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Numerical Solution of Singular IVPs of Lane-Emden Type Using Integral Operator and Radial Basis Functions

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Abstract
A numerical method for solving the Lane-Emden equations as singular initial value problems is presented. The method is based on using integral operator and convert Lane-Emden equations to integral equations and interpolation by radial basis functions (RBFs). Also, Legendre-Gauss quadrature integration method utilized to reduce the solution of integral equations to the solution of algebraic equations. Several examples are given and numerical examples are presented to demonstrate the validity and applicability of the method.
Keywords : Lane-Emden equations; Strictly positive functions; Radial basis functions.

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1 Introduction

Recently, a lot of attention has been focused on the study of singular initial value problems (IVPs) in the second-order ordinary differential equations (ODEs). Many problems arising in the field of mathematical physics and astrophysics can be modelled by Lane-Emden type initial value problems, which can be written in the form:

$$y'' + \frac{\alpha}{x} y' + f(y) = 0, \qquad 0 < x \leq 1, \qquad \alpha \geq 0, \tag{1.1}$$

subject to conditions

$$y(0) = A, \qquad y'(0) = B, \tag{1.2}$$

where A and B are constants and $f(y)$ is a real-valued continuous function. This equation was used to model various phenomena such as the theory of stellar structure, the thermal

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behaviour of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents [12, 16, 33].

On the other hand, another class of singular initial value problems of Lane-Emden type can also be given in the form:

$$y'' + \frac{\alpha}{x}y' + f(x,y) = g(x), \quad 0 < x \leq 1, \quad \alpha \geq 0, \tag{1.3}$$

subject to conditions given in Eq. (1.2), where A and B are constants, $f(x,y)$ is a continuous real valued function, and $g(x) \in C[0,1]$. Eq. (1.3) differs from the classical Lane-Emden type Eq. (1.2), for the function $f(x,y)$ and for the inhomogeneous term $g(x)$.

Since, Lane-Emden type equations have significant applications in many fields of the scientific and technical world, a variety of forms of $f(y)$ have been investigated by many researchers. A discussion of the formulation of these models and the physical structure of the solutions can be found in the literature. For example, it models the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [16, 38, 41] when $f(y) = y^m$, the gravitational potential of the degenerate white-dwarf stars [12] when $f(y) = (y^2 - C)^{\frac{3}{2}}$, the isothermal gas spheres [16] when $f(y) = e^y$ and so on.

Recently many analytical methods have been used to solve Lane-Emden equations, the main difficulty arises in the singularity of the equation at $x = 0$. Currently most techniques in use for handling the Lane-Emden type problems are based on either series solutions or perturbation techniques. Bender et al. [5] handled the solution of Lane-Emden equations as well as those of a variety of nonlinear problems in quantum mechanics and astrophysics by means of perturbation methods based on the existence of a small parameter. Approximate solutions to the above problems were presented by Shawagfeh [38] and Wazwaz [41, 42] by applying the Adomian method which provides a convergent series solution. Nouh [29] accelerated the convergence of a power series solution of the LaneEmden equation by using an Euler-Abel transformation and Pade approximation. Mandelzweig and Tabakin [26] applied Bellman and Kalaba's quasilinearization method and Ramos [31] used an piecewise linearization technique based on the piecewise linearization of the Lane-Emden equation. Bozkhov and Martins [7] and later Momoniat and Harley [28] applied the Lie Group method successfully to generalized Lane-Emden equations of the first kind. Exact solutions of generalized Lane-Emden solutions of the first kind are investigated by Goenner and Havas [17]. Liao [22] solved Lane-Emden type equations by applying a homotopy analysis method. He [18] obtained an approximate analytical solution of the Lane-Emden equation by applying a variational approach which uses a semi inverse method. Ramos [32] presented a series approach to the Lane-Emden equation and gave the comparison with He's homotopy perturbation method. The authors of this paper, Yldrm and Ozis [30] and also Chowdhury and Hashim [14] gave the solutions of a class of singular second-order IVPs of Lane-Emden type by using He's homotopy perturbation method. Youseffi [44] converted the Lane-Emden equation to an integral equation and then using Legendre wavelets, obtained an approximate solution for $0 < x \leq 1$.

