, the solution $u(x,t)$ of Eq. (2.1) can be approximated by truncated Taylor series with respect to u by

$$U(m,n) - \sum_{i=1}^{p}\sum_{j=0}^{N}\sum_{l=0}^{m-1}\sum_{k=0}^{n-1} V_i(k,l)\frac{G_{ij}(m-k-1,n-l-1)}{(m-k)(n-l)} = F(m,n), \quad m,n = 0,1,2,\cdots,N \tag{2.7}$$

where U, V_i, F and G_{ij} are differential transforms of u, v_i, f and $g_{ij} = w_{ij}u^j$, respectively.

We also obtain

$$G_{ij}(m-k-1,n-l-1) = \frac{1}{(m-k-1)}.\frac{1}{(n-l-1)}\sum_{r=0}^{m-k-2}\sum_{s=0}^{n-l-2} w(r,s)u^j(m-k-r-2,n-l-s-2) \tag{2.8}$$

where W_{ij} and U^j are differential transforms of w_{ij} and u^j, respectively. Therefore a recurrence relation for $U(m,n)$ is obtained with the starting values $U(0,0)$, $U(m,0)$, $U(0,n)$ for $m,n = 1,2,\cdots$. Then we use the truncated form

$$u(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} U(m,n).(x-x_0)^m(t-t_0)^n. \tag{2.9}$$

to obtain $u(x,t)$. For details about the DTM and it's application for solving problem (2.1), we refer the reader to [25, 28].

3 Numerical Examples

In this section, we apply HPM to solve three linear and nonlinear Volterra integral equations. These examples are solved numerically in [28] by DTM. The main objective here is to solve these examples by using the HPM given in section 2 and comparing the results with the results in [28].

Example 3.1. *Consider the linear Volterra integral equation [28]*

$$u(x,t) - \int_0^t \int_0^x (xy^2 + \cos z).u(y,z)dydz = x\sin t - \frac{1}{4}x^5 + \frac{1}{4}x^5\cos t - \frac{1}{4}x^2\sin^2 t, \quad x,t \in [0,1], \tag{3.10}$$

with the exact solution $u(x,t) = x\sin t$.

By HPM, let $N(u) = u(x,t) - f(x,t)$. Hence, we may choose a convex homotopy as

$$H(u,p) = u(x,t) - f(x,t) - p\int_0^t\int_0^x k(x,t,y,z,u(y,z))dydz = 0 \tag{3.11}$$

substituting (2.5) into (3.11), and equating the terms with identical powers of p, we have

$$p^0 : u_0(x,t) = f(x,t) \Rightarrow u_0(x,t) = x\sin t - \frac{1}{4}x^5 + \frac{1}{4}x^5\cos t - \frac{1}{4}x^2\sin^2 t.$$

$$p^1 : u_1(x,t) = \int_0^t\int_0^x (xy^2 + \cos z)u_0(y,z)dydz$$

$$\Rightarrow u_1(x,t) = \frac{1}{4}x^5 + \frac{1}{4}x^2 - \frac{1}{32}x^9 t + \frac{1}{32}x^9\sin t - \frac{1}{24}x^6\sin t - \frac{1}{240}x^6 t + \frac{11}{240}x^6\cos t\sin t$$
$$- \frac{1}{4}x^5\cos t - \frac{1}{36}x^3\sin^3 t - \frac{1}{4}x^2\cos^2 t.$$

$$p^2 : u_2(x,t) = \int_0^t\int_0^x (xy^2 + \cos z)u_1(y,z)dydz$$

$$\Rightarrow u_2(x,t) = \frac{1}{648}x^7\cos t\sin^2 t - \frac{1}{1680}x^7 t\sin t + \frac{1}{240}x^6 t + \frac{1}{24}x^6\sin t - \frac{11}{240}x^6\cos t\sin t$$
$$- \frac{149}{45360}x^7 - \frac{1}{320}x^{10}t\sin t - \frac{1}{36}x^3\cos^2 t\sin t - \frac{1}{32}x^9 t - \frac{1}{32}x^9\sin t + \frac{1}{384}x^{10}$$
$$- \frac{1}{576}x^4\sin^4 t - \frac{1}{384}x^{13}\cos t - \frac{11}{5040}x^7\cos^3 t + \frac{13}{8640}x^{10}\cos^t - \frac{71}{17280}x^{10}\cos^2 t$$
$$+ \frac{1}{384}x^{13} - \frac{1}{4320}x^{10}t^2 + \frac{113}{45360}x^7\cos t + \frac{1}{336}x^7\cos^2 t - \frac{1}{768}x^{13}t^2 + \frac{1}{36}x^3\sin t.$$

and in general

$$P^{N+1} : u_{N+1}(x,t) = \int_0^t\int_0^x (xy^2 + \cos z)H_N(y,z)dydz, \qquad N = 0,1,2,\cdots. \tag{3.12}$$

Therefore, the approximate solution of Example 3.1. can be readily obtained by

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{3.13}$$

In practice, all terms of series (3.13) can not be determined and so we use an approximation of the solution by the following truncated series:

$$u_N(x,t) = \sum_{i=0}^{N-1} u_i(x,t), \qquad \text{with} \qquad u(x,t) = \lim_{N\to\infty} u_N(x,t). \tag{3.14}$$

As we report absolute error which is defined by

$$e_N(x,t) = |u(x,t) - u_N(x,t)|.$$

The results for Example (3.1) are shown in Table 1 (with five and six terms).

