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Linear optimization of fuzzy relation inequalities with max-Lukasiewicz composition

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Abstract

In this paper, we study the finitely many constraints of fuzzy relation inequalities problem and optimize the linear objective function on this region which is defined with fuzzy max-Lukasiewicz operator. In fact Lukasiewicz t-norm is one of the four basic t-norms. A new simplification technique is given to accelerate the resolution of the problem by removing the components having no effect on the solution process. Also, an algorithm and one numerical example are offered to abbreviate and illustrate the steps of the problem resolution process.

Keywords : Linear objective function optimization; Fuzzy relation equations; Fuzzy relation inequalities; max-Lukasiewicz composition.

1 Introduction

Fuzzy relation equations (FRE), fuzzy relation inequalities (FRI) and their connected problems have been investigated by many researchers in both theoretical and applied areas [4, 5, 8, 11, 13, 18, 32, 33, 35, 42]. Sanchez [34] started a development of the theory and applications of FRE treated as a formalized model for non-precise concepts. Generally, FRE and FRI has a number of properties that make it suitable for formulizing the uncertain information upon which many applied concepts are usually based. The application of (FRE) and (FRI) can be seen in many areas, for instance, fuzzy control, fuzzy decision making, system analysis, fuzzy modeling, fuzzy arithmetic, fuzzy symptom, diagnosis, and especially fuzzy medical diagnosis and so on (see [1, 2, 5, 7, 8, 9, 23, 27, 30, 31, 32, 41, 44]).

An interesting extensively investigated kind of such these problems is the optimization of the objective functions on the region whose feasible solutions sets have been defined as FRE or FRI constraints [3, 10, 14, 16, 18, 21, 22, 25, 26, 37, 38, 39, 40]. Fang and Li solved the linear optimization problem with respect to the FRE constraints by considering the max-min composition [10]. The max-min composition is commonly used when a system requires conservative solutions in sense that the goodness of one value can not compensate the badness of another value [21]. Recent results in the literature, however, show that the min operator is not always the best choice for intersection operation. Instead, the max-product composition provided results better or equivalent to the max-min composition in some application [1].

The fundamental result for fuzzy relation equations with max-product composition goes back to Pedrycz [30]. Recent study in this regard can be found in Bourk and Fisher [3]. They extended the study of an inverse solution of a system of fuzzy relation equations with max-product

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composition. They provided theoretical results for determining the complete solution sets as well as the conditions for the existence of resolutions. Their results showed that such complete solution sets can be characterized by one maximum solution and a number of minimal solutions. Furthermore, the monograph by Di Nola, Sessa, Pedrycz and Sanchez [8] contains a thorough discussion of this class of equations. Nonetheless, recent published literatures show that max-L composition in which L is Lukasiewicz t-norm is played important role in applications [6, 7, 24, 28, 29, 36].

In this paper, we consider the linear optimization problem of the fuzzy relation inequalities (FRI) with max-Lukasiewicz operator which, we show max-L for simplicity [20]. This problem can be formulated as following:

$$\begin{aligned} &\min c^t x \\ s.t. \quad & A \circ_L x \geq d^1 \\ & B \circ_L x \leq d^2 \\ & x \in [0,1]^n \end{aligned} \qquad (1.1)$$

where $A = (a_{ij})_{m\times n}$, $a_{ij} \in [0,1]$, $B = (b_{ij})_{l\times n}$, $b_{ij} \in [0,1]$, are fuzzy matrices, $d^1 = (d_i^1)_{m\times 1} \in [0,1]^m$,$d^2 = (d_i^2)_{l\times 1} \in [0,1]^l$ are fuzzy vectors and,$x = (x_j)_{n\times 1} \in [0,1]^n$ is unknown fuzzy vector and, $c = (c_j)_{n\times 1} \in R^n$is vector of cost coefficients, and "$\circ_L$" denotes the fuzzy max-L operator. Problem (1.1) can be rewritten as following problem:

$$\begin{aligned} &\min c^t x \\ s.t. \quad & a_i \circ_L x \geq d_i^1, \quad i \in I^1 = \{1,2,...,m\}, \\ & b_i \circ_L x \leq d_i^2, \quad i \in I^2 = \{1,2,...,l\}, \\ & 0 \leq x_j \leq 1, \quad j \in J = \{1,2,...,n\}, \end{aligned} \qquad (1.2)$$

where a_i and b_i are i'th row of the matrices A and B, respectively and the constraints are expressed by the max-L operator definition as:

$$\begin{aligned} &\forall i \in I^1: \\ &a_i \circ_L x = \max_{j\in J}\{\max(a_{ij} + x_j - 1, 0)\} \geq d_i^1 \\ &\forall i \in I^2: \\ &b_i \circ_L x = \max_{j\in J}\{\max(a_{ij} + x_j - 1, 0)\} \leq d_i^2, \end{aligned} \qquad (1.3)$$

In Section 2, the feasible solutions set of the problem (1.2) and its properties are studied also, necessary and sufficient conditions are given to realize the feasibility of the problem (1.2). In Section 2, some simplification operations are presented to accelerate the resolution process. Also, in Section 4 an algorithm is introduced to solve the problem and one example is given to illustrate the algorithm. Finally, a conclusion is stated in Section 5.