In the present article, we are concerned with the application of radial basis functions to the numerical solution of Eq. (1.3). The method consists of convert of Lane-Emden equations to integral equations and expanding the solution by radial basis functions with unknown coefficients. The properties of radial basis functions together with the Gaussian integration formula are then utilized to evaluate the unknown coefficients and find an approximate solution to Eq. (1.3).

2 Radial basis functions

In this section the RBFs method is defined as a technique for interpolation of the scattered data. Some well-known radial basis functions (RBFs) are listed in Table 1. Let r be the Euclidean distance between a fixed point $x^* \in R^d$ and any $x \in R^d$ i.e. $\|x - x^*\|_2$. A radial function $\phi^* = \phi(\|x - x^*\|_2)$ depends only on the distance between $x \in R^d$ and fixed point $x^* \in R^d$. This property results that the radial basis function ϕ^* is radially symmetric about x^*. It is clear that the functions in Table 1 are globally supported, infinitely differentiable and depend on a free parameter c.

Let $\{x_0, x_1, ..., x_N\}$ be a given set of distinct points in R^d. The main idea behind the use of RBFs is interpolation by translation of a single function i.e. the interpolating RBFs approximation is considered as

$$F(x) = \sum_{i=0}^{N} \lambda_i \phi_i(x), \tag{2.4}$$

Table 1.
Some well-known functions that generate RBFs.

Name of function	Definition
Gaussian (GA)	$\phi(r) = \exp\ (-c^2 r^2)$
Hardy Multiquadric (MQ)	$\phi(r) = \sqrt{r^2 + c^2}$
Inverse Multiquadrics (IMQ)	$\phi(r) = (\sqrt{r^2 + c^2})^{-1}$
Inverse Quadric (IQ)	$\phi(r) = (r^2 + c^2)^{-1}$

where $\phi_i(x) = \phi(\|x - x_i\|_2)$ and λ_i are unknown scalars for $i = 0, 1, \cdots, N$. Assume that we want to interpolate the given values $f_i = f(x_i)$, $i = 0, 1, \cdots, N$. The unknown scalars λ_i are chosen so that $F(x_i) = f_i$ for $i = 0, 1, \cdots, N$ which results in the following linear system of equations

$$AZ = f, \tag{2.5}$$

where $A_{i,j} = \phi_i(x_j)$, $Z = [\lambda_0, \lambda_1, \cdots, \lambda_N]$ and $f = [f_0, f_1, \cdots, f_N]$. Since all applicable ϕ have global support, this method produces a dense matrix A. The matrix A can be shown to be positive definite (and therefore nonsingular) for distinct interpolation points for GA, IMQ and IQ by Schoenberg's Theorem [37]. Also using the Micchelli Theorem [27] we can show that A is invertible for distinct sets of the scattered points in the case of MQ.

Although the matrix A is nonsingular in the above cases, usually it is very ill-conditioned i.e. the condition number of A

$$k_s(A) = \|A\|_s \|A^{-1}\|_s, \qquad s = 1, 2, \infty, \tag{2.6}$$

is a very large number. Therefore a small perturbation in initial data may produce a large amount of perturbation in the solution. Thus we have to use more precision arithmetic than the standard floating point arithmetic in our computation. For a fixed number of interpolation points the condition number of A depends on the shape parameter c, support of the RBFs and minimum separation distance of interpolation points. Also the condition number grows with N for fixed values of shape parameter c. In practice, the shape parameter c must be adjusted with the number of interpolating points in order to produce an interpolation matrix which is well conditioned enough to be inverted in finite precision arithmetic [35].

Despite research done by many scientists to develop algorithms for selecting the values of c which produce the most accurate interpolation (e.g. see [11, 34]), the optimal choice of shape parameter is still an open question.

3 Legendre-Gauss nodes and weights

Let $L_{M+1}(x)$ be the Legendre polynomial of order $M+1$ on $[-1,1]$. Then the Legendre-Gauss nodes are

$$-1 < x_0 < x_1 < \cdots < x_M < 1, \tag{3.7}$$

where $\{x_i\}_{i=0}^{M}$ are the zeros of $L_{M+1}(x)$. No explicit formulas are known for the points x_i, and so they are computed numerically using subroutines [10]. Also we approximate the integral of f on $[-1,1]$ as

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=0}^{M} w_i f(x_i), \tag{3.8}$$

where x_i are Legendre-Guass nodes in Eq. (3.7) and the weights w_i given in [10]

$$w_i = \frac{2}{(1-x_i^2)[L'_{M+1}(x_i)]^2}, \qquad i = 0, 1, \cdots, M. \tag{3.9}$$

It is well known [21] that the integration in Eq. (3.8) is exact whenever $f(x)$ is a polynomial of degree $\leq 2M+1$.