Table 1
Numerical results of Example (3.1)

(x,t)	$(e_N(x,t)\, for N = 10)_{DTM}$	$(e_N(x,t)\, for N = 12)_{DTM}$	$(e_N(x,t)\, for N = 5)_{HPM}$	$(e_N(x,t)\, for N = 6)_{HPM}$
(0.2,0.2)	0.102586e-15	0.270000e-19	0.6255e-17	0.1e-19
(0.2,0.8)	0.428632e-9	0.176034e-11	0.502058e-13	0.11286e-15
(0.4,0.6)	0.362716e-10	0.837533e-13	0.135548e-11	0.487378e-14
(0.4,1)	0.995691e-8	0.639314e-10	0.227857e-10	0.122655e-12
(0.6,0.2)	0.307760e-15	0.800000e-19	0.204143e-13	0.4019e-16
(0.6, 0.8)	0.128589e-8	0.528102e-11	0.176085e-9	0.126656e-11
(0.8,0.4)	0.839748e-12	0.861510e-15	0.296845e-10	0.164857e-12
(0.8,0.8)	0.171453e-8	0.704137e-11	0.242572e-8	0.253020e-10
(1,0.6)	0.906789e-10	0.209383e-12	0.462107e-8	0.526451e-10
(1,1)	0.248923e-7	0.159828e-9	0.101211e-6	0.179715e-8

Example 3.2. *We consider the second example as*

$$u(x,t)-\int_0^t\int_0^x (xy+te^z)u(y,z)dydz$$
$$=xe^{-t}+t-\frac{1}{3}x^4-xt+\frac{1}{3}x^4e^{-t}-\frac{1}{2}x^2t^2-\frac{1}{4}x^3t^2-xt^2e^t+xte^t,\quad x,t\in[0,1], \tag{3.15}$$

with the exact solution $u(x,t)=xe^{-t}+t$.

Substituting (2.5) into (3.11), and equating the terms with identical powers of p*, we have*

$$\begin{aligned}
&p^0: u_0(x,t)=f(x,t)\\
&\quad\Rightarrow u_0(x,t)=xe^{-t}+t-\frac{1}{3}x^4-xt+\frac{1}{3}x^4e^{-t}-\frac{1}{2}x^2t^2-\frac{1}{4}x^3t^2-xt^2e^t+xte^t.\\
&p^1: u_1(x,t)=\int_0^t\int_0^x (xy^2+te^2)u_0(y,z)dydz\\
&\quad\Rightarrow u_1(x,t)=\frac{1}{18}x^7+\frac{1}{15}x^5t+\frac{1}{2}x^2te^t-\frac{1}{2}x^2t^2e^t-xte^t+e^tt^2x-\frac{1}{4}x^2t+\frac{1}{2}x^2t^2+\frac{1}{3}x^3t\\
&\qquad+\frac{1}{4}x^3t^2+\frac{1}{8}x^4t-\frac{1}{24}x^5t^3-\frac{1}{6}x^4t^2+\frac{4}{3}x^4-\frac{1}{16}x^4t^3e^t+\frac{7}{8}x^4te^t-\frac{5}{24}x^4t^2e^t\\
&\qquad+\frac{7}{8}x^4te^t-\frac{5}{24}x^4t^2e^t+\frac{1}{3}x^3t^2e^t-\frac{1}{3}x^3te^t-\frac{1}{60}x^6t^3+\frac{1}{2}x^2t^2e^{2t}-\frac{1}{4}x^2te^{2t}+xt\\
&\qquad-\frac{1}{4}x^2e^{2t}t^3-\frac{1}{18}x^7t+\frac{1}{15}x^5t^2-\frac{1}{15}x^5e^tt-\frac{1}{6}x^3e^tt^3-x^4e^t-\frac{1}{18}x^7e^{-t}-\frac{1}{3}x^4e^{-t}.\\
&p^2: u_2(x,t)=\int_0^t\int_0^x (xy^2+te^z)u_1(y,z)dydz.
\end{aligned} \tag{3.16}$$

Therefore, the approximate solution of Example (3.2) can be readily obtained by (3.14). The results are presented in Table 2.