2 The characteristics of the feasible solution set

Definition 2.1 *Define* $S(A,d^1)_i = \{x \in [0,1]^n : a_i \circ_L x \geq d_i^1\}$ *for each* $i \in I^1$, $S(B,d^2)_i = \{x \in [0,1]^n : b_i \circ_L x \leq d_i^2\}$ *for each* $i \in I^2$, *and* $S(A,B,d^1,d^2) = S(A,d^1) \bigcap S(B,d^2) = \{x \in [0,1]^n : A \circ_L x \geq d^1, B \circ_L x \leq d^2\}$.

Lemma 2.1 *(a)* $S(A,d^1) \neq \emptyset$ *if only if for each* $i \in I^1$ *there exists some* $j \in J$ *such that* $d_i^1 \leq a_{ij}$.

(b) If $S(A,d^1) \neq \emptyset$ *then* $\bar{1} = [1,1,...,1]^t_{1\times n}$ *is the single maximum solution of* $S(A,d^1)$.

Proof.

(a) Suppose $x \in S(A,d^1)$. *Thus,* $x \in S(A,d^1)_i$, $\forall i \in I^1$ *by Definition 2.1, and thus, by Relation (1.3), for each* $i \in I^1$ *there are some* $j_i \in J$ *such that*$\max(a_{ij_i} + x_{j_i} - 1, 0) \geq d_i^1$. *Since for each* $i \in I^1$ *and* $j \in J$ *we have* $d_i^1 \geq 0$ *and*$x_j \leq 1$, *then* $d_i^1 \leq a_{ij_i} + x_{j_i} - 1 \leq a_{ij_i}$ *therefore for each* $i \in I^1$ *there exists some* $j \in J$ *such that* $d_i^1 \leq a_{ij}$. *Conversely, suppose there exist some* $j_i \in J$ *such that* $d_i^1 \leq a_{ij_i}$, $\forall i \in I^1$. *Set* $x = \bar{1} = [1,1,...,1]^t_{1\times n}$. *Since* $x \in [0,1]^n$ *and*$\max_{j\in J}\{\max(a_{ij_i} + x_{j_i} - 1, 0)\} \geq \max(a_{ij_i} + x_{j_i} - 1, 0) \geq a_{ij_i} + x_{j_i} - 1 = a_{ij_i} \geq d_i^1$, $\forall i \in I^1$ *hence, by Relation (1.3),* $x \in S(A,d^1)_i$, $\forall i \in I^1$ *and as a result* $x \in S(A,d^1)$.

(b) Proof of this part is easily attained from the part (a) and this fact that $x_j \leq 1, \forall j \in J$.

Lemma 2.2 *(a)* $S(B,d^2) \neq \emptyset$

(b) The single minimum solution of $S(B,d^2)$ *is* $\bar{0} = [0,0,...,0]^t_{1\times n}$.

Proof.

Set$x = \bar{0} = [0,0,...,0]^t_{1\times n}$. *We select* $i \in I^2$ *arbitrary and constant hereafter. Since,* $0 \leq b_{ij}, d_i^2 \leq 1$ *we have* $b_{ij} + x_j - 1 \leq d_i^2$,$\forall j \in J$ *then*$\max(b_{ij} + x_j - 1, 0) \leq d_i^2$,$\forall j \in J$ *and hence* $\max_{j\in J}\{\max(b_{ij} + x_j - 1, 0)\} \leq d_i^2$ *therefore* $x \in S(B,d^2)$ *and then part (a) and (b) are proved.*

Theorem 2.1 ***(Necessary condition)***

If $S(A,B,d^1,d^2) \neq \emptyset$ then,$\forall i \in I^1 \exists j \in J$ Such that $d_i^1 \leq a_{ij}$.

Proof.

This Theorem is clearly proved from Lemmas 2.1 and 2.2 and Definition 2.1.

Definition 2.2 *Set $\overline{x} = (\overline{x}_j)_{n\times 1}$, where $\overline{x}_j = \min_{i\in I^2}\{\min\{1+d_i^2 - b_{ij}, 1\}\} = \min\{1, \min_{i\in I^2}\{1+d_i^2 - b_{ij}\}\}$.*

Lemma 2.3 *$\overline{x}$ is the single maximum solution of $S(B,d^2)$.*

Proof.

Suppose $x \in S(B,d^2)$ then, $x \in S(B,d^2)_i, \forall i \in I^2$, and then, $\max_{j\in J}\{\max(b_{ij}+x_j-1,0)\} \leq d_i^2, \forall i \in I^2$ by definition 2.1 and Relation 3, therefore $b_{ij}+x_j-1 \leq d_i^2, \forall i \in I^2$ and $\forall j \in J$, and hence, for each $j \in J$, we have $x_j \leq 1+d_i^2 - b_{ij}, \forall i \in I^2$, and then, $x_j \leq \min_{i\in I^2}\{1+d_i^2-b_{ij}\}, \forall j \in J$. By the way, since$x_j \leq 1, \forall j \in J$, therefore we have$x_j \leq \min\{1, \min_{i\in I^2}\{1+d_i^2-b_{ij}\}\} = \overline{x}_j$, and then $x \leq \overline{x}$, because of being arbitrary $j \in J$, and the proof is completed.