4 Solution of the problem via radial basis functions

Consider the Lane-Emden equations given in Eq. (1.3). Define integral operator

$$L(.) = \int_0^x t^{-\alpha} \int_0^t s^{\alpha}(.)dsdt. \tag{4.10}$$

Operating with L on Eq. (1.3), it then follows:

$$y(x) = A + L(g(x)) - L(f(x,y)), \tag{4.11}$$

where

$$A = y(0). \tag{4.12}$$

Let

$$G(x) = A + L(g(x)), \tag{4.13}$$

and

$$F(t, y(t)) = -t^{-\alpha} \int_0^t s^{\alpha} f(s, y(s))ds. \tag{4.14}$$

So, we can get

$$y(x) = G(x) + \int_0^x F(t, y(t))dt, \tag{4.15}$$

which is nonlinear Volterra integral equation. Let $\phi(x)$ be a radial basis function and we approximate $y(x)$ with interpolation by function $\phi(x)$ i.e.,

$$y(x) \simeq \sum_{j=0}^{N} c_j \phi(x - x_j) = C^T \Psi(x), \tag{4.16}$$

where C and $\Psi(x)$ are $(N+1) \times 1$ matrices given by

$$C = [c_0, c_1, \cdots c_N]^T, \qquad \Psi(x) = [\phi(x - x_0), \phi(x - x_1), \cdots \phi(x - x_N)]^T.$$

Also x_j are shifted the Chebyshev-Gauss-Radau nodes on $[0, 1]$

$$x_j = \frac{1}{2} \cos\left(\frac{2\pi j}{2N+1}\right) + \frac{1}{2}, \quad j = 0, 1, \cdots, N. \tag{4.17}$$

Now by substituting Eq. (4.16) in Eq. (4.15) we have:

$$C^T \Psi(x) = G(x) + \int_0^x F(t, C^T \Psi(t))dt. \tag{4.18}$$

For obtaining c_j, $j = 0, 1, \cdots, N$ in the above equation, by collocating at the points $x = x_i$ for $i = 0, 1, \cdots, N$ we have:

$$C^T \Psi(x_i) = G(x_i) + \int_0^{x_i} F(t, C^T \Psi(t))dt. \tag{4.19}$$

By change of variable $t = \frac{x_i}{2}(\tau + 1)$, Eq. (4.19) can be written as:

$$C^T \Psi(x_i) = G(x_i) + \frac{x_i}{2} \int_{-1}^{1} F(\frac{x_i}{2}(\tau + 1), C^T \Psi(\frac{x_i}{2}(\tau + 1)))d\tau. \tag{4.20}$$

By applying numerical integration method given in Eq. (3.8), we can approximate the integral in Eq. (4.20) and hence the above equation can be written as follow:

$$C^T \Psi(x_i) = G(x_i) + \frac{x_i}{2} \sum_{k=0}^{M} w_k F(\frac{x_i}{2}(\tau_k + 1), C^T \Psi(\frac{x_i}{2}(\tau_k + 1))), \tag{4.21}$$

for $i = 0, 1, \cdots, N$ and w_k are given in Eq. (3.9). From (4.14) and numerical integration method given in Eq. (3.8), we obtain:

$$F(\frac{x_i}{2}(\tau_k + 1), C^T \Psi(\frac{x_i}{2}(\tau_k + 1))) \simeq -\frac{[\frac{x_i}{2}(\tau_k + 1)]^{1-\alpha}}{2} \sum_{p=0}^{M} w_p [\xi_{k,p}^{(i)}]^\alpha f(\xi_{k,p}^{(i)}, C^T \Psi(\xi_{k,p}^{(i)})) \tag{4.22}$$

where $\xi_{k,p}^{(i)} = \frac{x_i}{4}(\tau_k + 1)(\tau_p + 1)$. So by substituting Eq. (4.22) in Eq. (4.21) we have:

$$C^T \Psi(x_i) = G(x_i) + \frac{x_i}{2} \sum_{k=0}^{M} \sum_{p=0}^{M} w_k w_{k,p}^{(i)} f(\xi_{k,p}^{(i)}, C^T \Psi(\xi_{k,p}^{(i)})), \qquad i = 0, 1, \cdots N, \tag{4.23}$$

where

$$w_{k,p}^{(i)} = -\frac{[\frac{x_i}{2}(\tau_k+1)]^{1-\alpha}}{2}[\xi_{k,p}^{(i)}]^{\alpha} w_p. \tag{4.24}$$

