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Research Article

Revision of a fuzzy distance measure

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Abstract

In this paper, we show that the computed distance between two fuzzy numbers using the fuzzy distance measure proposed by Chakraborty et al. [Mathematical and Computer Modelling 43 (2006) 254-261] is not always a fuzzy number. Then, we improve it and prove that the result of the revised distance is always a nonnegative fuzzy number.

Keywords : Fuzzy distance measure; Generalized fuzzy number; LR-type fuzzy number.

1 Introduction

Human intuition says that the distance between two uncertain numbers should be also an uncertain number. It can also be assumed that if a fuzzy number is defined as a collection of points with different degrees of belongingness, then the distance between two fuzzy numbers is noting but the collection of pairwise distance between the elements of the respective fuzzy numbers [2, 4, 5]. In view of this, Voxman [5] first introduced a fuzzy distance for generalized fuzzy numbers using the concept of α-cut.
In 2006, Chakraborty et al. [2] have been proposed a fuzzy distance measure for the generalized fuzzy numbers and also for the LR-type fuzzy numbers, based on the interval difference. In 2012, Allahviranloo et al. [1] introduced a new fuzzy distance measure for fuzzy numbers and presented the metric properties of the porposed maesure. Afterwards, in 2010, Guha et al. [3] have been presented a new fuzzy distance measure which can calculate the distance measure between two fuzzy numbers with different confidence level. Also, Guha et al. in [3] showed that the distance measure proposed by Chakraborty et al. [2], is not able to calculate the distance properly for some cases. To this end, they have illustrated that the computed distance between two distinct fuzzy numbers by Chakraborty et al.'s distance may be zero. Unfortunately, Guha et al. in [3] did not state that the result of the distance measure proposed by Chakraorty et al. [2] may not be a fuzzy number.
In this paper, we show by two simple examples, that the computed distance using the fuzzy distance measure provided by Chakraborty et al. [2] is not always a fuzzy number. Also, in this paper, Chakraorty et al.'s distance is revised so that we obtain always a nonnegative fuzzy number as distance between two fuzzy numbers. Here, we present the basic notations and definitions of generalized fuzzy numbers and LR-type fuzzy numbers which are presented in [2].

A fuzzy set $\widetilde{A}$ defined on the universe X is characterized by a membership function such that $\mu_{\widetilde{A}} : X \to [0, 1]$. The support of $\widetilde{A}$, say

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$supp(\widetilde{A})$, is defined by the set $\{x \in X | \mu_{\widetilde{A}}(x) > 0\}$ and the α-cut set of $\widetilde{A}$ leads to a set such that $\{x \in X | \mu_{\widetilde{A}}(x) \geqslant \alpha\}$ for all $\alpha \in (0, 1]$. A generalized fuzzy number $\widetilde{A}$, conventionally represented by $\widetilde{A} = (a_1, a_2; \beta, \gamma)$, i.e., *(left point, right point; left spread, right spread)*, is a normalized convex fuzzy set [6] on the real line $\mathbb{R}$ if

(1) $supp(\widetilde{A})$ is a closed and bounded interval, i.e., $supp(\widetilde{A}) = [a_1 - \beta, a_2 + \gamma]$.

(2) $\mu_{\widetilde{A}}$ is an upper semi-continuous function.

(3) $a_1 - \beta \leqslant a_1 \leqslant a_2 \leqslant a_2 + \gamma$.

(4) The membership function is in the following form:

$$\mu_{\widetilde{A}}(x) = \begin{cases} f(x), & x \in [a_1 - \beta, a_1], \\ 1, & x \in [a_1, a_2], \\ g(x), & x \in [a_2, a_2 + \gamma], \end{cases}$$

where $f(x)$ and $g(x)$ are the monotonic increasing and decreasing functions on $[a_1 - \beta, a_1]$ and $[a_2, a_2 + \gamma]$ respectively.

Specifically, an LR-type fuzzy number is obtained form a generalized fuzzy number if $\beta, \gamma \neq 0$ and the shape functions $f(x)$ and $g(x)$ are approximated by $L(\frac{a_1 - x}{\beta})$ and $R(\frac{x - a_2}{\gamma})$ respectively. In particular, $\widetilde{A}$ is trapezoidal in shape if $a_1 < a_2$ and $L(x) = R(x) = \max\{1 - |x|, 0\}$ and triangular if $a_1 = a_2$ and $L(x) = R(x) = \max\{1 - |x|, 0\}$. Also, the fuzzy number $\widetilde{A} = (a_1, a_2; \beta, \gamma)$ is nonnegative if $a_1 - \beta \geqslant 0$. In the next section, we briefly describe the fuzzy distance measure proposed by Chakraborty et al. [2].