Table 2
Numerical results of Example (3.2)

(x,t)	$(e_N(x,t) for N=10)_{DTM}$	$(e_N(x,t) for N=12)_{DTM}$	$(e_N(x,t) for N=5)_{HPM}$	$(e_N(x,t) for N=6)_{HPM}$
(0.2,0.6)	0.402100e-13	0.173090e-10	0.382279e-13	0.426960e-16
(0.2,1)	0.299678e-10	0.462285e-8	0.139418e-9	0.571044e-12
(0.4,0.4)	0.419090e-15	0.406714e-12	0.148118e-13	0.140259e-16
(0.4,0.8)	0.333987e-11	0.806738e-9	0.300519e-9	0.140561e-11
(0.6,0.8)	0.500981e-11	0.121011e-8	0.541918e-8	0.541918e-8
(0.6,1)	0.899036e-10	0.138686e-7	0.173491e-6	0.223797e-8
(0.8,0.4)	0.838170e-15	0.813427e-12	0.125384e-10	0.330774e-13
(0.8,0.8)	0.667975e-11	0.161348e-8	0.554773e-7	0.582993e-9
(1,0.6)	0.201100e-12	0.865450e-10	0.137047e-7	0.111409e-9
(1,1)	0.149839e-9	0.231143e-7	0.941524e-5	0.224946e-6

Example 3.3. *We consider the nonlinear Volterra integral equation*

$$u(x,t) - \int_0^t \int_0^x (y^2 + e^{-2z}) u^2(y,z) dydz = x^2 e^t + \frac{1}{14} x^7 - \frac{1}{14} x^7 e^{2t} - \frac{1}{5} x^5 t, \quad x,t \in [0,1] \tag{3.17}$$

with the exact solution $u(x,t) = x^2 e^t$.

By HPM, we may choose a convex homotopy such as

$$H(u,p) = u(x,t) - f(x,t) - p \int_0^t \int_0^x (y^2 + e^{-2z}) u^2(y,z) dydz = 0 \tag{3.18}$$

substituting (2.5) into (3.18) and equating the terms with identical powers of p, *we have*

$$p^0 : u_0(x,t) = f(x,t) \Rightarrow u_0(x,t) = x^2 e^t + \frac{1}{14} x^7 - \frac{1}{14} x^7 e^{2t} - \frac{1}{5} x^5 t.$$

$$p^1 : u_1(x,t) = \int_0^t \int_0^x (y^2 + e^{-2z}) u_0^2(y,z) dydz$$

$$\begin{aligned} \Rightarrow u_1(x,t) = \frac{1}{428828400} &(96525x^{17} + 204204x^{15} - 235620x^{13} - 30630600x^7 + 389844x^{11} \\ &+ 439824x^{13}t^3 + 408408x^{15}te^{2t} + 5105100x^{12}e^t + 32175x^{17}e^{4t} - 408408x^{15}t^2 \\ &+ 471240x^{13}te^{-2t} + 21441420x^8te^{-t} + 21441420x^8e^{-t} - 6126120x^10e^{-t} \\ &+ 235620x^{13}e^{-2t} - 779688x^{11}te^{-2t} - 779688e^{-2t}t^2x^{11} + 471240x^{13}t^2 \\ &+ 128700x^{17}t - 291720x^{15}t - 1701700x^{12}e^{3t} + 11027016x^{10}e^t - 4900896x^{10}). \end{aligned}$$

$$p^2 : u_2(x,t) = \int_0^t \int_0^x (y^2 + e^{-2z})(2u_0(y,z).u_1(y,z) dydz.$$

Table 3 shows the numerical results for Example (3.3).

Tables 1, 2 and 3 contain a numerical comparison between our solution using HPM (with $N = 2, \cdots, N = 6$) and the solutions of the same problems presented in [28] using DTM (with $N = 10, N = 12$).

4 Conclusion

In this work, we have proposed the HPM for solving two-dimensional Volterra integral equations. Several illustrative examples have shown that HPM is very efficient and simple for solving two-dimensional Volterra integral equations. Also, comparison with DTM method [28], which involves complicated computations, has shown that HPM gives comparable results with simple computation.

Table 3
Numerical results of Example (3.3)

(x,t)	$(e_N(x,t) \, for N=10)_{DTM}$	$(e_N(x,t) \, for N=12)_{DTM}$	$(e_N(x,t) \, for N=2)_{HPM}$	$(e_N(x,t) \, for N=3)_{HPM}$
(0.1,0.7)	0.525896e-11	0.163754e-13	0.392334e-14	0.178676e-18
(0.2,0.3)	0.182059e-14	0.104800e-17	0.112743e-11	0.2320088e-15
(0.3,0.9)	0.764536e-9	0.392498e-11	0.174116e-8	0.277928e-11
(0.4,1)	0.437002e-8	0.276603e-10	0.736977e-7	0.329421e-9
(0.5,0.8)	0.576196e-9	0.234034e-11	0.665440e-6	0.555280e-8
(0.6,1)	0.983256e-8	0.622356e-10	0.139074e-4	0.268358e-6
(0.7,0.6)	0.468694e-10	0.107361e-12	0.226745e-4	0.509410e-6
(0.8,1)	0.174801e-7	0.110641e-9	0.738018e-3	0.435742e-4
(0.9,0.5)	0.103376e-10	0.164652e-13	0.385189e-3	0.191449e-4
(1,1)	0.273127e-7	0.72877e-9	0.180458e-1	0.262892e-2

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