Theorem 2.2

$$S(B,d^2) = [\overline{0}, \overline{x}].$$

Proof.

It is clearly proved from part (b) of Lemma 2.2 and Lemma 2.3.

Definition 2.3 *Let $J_i = \{j \in J : d_i^1 \leq a_{ij}\}, \forall i \in I^1$. For each $j \in J_i$, we define $i_{x(j)} = (i_{x(j)_k})_{n\times 1}$ such that*

$$i_{x(j)_k} = \begin{cases} 1+d_i^1 - a_{ij} & k = j \\ 0 & k \neq j \end{cases}$$

Lemma 2.4 *Assume $i \in I^1$ is a fixed number.*

(a) For each $j \in J_i$, the vectors $i_{x(j)}$ are the minimal solutions of $S(A,d^1)_i$.

(b) If $d_i^1 = 0$ then $\overline{0}$ is the single minimum solution of $S(A,d^1)_i$.

Proof.

(a) Suppose $j \in J_i$ and$i \in I^1$, since $i_{x(j)_j} = 1+d_i^1 - a_{ij}$ then, $i_{x(j)} \in S(A,d^1)_i$ through relation 3. Now by contrary, let there exist $x \in S(A,d^1)_i$ such that$x < i_{x(j)}$, as a result $x_j < 1+d_i^1 - a_{ij}$ and $x_k = 0$ for$k \in J-\{j\}$. After that$a_{ij}+x_j-1 < d_i^1$, $\forall j \in J$ and followed that$x \notin S(A,d^1)_i$, by means of Relation 1.3, that is a contradiction.

(b) The proof of this part of lemma is clear because the one of the minimal solutions will be$i_{x(j)} = \overline{0}$.

Corollary 2.1 *If $S(A,d^1)_i \neq \emptyset$, then $S(A,d^1)_i = \bigcup_{j\in J_i}[i_{x(j)}, \overline{1}]$, where $i \in I^1$.*

Proof.

Take into account $S(A,d^1)_i \neq \emptyset$ means the vector $\overline{1}$ is the maximum solution and the vectors$i_{x(j)}$, $\forall j \in J_i$ are the minimal solutions in $S(A,d^1)_i$ as a result of Lemmas 2.1 and 2.4, respectively. Now, let$x \in \bigcup_{j\in J_i}[i_{x(j)}, \overline{1}]$, so, for some$j \in J_i$, $x \in [i_{x(j)}, \overline{1}]$ and also $x \in [0,1]^n$ and $x_j \geq i_{x(j)_j} = 1+d_i^1 - a_{ij}$ via Definition 2.3, Hence, $x \in S(A,d^1)_i$ through Relation 1.3. Conversely, let$x \in S(A,d^1)_i$. Then there exits some $j' \in J$ such that $x_{j'} \geq 1+d_i^1 - a_{ij'}$ as a result of Relation 1.3. Since,$x \in [0,1]^n$ so,$1+d_i^1 - a_{ij'} \leq 1$, then$d_i^1 \leq a_{ij'}$, and for that reason $j' \in J_i$. Therefore, $i_{x(j')} \leq x \leq \overline{1}$ that implies $x \in \bigcup_{j\in J_i}[i_{x(j)}, \overline{1}]$.

Definition 2.4 *Let $e = (e(1), e(1.2), ...e(m)) \in J_1 \times J_2 \times ... \times J_m$ such that $e(i) = j \in J_i$. We define $x(e)_j = \max_{i\in I_j^e}\{1+d_i^1 - a_{ij}\}$ if $I_j^e \neq \emptyset$ and $x(e)_j = 0$ if $I_j^e = \emptyset$, where $I_j^e = \{i \in I^1 : e(i) = j\}$.*

Lemma 2.5 *Let $S(A,d^1) \neq \emptyset$, then $S(A,d^1) = \bigcup_{x(e)\in X(e)}[x(e), \overline{1}]$, where$X(e) = \{x(e) : e \in J_I\}$.*

Proof.

If $S(A,d^1) \neq \emptyset$ then, $S(A,d^1)_i \neq \emptyset$, $\forall i \in I^1$. Hence, by Corollary 2.1 and Definitions 2.1 and 2.4, we have

$$\begin{aligned} S(A,d^1) &= \bigcap_{i\in I^1} S(A,d^1)_i = \\ &\bigcap_{i\in I^1}[\bigcup_{j\in J_i}[i_{x(j)}, \overline{1}]] = \bigcap_{i\in I^1}[\bigcup_{e(i)\in J_i}[i_{x(e(i))}, \overline{1}]] \\ &= \bigcup_{e\in J_I}[\bigcap_{i\in I^1}[i_{x(e(i))}, \overline{1}]] = \bigcup_{e\in J_I}[x(e), \overline{1}] \\ &= \bigcup_{x(e)\in X(e)}[x(e), \overline{1}] \end{aligned} \tag{2.4}$$

From Lemma 2.5, it is obvious that $S(A,d^1) = \bigcup_{x(e)\in X_0(e)}[x(e), \overline{1}]$ and $X_0(e) = S_0(A,d^1)$, where $X_0(e)$ and $S_0(A,d^1)$ are the set of minimal solutions of $X(e)$ and $S(A,d^1)$, respectively.