This is a nonlinear system of equations that can be solved via Newton's iteration method to obtain unknown vector C^T.

5 Numerical examples

We use the method presented in this paper to solve four examples given in [30, 14, 43, 20, 1, 15, 4]. By choosing appropriate Radial basis function $\phi(x)$ and shape parameter c we can get high accurate solution and best choice of the $\phi(x)$ depends on the form of the problem. In all examples, we use GA-RBF and MQ-RBF and also we use the maximum errors for different N which is given as

$$E_\infty = max\{|y(x) - \sum_{j=0}^{N} c_j\phi(x-x_j)| : x \in [0,1]\}.$$

All of the computations have been done using the Maple 13 with 150 digits precision, $M = 10$ and we solved obtained system by Newton's iteration method with start point $[0, 0, ..., 0]$.

Error Analysis: Madych have proven exponential convergence property of multiquadratic approximation [23]. He has shown that under certain conditions, the interpolation error is $\varepsilon = O(\lambda^{\frac{c}{h}})$ where c is the shape parameter, h is the mesh size and $0 < \lambda < 1$ is a constant. It implies we can improve the approximated solution either by reducing the size of h or by increasing the magnitude of c. It means that if $c \to \infty$ then $\varepsilon \to 0$. Since increasing of c can improve the accuracy exponentially without extra computation [13, 19, 23, 24], it is preferred to decrease error rather than reducing h.

However, according to 'uncertainty principle' of Schaback [36], as the error becomes smaller, the matrix becomes more ill-conditioned; hence the solution will break down as c becomes too large. The experimental results confirm such behavior of the error values as c becomes larger. The numerical results for Examples 1,2 and 3 on interval $[0, 1]$ are demonstrated in Fig.s 2 and 3, which show according to the findings of Madych, the error functions decrease exponentially as c becomes larger in bounded interval. After that according to the research of Schaback the error values decline as c becomes too large. The best c is different for various problems and not the same RBFs.

Example 5.1. *Consider the homogeneous Lane-Emden type equation [14, 43, 20, 1]*

$$y'' + \frac{2}{x}y' - (4x^2+6)y = 0, \qquad 0 < x \leq 1, \tag{5.25}$$

subject to conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{5.26}$$

The exact solution of this test problem is $y(x) = e^{x^2}$.

This type of equation has been solved by [14, 43, 20, 1] with the homotopy-perturbation method, variational iteration method, power series with Padé approximation and modified Legendre-spectral method respectively.

In Table 2, we list the results obtained by the RBF collocation method proposed in this paper with GA-RBF ($c = 0.001$) and MQ-RBF ($c = 2$). Also we contrast our results with the corresponding results reported by Karimi Vanani et al. [20] and Adibi et al. [1].

Table 2.
Maximum absolute errors for different values of N for Example 5.1.

N	GA-RBF ($c = 0.001$)	MQ-RBF ($c = 2$)	Method [1]	Method [20]
5	6.51×10^{-4}	1.35×10^{-3}	7.80×10^{-4}	—
10	2.15×10^{-8}	7.80×10^{-7}	1.17×10^{-5}	—
12	2.51×10^{-10}	3.52×10^{-8}	—	2.26×10^{-4}
15	2.44×10^{-13}	3.58×10^{-10}	1.40×10^{-8}	—
16	2.30×10^{-14}	7.54×10^{-11}	—	3.05×10^{-6}
20	5.38×10^{-15}	1.53×10^{-13}	3.59×10^{-10}	2.73×10^{-8}
24	7.49×10^{-16}	2.80×10^{-16}	—	1.72×10^{-10}
25	4.52×10^{-16}	5.59×10^{-17}	3.96×10^{-13}	—
28	1.42×10^{-18}	1.42×10^{-18}	—	8.15×10^{-13}
30	1.41×10^{-18}	1.41×10^{-18}	5.35×10^{-14}	—

From the contents of Table 3, it is clear that the choice of the shape parameter has an auxiliary role in the stability of the problem.The dimension of matrix A should be small sufficiently to guarantee the stability of the solution of the resulted linear system.