2 Chakraborty et al.'s distance

Let us consider two fuzzy numbers $\widetilde{A_1}$ and $\widetilde{A_2}$, denoted as $\widetilde{A_1} = (a_1, a_2; \beta_1, \gamma_1)$ and $\widetilde{A_2} = (a_3, a_4; \beta_2, \gamma_2)$, where a_1 and a_3 are left points, a_2 and a_4 are right points, β_1 and β_2 are left spreads and γ_1 and γ_2 are right spreads. Suppose that the α-cut of $\widetilde{A_1}$ and $\widetilde{A_2}$ be as follows:

$$[\widetilde{A_1}]_\alpha = [A_1^L(\alpha), A_1^R(\alpha)], \qquad \text{and}$$
$$[\widetilde{A_2}]_\alpha = [A_2^L(\alpha), A_2^R(\alpha)],$$

for all $\alpha \in (0, 1]$. Chakraborty et al. [2] defined the fuzzy distance between $\widetilde{A_1}$ and $\widetilde{A_2}$ as:

$$\widetilde{d}\,(\widetilde{A_1}, \widetilde{A_2}) = (d^L_{\alpha=1}, d^R_{\alpha=1}; \theta, \sigma), \tag{2.1}$$

where

$$\theta = d^L_{\alpha=1} - \max\left\{\int_0^1 d^L_\alpha \, d\alpha, 0\right\}, \qquad \text{and}$$
$$\sigma = \int_0^1 d^R_\alpha \, d\alpha - d^R_{\alpha=1},$$

and d^L_α, d^R_α are defined as follows:

$$d^L_\alpha = \lambda[A_1^L(\alpha) - A_2^L(\alpha) + A_1^R(\alpha) - A_2^R(\alpha)] + [A_2^L(\alpha) - A_1^R(\alpha)],$$

and

$$d^R_\alpha = \lambda[A_1^L(\alpha) - A_2^L(\alpha) + A_1^R(\alpha) - A_2^R(\alpha)] + [A_2^R(\alpha) - A_1^L(\alpha)],$$

for

$$\lambda = \begin{cases} 1, & \text{if } \frac{A_1^L(1) + A_1^R(1)}{2} \geqslant \frac{A_2^L(1) + A_2^R(1)}{2}, \\ 0, & \text{if } \frac{A_1^L(1) + A_1^R(1)}{2} < \frac{A_2^L(1) + A_2^R(1)}{2}. \end{cases}$$

In continuation, we will show that the result of the Eq. (2.1) may not be a fuzzy number.

3 The revised distance

In this section, we first show by two examples that $\widetilde{d}\,(\widetilde{A_1}, \widetilde{A_2})$ is not always a fuzzy number. Then, we revise the fuzzy distance measure proposed by Chakraborty et al. [2] so that we get always a fuzzy number as the distance between $\widetilde{A_1}$ and $\widetilde{A_2}$.

Example 3.1 *Consider the trapezoidal fuzzy numbers* $\widetilde{A_1} = (0, 6; 1, 1)$ *and* $\widetilde{A_2} = (2, 7; 1, 1)$. *Therefore, we will have*

$$A_1^L(\alpha) = \alpha - 1, \qquad A_1^R(\alpha) = 7 - \alpha,$$
$$A_2^L(\alpha) = \alpha + 1, \qquad A_2^R(\alpha) = 8 - \alpha.$$

Since $\frac{A_1^L(1) + A_1^R(1)}{2} = 3$ *and* $\frac{A_2^L(1) + A_2^R(1)}{2} = \frac{9}{2}$, *then* $\lambda = 0$ *and consequently*

$$d^L_\alpha = -6 + 2\alpha, \qquad d^R_\alpha = 9 - 2\alpha.$$

Thus, we obtain

$$d^L_{\alpha=1} = -4, \qquad d^R_{\alpha=1} = 7,$$
$$\theta = -4, \qquad \sigma = 1.$$

Obviously, since $\theta < 0$ *then* $\widetilde{d}\,(\widetilde{A_1}, \widetilde{A_2})$ *can not be a fuzzy number, in this case.*