Corollary 2.2 *(a) If $d_i^1 = 0$ for $i \in I^1$, then we can remove the i'th row of the matrix A.*
(b) If $j \notin J_i$, $\forall i \in I^1$ then we can omit j'th column of the matrix A for the purpose of finding $x(e)$.
Proof.
(a) It is proved from Definition 2.4 and the part (b) of the Lemma 2.4, because we will get minimal elements of $S(A, d^1)$
(b) It is proved only by using Definition 2.4.

It is recalled that in part (a), by Definition 2.4 and the part (b) of the Lemma 2.4, the i'th row of the matrix A has no effect in the calculation of the vectors $x(e)$ belong to $X_0(e) = S_0(A, d^1)$, and also in part (b), before calculating the vectors $x(e)$, $\forall e \in I_J$, we can remove j'th column of the matrix A by the use of Definition 2.4 and set $x(e)_j = 0$.

Theorem 2.3 *If $S(A, B, d^1, d^2) \neq \emptyset$, then $S(A, B, d^1, d^2) = \bigcup_{x(e)\in X_0(e)} [x(e), \overline{x}]$.*
Proof.
It is obvious from Definition 2.1, Theorem 2.2 and Lemma 2.5.

Corollary 2.3 ***(Necessary and Sufficient Condition)***
$S(A, B, d^1, d^2) \neq \emptyset$ if only if $\overline{x} \in S(A, d^1)$ or, equivalently, $S(A, B, d^1, d^2) \neq \emptyset$ if only if there exists some $e \in J_I$ such that$x(e) \leq \overline{x}$.
Proof:
It is clearly resulted from Theorem 2.2, Lemma 2.2 and Lemma 2.3.

3 Simplification operations and resolution algorithm

In order to solve the problem (1.2), it is initially converted into two follow sub-problems

$$\begin{aligned} &\min c^{+t}x \\ s.t.\quad & A \circ_L x \geq d^1 \qquad (4a) \\ & B \circ_L x \leq d^2 \\ & x \in [0,1]^n \end{aligned}$$

$$\begin{aligned} &\min c^{-t}x \\ s.t.\quad & A \circ_L x \geq d^1 \qquad (4b) \\ & B \circ_L x \leq d^2 \\ & x \in [0,1]^n \end{aligned}$$

where, $c_j^+ = \max(0, c_j)$ and $c_j^- = \min(0, c_j)$.
It is understandable that $\overline{x}$ is an optimal solution of (4b). Also, (4a) achieves its optimal points at some$x(e) \in X_0(e)$. Once $x(e_0)$ optimizes (4a), we set $x^* = (x_j^*)_{n\times 1}$ such that

$$x_j^* = \begin{cases} \overline{x}_j & , \quad c_j \leq 0 \\ x(e_0)_j & , \quad c_j > 0 \end{cases}$$

Now following lemma gives us an optimal point of the problem (1.2).

Lemma 3.1 *x^* is an optimal solution of the problem (1.2).*
Proof. *See the Theorem 2.1 in [14].*

In order to calculate x^*, it is enough to find $\overline{x}$ and $x(e_0)$. Although $\overline{x}$ is easily attained through Definition 2.2, but $x(e_0)$ is not so, because, $X_0(e)$ is attained by pairwise comparison of $X(e)$ members. For that reason, having complete set of $X_0(e)$ is time-consuming, especially, while $X(e)$ has several members. Therefore, simplification operations can hasten the resolution of the problem (4a). With the intention of simplification the vectors $e \in J_I$ is removed at what time $x(e)$ is not optimal of (4a). One of such these operations is given by Corollary 2.2. Other operations are attained by follow theorems.

Definition 3.1 *Let$\overline{J}_i = \{j \in J_i : 1 + d_i^1 - a_{ij} \leq \overline{x}_j\}$,$\forall i \in I^1$ where $\overline{x}$ comes from Definition 2.2.*

Theorem 3.1 *$S(A, B, d^1, d^2) \neq \emptyset$ if only if$\overline{J}_i \neq \emptyset$,$\forall i \in I^1$.*
Proof.
Suppose $S(A, B, d^1, d^2) \neq \emptyset$. Therefore by Corollary 2.3, $\overline{x} \in S(A, B, d^1, d^2)$ and so we have$\overline{x} \in S(A, d^1)_i$,$\forall i \in I^1$. Thus, for each $i \in I^1$there exists some $j \in J$ such that $\overline{x}_j \geq 1 + d_i^1 - a_{ij}$, as a result of Corollary 2.12.1, consequently $\overline{J}_i \neq \emptyset$, $\forall i \in I^1$. Conversely, suppose$\overline{J}_i \neq \emptyset$,$\forall i \in I^1$. It means that, $\forall i \in I^1$ there exists some $j \in J$ such that$\overline{x}_j \geq 1 + d_i^1 - a_{ij}$. Hence, $\overline{x} \in S(A, d^1)_i$, $\forall i \in I^1$ through Corollary 2.1, as a result $\overline{x} \in S(A, d^1)$. This fact go with Lemma 2.3 implies$\overline{x} \in S(A, B, d^1, d^2)$, therefore, $S(A, B, d^1, d^2) \neq \emptyset$.