Table 3.
Some values of shape parameter c, $k_\infty(A)$ using GA-RBF with $N = 18$ for Example 5.1.

Shape parameter c	E_∞	$k_\infty(A)$
0.05	0.1672×10^{-15}	0.5149×10^{79}
0.1	0.1872×10^{-15}	0.7492×10^{68}
1	0.4748×10^{-13}	0.8409×10^{32}
5	0.9163×10^{-4}	0.1671×10^{10}
10	0.1947×10^{-1}	0.7006×10^{6}
30	0.8709	0.2528×10^{3}

Example 5.2. *Consider the nonhomogeneous Lane-Emden equation [30, 15, 4, 6]*

$$y'' + \frac{8}{x}y' + xy = x^5 - x^4 + 44x^2 - 30x, \qquad 0 < x \le 10, \tag{5.27}$$

subject to conditions

$$y(0) = 0, \qquad y'(0) = 0, \tag{5.28}$$

for which the exact solution is $y(x) = x^4 - x^3$.

In Table 4, we list the results obtained by the RBF collocation method proposed in this paper with GA-RBF ($c = 0.001$) and MQ-RBF ($c = 1000$). Also we contrast our results with the corresponding results reported by Bhrawy et al. [6]. The displayed results show that the RBF method is more accurate than shifted Jacobi collocation method (SJC) [6].

Table 4.
Some values of shape parameter c, $k_\infty(A)$ using GA-RBF with $N = 18$ for Example 5.2.

x	GA-RBF ($c = 0.001$)	MQ-RBF ($c = 1000$)	SJC($\alpha = \beta = \frac{-1}{2}$)	SJC($\alpha = \frac{1}{2}, \beta = \frac{-1}{2}$)
0	3.46×10^{-20}	7.85×10^{-18}	4.54×10^{-13}	1.13×10^{-13}
1	2.78×10^{-20}	7.52×10^{-18}	4.54×10^{-13}	1.13×10^{-13}
2	2.53×10^{-20}	7.82×10^{-18}	0	5.68×10^{-14}
3	2.60×10^{-20}	8.84×10^{-18}	6.82×10^{-13}	2.86×10^{-14}
4	1.84×10^{-20}	6.68×10^{-18}	0	5.68×10^{-14}
5	7.47×10^{-22}	1.18×10^{-19}	1.13×10^{-13}	0
6	2.12×10^{-20}	8.95×10^{-18}	4.54×10^{-13}	2.27×10^{-13}
7	3.82×10^{-20}	1.64×10^{-17}	0	0
8	4.46×10^{-20}	1.96×10^{-17}	0	0
9	5.20×10^{-20}	2.33×10^{-17}	0	0
10	6.65×10^{-20}	3.04×10^{-17}	0	0

Example 5.3. *(The isothermal gas spheres equation) Consider the nonlinear, homogeneous Lane-Emden type equation [15, 4, 2, 3]*

$$y'' + \frac{2}{x}y' + e^y = 0, \qquad 0 < x \leq 2.5, \tag{5.29}$$

subject to conditions

$$y(0) = 0, \qquad y'(0) = 0. \tag{5.30}$$

A series solution obtained by Wazwaz [41], Liao [22], Singh et al. [39] and Ramos [32] by using ADM, HAM, MHAM and series expansion respectively:

$$y(x) \simeq -\frac{1}{6}x^2 + \frac{1}{5.4!}x^4 - \frac{8}{21.6!}x^6 + \frac{122}{81.8!}x^8 - \frac{61.67}{495.10!}x^{10}. \tag{5.31}$$

Table 5 shows the comparison of $y(x)$ obtained by the RBF method proposed in this paper with (GA-RBF ,$N = 10$, $c = 0.3$) and those obtained by Wazwaz [41].

The resulting graph of the isothermal gas spheres equation in comparison to the presented method and those obtained by Wazwaz [41] is shown in Fig. 1.