Example 3.2 *Consider the trapezoidal fuzzy numbers* $\widetilde{A}_1 = (5, 8; 2, 1)$ *and* $\widetilde{A}_2 = (1, 9; 2, 2)$. *Therefore,*

$$A_1^L(\alpha) = 3 + 2\alpha, \qquad A_1^R(\alpha) = 9 - \alpha,$$

$$A_2^L(\alpha) = -1 + 2\alpha, \qquad A_2^R(\alpha) = 11 - 2\alpha.$$

Since $\frac{A_1^L(1)+A_1^R(1)}{2} = \frac{13}{2}$ *and* $\frac{A_2^L(1)+A_2^R(1)}{2} = 5$, *then* $\lambda = 1$ *and consequently*

$$d_\alpha^L = -8 + 4\alpha, \qquad d_\alpha^R = 10 - 3\alpha.$$

Hence, we obtain

$$d_{\alpha=1}^L = -4, \qquad d_{\alpha=1}^R = 7,$$

$$\theta = -4, \qquad \sigma = \frac{3}{2}.$$

Similarly, since $\theta < 0$ *then* $\widetilde{d}(\widetilde{A}_1, \widetilde{A}_2)$ *can not be a fuzzy number.*

Regarding to the above examples, we conclude that using the fuzzy distance measure proposed by Chakraborty et al. [2] the distance between two fuzzy numbers may fail to be a fuzzy number. In the following, we revise fuzzy distance measure proposed by Chakraborty et al. [2]. To this end, we correct only the definitions of left point and left spread of the distance. We define the revised fuzzy distance between $\widetilde{A}_1$ and $\widetilde{A}_2$ as follows:

$$\widetilde{d}^*(\widetilde{A}_1, \widetilde{A}_2) = (d_{\alpha=1}^{*L}, d_{\alpha=1}^R; \theta^*, \sigma),$$

where $d_{\alpha=1}^R$ and σ are defined as before, but

$$d_{\alpha=1}^{*L} = \max\left\{d_{\alpha=1}^L, 0\right\},$$

$$\theta^* = d_{\alpha=1}^{*L} - \max\left\{\int_0^1 d_\alpha^L \, d\alpha, 0\right\}.$$

In the following theorem, we show that by the improved fuzzy distance measure $\widetilde{d}^*$, the result of distance is always a fuzzy number.

Theorem 3.1 *For arbitrary fuzzy numbers* $\widetilde{A}_1$ *and* $\widetilde{A}_2$,

$$\widetilde{d}^*(\widetilde{A}_1, \widetilde{A}_2) = (d_{\alpha=1}^{*L}, d_{\alpha=1}^R; \theta^*, \sigma),$$

is always a nonnegative fuzzy number.

Proof. Firstly, we prove that $\widetilde{d}^*(\widetilde{A}_1, \widetilde{A}_2)$ is a fuzzy number. To this end, it is sufficient to show that

$$d_{\alpha=1}^{*L} \leqslant d_{\alpha=1}^R, \qquad \text{and} \qquad \theta^*, \sigma \geqslant 0. \tag{3.2}$$

Since $A_1^L(\alpha) \leqslant A_1^R(\alpha)$ and $A_2^L(\alpha) \leqslant A_2^R(\alpha)$, for all $\alpha \in [0,1]$, then

$$d_\alpha^L \leqslant d_\alpha^R, \qquad \forall \alpha \in [0,1]. \tag{3.3}$$

Now we show $d_\alpha^R \geqslant 0$, for all $\alpha \in [0,1]$. Suppose that $\lambda = 1$ and there exists an $\alpha^* \in [0,1]$ such that $d_{\alpha^*}^R < 0$. The proof for $\lambda = 0$ is similar and omitted. Then we have

$$d_{\alpha^*}^R = A_1^R(\alpha^*) - A_2^L(\alpha^*) < 0$$

$$\Rightarrow \quad A_1^R(\alpha^*) < A_2^L(\alpha^*). \tag{3.4}$$

Since $A_2^L(\alpha)$ and $A_1^R(\alpha)$ are increasing and decreasing functions with respect to α, respectively, then by (3.4) we get

$$A_1^R(1) \leqslant A_1^R(\alpha^*) < A_2^L(\alpha^*) \leqslant A_2^L(1), \tag{3.5}$$