Theorem 3.2 *Let$S(A, B, d^1, d^2) \neq \emptyset$, then $S(A, B, d^1, d^2) = \bigcup_{x(e)\in \overline{X}(e)} [x(e), \overline{x}]$ where, $\overline{X}(e) = \{x(e) : e \in \overline{J}_I = \overline{J}_1 \times \overline{J}_2 \times ... \times \overline{J}_m\}$.*
Proof.
By considering Theorem 3.1, it is sufficient to show $x(e) \notin S(A, B, d^1, d^2)$once $e \notin \overline{J}_I$. Supposee $\notin \overline{J}_I$. Thus, there exist $i' \in I^1$ and

$j' \in J_{i'}$ such that $e(i') = j'$ and $1+d^1_{i'}-a_{i'j'} > \overline{x}_{j'}$, Then $i' \in I^e_{j'}$ and by means of Definition 2.4, $x(e)_{j'} = \max_{i\in I^e_{j'}}\{1+d^1_i - a_{ij'}\} \geq 1+d^1_{i'}-a_{i'j'} > \overline{x}_{j'}$. Therefore, $x(e) \leq \overline{x}$ will not be correct, and as a consequence of Theorem 3.13.1, we can obtain$x(e) \notin S(A,B,d^1,d^2)$.

It is noticeable that as a result of Definition 3.1, we have$\overline{J}_i \subseteq J_i$, $\forall i \in I^1$ that means$\overline{X}(e) \subseteq X(e)$. Also, by Theorem 3.2, $S_0(A,B,d^1,d^2) \subseteq \overline{X}(e)$ in which $S_0(A,B,d^1,d^2)$ is minimal elements of $S(A,B,d^1,d^2)$. Thus, the region of search can be reduced to find the set $S_0(A,B,d^1,d^2)$.

Definition 3.2 *Let $J^*_i = \{j \in \overline{J}_i : c^-_j \neq 0\}$, $\forall i \in I^1$.*

Theorem 3.3 *Suppose $x(e_0)$is an optimal solution in (4a) and $J^*_{i'} \neq \emptyset$ for some$i' \in I^1$, then there exist$x(e')$ such that$e'(i') \in J^*_{i'}$ and also, $x(e')$ is the optimal solution in (4a).*

Proof.
Suppose $J^*_{i'} \neq \emptyset$ for some $i' \in I^1$ and$e_0(i') = j'$. Define $e' \in \overline{J}_I$ such that $e'(i') = k \in J^*_{i'}$ and $e'(i) = e_0(i)$ for each $i \in I^1$ and $i \neq i'$. By means of Definition 2.4, we have

$$x(e_0)_{j'} = \max_{i\in I^{e_0}_{j'}}\{1+d^1_i - a_{ij'}\}$$
$$\geq \max_{i\in I^{e_0}_{j'}\&i\neq i'}\{1+d^1_i - a_{ij'}\} = x(e')_{j'}$$

and $x(e_0)_j = x(e')_j$ for each $j \in J$ and $j \neq j', k$. Therefore, with noting $c^+_k = 0$ we have:

$$c^{+t}x(e_0) = c_{j'+}x(e_0)_{j'} + \sum_{j\in J\&j\neq j'} c_{j+}x(e_0)_j \geq c_{j'+}x(e')_{j'} + \sum_{j\in J\&j\neq j'} c_{j+}x(e')_j = c^{+t}x(e')$$

Therefore $x(e')$ is an optimal solution in (4a) then, proof is completed.

Corollary 3.1 *If $J^*_i \neq \emptyset$ for some $i \in I^1$ then by omitting i'th row we reach a reduced problem for which each optimal solution is an optimal solution for the previous (main) problem.*

Proof.
It is resulted from Theorem 3.3 and also, notes that $c^+_j = 0$ for each $j \in J^*_i$.

Definition 3.3 *Let$j_1, j_2 \in J$, $c_{j_1} > 0$ and $c_{j_2} > 0$. We say j_2 dominates j_1 if only if*
(a) $j_1 \in \overline{J}_i$ implies $j_2 \in \overline{J}_i$,$\forall i \in I^1$.
(b) For each $i \in I^1$such that $j_1 \in \overline{J}_i$, $c_{j_1}(1+d^1_i - a_{ij_1}) \geq c_{j_2}(1+d^1_i - a_{ij_2})$.