Table 5.
Comparison between present method and Wazwaz method [41].

x	GA-RBF ($c = 0.3$)	Wazwaz [41]	Error
0.0	0.0000000000	0.0000000000	0.0
0.1	-0.0016658328	-0.0016658338	9.91×10^{-10}
0.2	-0.0066533691	-0.0066533671	2.07×10^{-9}
0.5	-0.0411539552	-0.0411539568	1.55×10^{-9}
1.0	-0.1588276770	-0.1588273536	2.23×10^{-7}
1.5	-0.3380194261	-0.3380131102	6.31×10^{-6}
2.0	-0.5598230040	-0.5599626601	1.39×10^{-4}
2.5	-0.8063408705	-0.8100196713	3.67×10^{-3}

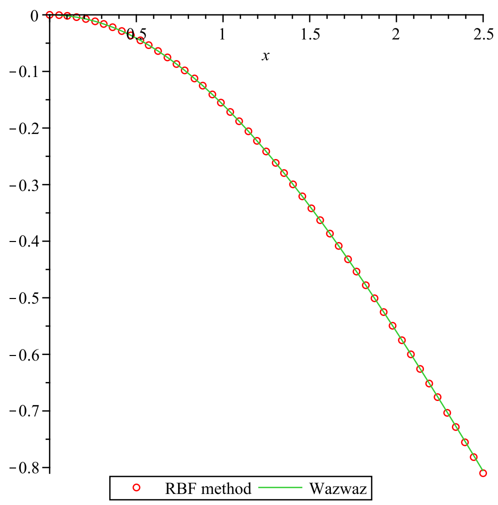

Fig. 1. Graph of isothermal gas sphere equation in comparison with Wazwaz solution [41].

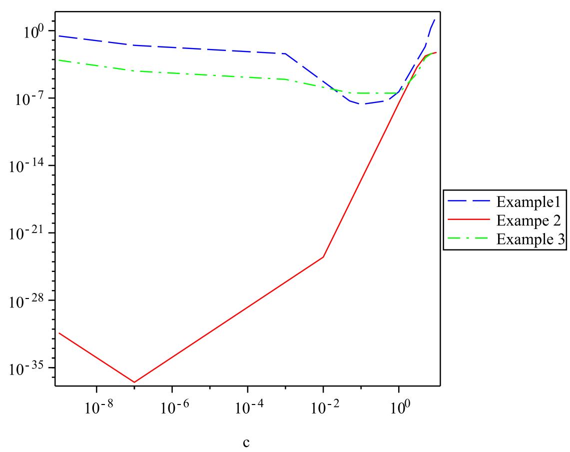

Fig. 2. Horizontal axis is related to shape parameter (c) and vertical axis shows error values with *log* mode when the solutions are approximated by using GA-RBF with $N = 10$ on interval $[0, 1]$.

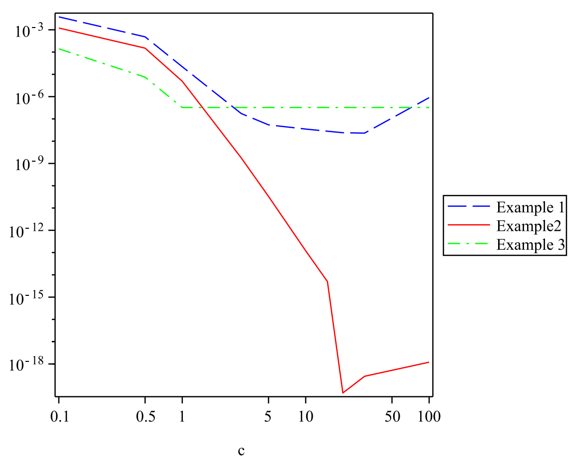

Fig. 3. Horizontal axis is related to shape parameter (c) and vertical axis shows error values with *log* mode when the solutions are approximated by using MQ-RBF with $N = 10$ on interval $[0, 1]$.

6 Conclusion

The aim of present work is to develop an efficient and accurate method for solving the Lane-Emden equations as singular initial value problems. The properties of the radial basis functions together with the Gaussian integration method are used to reduce the problem to the solution of nonlinear algebraic equations. This technique is very simple, the elements of system can be obtained easily and involve less computation. The illustrative example confirm the validity of the method.

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