and consequently

$$A_1^R(1) - A_2^L(1) < 0. \tag{3.6}$$

On the other hand, $\lambda = 1$ implies that

$$A_2^R(1) - A_1^L(1) \leqslant A_1^R(1) - A_2^L(1). \tag{3.7}$$

From (3.6) and (3.7) we conclude

$$A_2^R(1) < A_1^L(1). \tag{3.8}$$

Finally, by Eqs. (3.4) and (3.8) we have

$$A_1^R(\alpha^*) < A_2^L(\alpha^*) \leqslant A_2^L(1)$$

$$\leqslant A_2^R(1) < A_1^L(1) \leqslant A_1^R(1),$$

that implies

$$A_1^R(\alpha^*) < A_1^R(1). \tag{3.9}$$

Since $A_1^R(\alpha)$ is a decreasing function with respect to α, then Eq. (3.9) is a contradiction and thus

$$0 \leqslant d_\alpha^R, \qquad \forall \alpha \in [0,1]. \tag{3.10}$$

Consequently, from (3.3) and (3.10) we obtain

$$\max\{d_\alpha^L, 0\} \leqslant d_\alpha^R, \quad \forall \alpha \in [0,1],$$

specially

$$d^{*L}_{\alpha=1} = \max\left\{d^L_{\alpha=1}, 0\right\} \leqslant d^R_{\alpha=1}.$$

Therefore, the first part of Eq. (3.2) holds. Now we show that the left and right spreads are nonnegative. Since either $d^L_\alpha = A^L_1(\alpha) - A^R_2(\alpha)$ or $d^L_\alpha = A^L_2(\alpha) - A^R_1(\alpha)$, then d^L_α is always a increasing function with respect to α and consequently

$$\int_0^1 d^L_\alpha\, d\alpha \leqslant d^L_{\alpha=1} \leqslant d^{*L}_{\alpha=1}. \tag{3.11}$$

On the other hand, by the definition of $d^{*L}_{\alpha=1}$, we have

$$0 \leqslant d^{*L}_{\alpha=1}. \tag{3.12}$$

From (3.11) and (3.12) we conclude

$$\max\left\{\int_0^1 d^L_\alpha\, d\alpha, 0\right\} \leqslant d^{*L}_{\alpha=1},$$

that means $\theta^* \geqslant 0$. Similarly, since either $d^R_\alpha = A^R_1(\alpha) - A^L_2(\alpha)$ or $d^R_\alpha = A^R_2(\alpha) - A^L_1(\alpha)$, then d^R_α is always a decreasing function with respect to α and consequently

$$d^R_{\alpha=1} \leqslant \int_0^1 d^R_\alpha\, d\alpha, \tag{3.13}$$

that implies $\sigma \geqslant 0$. Therefore, the second part of Eq. (3.2) holds too, and thus $\widetilde{d^*}(\widetilde{A_1}, \widetilde{A_2})$ is always a fuzzy number.

Also, obviously

$$d^{*L}_{\alpha=1} - \theta^* = \max\left\{\int_0^1 d^L_\alpha\, d\alpha, 0\right\} \geqslant 0,$$

that concludes $\widetilde{d^*}(\widetilde{A_1}, \widetilde{A_2})$ is nonnegative. Therefore, the proof is completed. □

Remark 3.1 *It can be easily verified that the symmetry and tringle properties of metrics for $\widetilde{d^*}$ hold. The proof of the triangle property for $\widetilde{d^*}$ is exactly the same as that for $\widetilde{d}$ (see section 3.3 of [2]).*

In continuation, we consider two previous examples and apply the revised fuzzy distance measure to obtain the fuzzy distance between two fuzzy numbers.

Example 3.3 *Consider the trapezoidal fuzzy numbers $\widetilde{A_1} = (0, 6; 1, 1)$ and $\widetilde{A_2} = (2, 7; 1, 1)$ presented in Example 3.1. The fuzzy distance computed by our revised method is given by $\widetilde{d^*}(\widetilde{A_1}, \widetilde{A_2}) = (0, 7; 0, 1)$.*

Example 3.4 *Consider the trapezoidal fuzzy numbers $\widetilde{A_1} = (5, 8; 2, 1)$ and $\widetilde{A_2} = (1, 9; 2, 2)$ presented in Example 3.2. Then the fuzzy distance using our revised method is computed as $\widetilde{d^*}(\widetilde{A_1}, \widetilde{A_2}) = (0, 7; 0, \frac{3}{2})$.*

4 Conclusion

In this paper, we have corrected the fuzzy distance measure proposed by Chakraborty et al. [2]. It is found that by Chakraborty et al.'s method the fuzzy distance between two fuzzy numbers may fail to be a fuzzy number and then we are not able to calculate the distance properly for some cases, while by our revised method the fuzzy distance between two fuzzy numbers is always a nonnegative fuzzy number.

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