Theorem 3.4 *Suppose $x(e_0)$ is the optimal in (4a) and j_2 dominates j_1 for $j_1, j_2 \in J$, then, there exist $x(e')$ such that $I^{e'}_{j_1} = \emptyset$ and also, $x(e')$ is an optimal solution in (4a).*
Proof. *Define $e' = (e'(i))_{m\times 1}$ such that*

$$e'(i) = \begin{cases} e_0(i) & i \notin I^{e_0}_{j_1} \\ j_2 & i \in I^{e_0}_{j_1} \end{cases}$$

It is obvious that $I^{e'}_{j_1} = \emptyset$ and so $x(e')_{j_1} = 0$. Also, $x(e_0)_j = x(e')_j$ for each $j \in J$ and$j \neq j_1, j_2$, $x(e')_{j_2} = 1+d^1_{i_0} - a_{i_0j_2}$. Now, if $i_0 \notin I^{e_0}_{j_1}$ then:

$$x(e_0)_{j_2} = x(e')_{j_2} = 1+d^1_{i_0} - a_{i_0j_2},$$

so we have

$$c^{+t}x(e_0) = c^+_{j_1}x(e_0)_{j_1} + \sum_{j\in J\&j\neq j_1} c_{j+}x(e_0)_j$$
$$\geq \sum_{j\in J\&j\neq j_1} c_{j+}x(e')_j = c^{+t}x(e')$$

That proof is completed in this case. Otherwise, assume $i_0 \in I^{e_0}_{j_1}$. We show $c^{+t}x(e_0) \geq c^{+t}x(e')$. As a result of definition 2.4, let $x(e_0)_{j_2} = 1 + d^1_i - a_{ij_2}$. Therefore, we have $c^+_{j_2}x(e_0)_{j_2} \geq 0$ by Definition 3.3. Consequently, since

$$c^{+t}x(e_0) = c^+_{j_1}x(e_0)_{j_1} + c^+_{j_2}x(e_0)_{j_2}$$
$$+ \sum_{j\neq j_1,j_2} c^+_j x(e_0)_j,$$

and

$$c^{+t}x(e') = c^+_{j_2}x(e')_{j_2} + \sum_{j\neq j_1,j_2} c^+_j x(e')_j$$

It is sufficient to show $c^+_{j_1}x(e_0)_{j_1} \geq c^+_{j_2}x(e')_{j_2}$. Now, by definition 2.4, set

$$x(e_0)_{j_1} = 1 + d^1_{i'} - a_{i'j_1}$$

Since j_2 dominates j_1, so we have

$$c^+_{j_1}(1+d^1_{i'} - a_{i'j_1}) \geq c^+_{j_2}(1+d^1_{i_0} - a_{i_0j_2})$$

That means $c^+_{j_1}x(e_0)_{j_1} \geq c^+_{j_2}x(e')_{j_2}$once $i_0 = i'$. Otherwise, suppose $i_0 \neq i'$. Since $i_0 \in I^{e_0}_{j_1}$ and j_2 dominates j_1, thus

$$c^+_{j_1}(1+d^1_{i_0} - a_{i_0j_1}) \geq c^+_{j_2}(1+d^1_{i_0} - a_{i_0j_2})$$

Also, through definition 2.4 we have $x(e_0)_{j_1} = \max_{i \in I_{j_1}^{e_0}} \{1 + d_i^1 - a_{ij_1}\} = 1 + d_{i'}^1 - a_{i'j_1}$ *that implies*

$$1 + d_{i'}^1 - a_{i'j_1} \geq 1 + d_i^1 - a_{ij_1}, \forall i \in I_{j_1}^{e_0}$$

Therefore

$$c_{j_1}^+(1 + d_{i'}^1 - a_{i'j_1}) \geq c_{j_1}^+(1 + d_{i_0}^1 - a_{i_0 j_1})$$

$$\geq c_{j_2}^+(1 + d_{i_0}^1 - a_{i_0 j_2})$$

That results $c_{j_1}^+ x(e_0)_{j_1} \geq c_{j_2}^+ x(e')_{j_2}$*, hence* $c^{+t}x(e_0) \geq c^{+t}x(e')$*, therefore proof is completed.*

Corollary 3.2 *If* j_2 *dominates* j_1 *for* $j_1, j_2 \in J$*, then, by omitting* j_1*'th column we reach a reduced problem for which each optimal solution is an optimal solution for the previous (main) problem.*

4 An algorithm for finding an optimal solution and example

Definition 4.1 *Consider the Problem (1.1). We call* $\overline{A} = (\overline{a}_{ij})_{m\times n}$ *and* $\overline{B} = (\overline{b}_{ij})_{l\times n}$ *the characteristic matrices of the matrix A and matrix B, respectively, where* $\overline{a}_{ij} = 1 + d_i^1 - a_{ij}$ *for each* $i \in I^1$ *and* $j \in J$*, also* $\overline{b}_{ij} = 1 + d_i^2 - b_{ij}$ *for each* $i \in I^2$ *and* $j \in J$ *.*

Algorithm: Given problem (1.2),

1. Find the matrices $\overline{A}$ and $\overline{B}$ by Definition 2.4.
2. If there exists $i \in I^1$ such that $a_{ij} < d_i^1$, $\forall j \in J$ then stop. Problem 1.2 is infeasible (see Theorem 2.1).
3. Calculate $\overline{x}$ from $\overline{B}$ by Definition 2.2.
4. If there exists $i \in I^1$ such that $d_i^1 = 0$ then remove i'Th row of the matrix $\overline{A}$ (see the part (a) of the Corollary 2.2).
5. If $\overline{a}_{ij} > \overline{x}_j$ then, set $\overline{a}_{ij} = 0 \forall i \in I^1$ and $\forall j \in J$.
6. If there exists $i \in I^1$ such that $\overline{a}_{ij} = 0, \forall j \in J$ then stop. Problem (1.2) is infeasible (see Theorem 3.1 and 3.2)
7. If there exists $j' \in J$ such that, $\bar{a}_{ij'} = 0$, $\forall i \in I^1$ then remove j'th column of the matrix $\overline{A}$ (see Theorem 3.2) and set $x(e_0)_{j'} = 0$.
8. For each $i \in I^1$, if $J_i^* \neq \emptyset$ then remove i'th row of the matrix $\overline{A}$ (see Corollary 3.1).
9. Remove each column $j \in J$ from $\overline{A}$ such that $c_j < 0$ and set $x(e_0)_j = 0$.
10. If j_2 dominates j_1 then remove the column j_1 from $\overline{A}$, $\forall j_1, j_2 \in J$ (see Corollary 3.2) and set $x(e_0)_{j_1} = 0$.
11. Let $J_i^{new} = \{j \in \overline{J}_i : \bar{a}_{ij} \neq 0\}$ and $J_I^{new} = J_1^{new} \times J_2^{new} \times ... \times J_m^{new}$. Find the vectors $x(e)$, $\forall e \in J_I^{new}$ by Definition 2.4 from $\overline{A}$, and $x(e_0)$ by pairwise comparison between the vectors $x(e)$.
12. Find x^* via Lemma 3.1.

5 Numerical example

Consider the problem in below:

$$\begin{aligned} \min \quad & 2x_1 - x_2 - 3x_3 + 2.5x_4 - x_5 \\ & +6x_6 - 3x_7 + 2x_8 + x_9 + 5x_{10} \end{aligned}$$

$$\left[\begin{array}{cccccccccc} 1 & 0.16 & 0.37 & 0.95 & 0.17 & 0.07 & 0.77 & 0.14 & 0.8 & 0.6 \\ 0.08 & 0.51 & 0.26 & 0.1 & 0.3 & 0.4 & 0.3 & 0.35 & 0.9 & 1 \\ 0.99 & 0.59 & 0.28 & 0.34 & 0.34 & 0.74 & 0.19 & 0.21 & 0.7 & 0.65 \\ 0.83 & 0.75 & 0.25 & 0.35 & 0.2 & 0.5 & 0.2 & 0.2 & 0.95 & 0.85 \\ 0.73 & 0.84 & 0.94 & 0.44 & 0.54 & 0.84 & 0.99 & 0.44 & 0.1 & 0.5 \\ 0.37 & 0.7 & 0.55 & 0.4 & 0.2 & 0.2 & 0.73 & 0.24 & 0.98 & 0.9 \end{array}\right] \circ_L \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{array}\right] \geq \left[\begin{array}{c} 0 \\ 0.5 \\ 0.57 \\ 0.6 \\ 0.72 \\ 0.6 \end{array}\right]$$

$$\begin{bmatrix} 0 & 0.94 & 0.69 & 0.49 & 0.5 & 0.51 \\ 0.02 & 1 & 0.82 & 0.59 & 0.89 & 0.76 \\ 0.1 & 0.73 & 0.4 & 0.37 & 0.45 & 0.7 \\ 0.1 & 0.1 & 0.3 & 0.2 & 0.25 & 0.49 \end{bmatrix}$$

$$\begin{matrix} 0.87 & 0.43 & 0.2 & 0.5 \\ 0.74 & 0.52 & 0.56 & 0.36 \\ 0.69 & 0.2 & 0.95 & 0.85 \\ 0.58 & 0.01 & 0.8 & 0.98 \end{matrix} \Bigg] \circ_L \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}$$

$$\leq \begin{bmatrix} 0.7 \\ 0.76 \\ 0.6 \\ 0.5 \end{bmatrix}$$

$x_j \in [0,1]^n$

Step 1:
The matrices $\overline{A}$ and $\overline{B}$ are as following

$$\overline{A} = \begin{bmatrix} 0 & 0.84 & 0.63 & 0.05 & 0.83 \\ 1.42 & 0.99 & 1.24 & 1.4 & 1.2 \\ 0.58 & 0.98 & 1.29 & 1.23 & 1.23 \\ 0.77 & 0.85 & 1.35 & 1.25 & 1.4 \\ 0.99 & 0.88 & 0.78 & 1.28 & 1.18 \\ 1.23 & 0.9 & 1.05 & 1.2 & 1.4 \end{bmatrix}$$

$$\begin{matrix} 0.93 & 0.23 & 0.86 & 0.2 & 0.4 \\ 1.1 & 1.2 & 1.15 & 0.6 & 0.5 \\ 0.83 & 1.38 & 1.36 & 0.87 & 0.92 \\ 1.1 & 1.4 & 1.4 & 0.65 & 0.75 \\ 0.88 & 0.73 & 1.28 & 1.62 & 1.22 \\ 1.4 & 0.87 & 1.36 & 0.62 & 0.7 \end{matrix} \Bigg]$$

$$\overline{B} = \begin{bmatrix} 1.7 & 0.76 & 1.01 & 1.21 & 1.2 \\ 1.74 & 0.76 & 0.94 & 1.17 & 0.87 \\ 1.5 & 0.87 & 1.2 & 1.23 & 1.15 \\ 1.4 & 1.4 & 1.2 & 1.3 & 1.25 \end{bmatrix}$$

$$\begin{matrix} 1.19 & 0.83 & 1.27 & 1.5 & 1.2 \\ 1 & 1.02 & 1.24 & 1.2 & 1.4 \\ 0.9 & 0.91 & 1.4 & 0.65 & 0.75 \\ 1.01 & 0.92 & 1.49 & 0.7 & 0.52 \end{matrix} \Bigg]$$

Step 2:
There is no $i \in I^1$ such that $a_{ij} < d_i^1, \forall j \in J$ therefore we can go to step 3.

Step 3:

$$\overline{x} = \begin{bmatrix} 1 & 0.76 & 0.94 & 1 & 0.87 \\ 0.9 & 0.83 & 1 & 0.65 & 0.52 \end{bmatrix}$$

Step 4:
Since $d_1^1 = 0$, then first row from matrix $\overline{A}$ is removed.

Step 5:
In according to this step, $\overline{A}$ is converted to as following:

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.58 & 0 & 0 & 0 & 0 \\ 0.77 & 0 & 0 & 0 & 0 \\ 0.99 & 0 & 0.78 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 0 & 0 & 0 & 0.6 & 0.5 \\ 0.83 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.65 & 0 \\ 0.88 & 0.73 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.62 & 0 \end{matrix} \Bigg]$$

Step 6:
There is no $i \in I^1$ such that $\overline{a}_{ij} = 0, \forall j \in J$ therefore we can go to step 7.

Step 7:
The second, fourth, fifth and eighth columns in according with this step are removed and we have $x(e_0)_2 = x(e_0)_4 = x(e_0)_5 = x(e_0)_8 = 0$, by the way matrix $\overline{A}$ is converted to following:

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.6 & 0.5 \\ 0.58 & 0 & 0.83 & 0 & 0 & 0 \\ 0.77 & 0 & 0 & 0 & 0.65 & 0 \\ 0.99 & 0.78 & 0.88 & 0.73 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.62 & 0 \end{bmatrix}$$

Step 8:
Since$J_5^* \neq \emptyset$, then we can delete fifth row, then we get to

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.6 & 0.5 \\ 0.58 & 0 & 0.83 & 0 & 0 & 0 \\ 0.77 & 0 & 0 & 0 & 0.65 & 0 \\ 0 & 0 & 0 & 0 & 0.62 & 0 \end{bmatrix}$$

Step 9:
Since $c_3, c_7 < 0$ then, we can remove third and seventh columns and we get to

$$\overline{A} = \begin{bmatrix} 0 & 0 & 0.6 & 0.5 \\ 0.58 & 0.83 & 0 & 0 \\ 0.77 & 0 & 0.65 & 0 \\ 0 & 0 & 0.62 & 0 \end{bmatrix}$$

Also, we have $x(e_0)_3 = x(e_0)_7 = 0$.

Step10:

In the attained matrix, first and ninth columns dominate sixth and tenth columns, respectively. By removing sixth and tenth columns, matrix $\overline{A}$ is converted to

$$\overline{A} = \begin{bmatrix} 0 & 0.6 \\ 0.58 & 0 \\ 0.77 & 0.65 \\ 0 & 0.62 \end{bmatrix}$$

Also, we have$x(e_0)_6 = x(e_0)_{10} = 0$.

Step11:

In the new matrix, we have $J_2^{new} = \{9\}$, $J_3^{new} = \{1\}$, $J_4^{new} = \{1, 9\}$ and $J_6^{new} = \{9\}$. For $e_1 = (9, 1, 1, 9)$, $x(e_1)_1 = 0.77$ and $x(e_1)_9 = 0.62$, then

$$x(e_1) = (0.77, 0, 0, 0, 0, 0, 0, 0, 0.62, 0)$$

Also, $e_2 = (9, 1, 9, 9)$ results in $x(e_2)_1 = 0.58$ and $x(e_2)_9 = 0.65$, then

$$x(e_2) = (0.58, 0, 0, 0, 0, 0, 0, 0, 0.65, 0)$$

Therefore minimal solutions are $x(e_1)$ and $x(e_2)$. Since $c^{+t}x(e_1) \geq c^{+t}x(e_2)$, then $x(e_0) = x(e_2) = (0.58, 0, 0, 0, 0, 0, 0, 0, 0.65, 0)$ is optimal solution for (4a).

Step12:

Since $x(e_0)$ optimizes the problem with objective function $c^{+t}x$ then

$$x^* = (0.58, 0.76, 0.94, 0, 0.87, 0, 0.83, 0, 0.65, 0)$$

6 Conclusion

In this paper, we studied the linear optimization problem with fuzzy relational inequalities constraints defined by max-Lukasiewicz operator. First, we discussed the feasibility region characterization, then; by introducing a new simplification technique the usual difficulty of finding the minimal solutions that optimize the problem with objective function $c^{+t}x$ was solved. In this relation an algorithm together with some simplification operations to accelerate the problem resolution was presented. At last, we gave an example to more illustrate of the problem.

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