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The Taylor Method for Numerical Solution of Fuzzy Generalized Pantograph Equations with Linear Functional Argument

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Abstract

In this paper, we introduce a numerical method based on the Taylor polynomials for the approximate solution of the pantograph equation with linear functional argument, with the fuzzy initial conditions . This method is illustrated by solving two examples.

Keywords : Fuzzy number; Fuzzy distance; Fuzzy differential equations; Pantograph equations; Taylor method

1 Introduction

The topic of fuzzy differential equations (FDEs) has been rapidly growing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [13]; it was followed up by Dubois and Prade [19], who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [32] and Goetschel and Voxman [22]. Kandel and Byatt [27, 28] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by Kaleva [25,26], Seikkala [33], He and Yi [23], Kloeden [29] and Menda [30], and by other researchers (see [7, 9, 10, 11, 14, 18, 24]). The numerical methods for solving fuzzy differential equations are introduced in [1]-[4]. Buckley and Feuring [8] introduced two analytical methods for solving nth-order linear differential equations with fuzzy initial conditions. Their first method of solution was to fuzzify the crisp solution and then check to see if it satisfies the differential equation with fuzzy initial conditions; and the second method was the reverse of the first method, in that they first solved the fuzzy initial value problem and the checked to see if it defined

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a fuzzy function ;also Allahviranloo et al. [5] introduced other numerical methods for solving nth-order linear differential equations with fuzzy initial conditions.
Our purpose in this study is to develop and to apply Taylor method to the generalized pantograph equation with linear functional argument, with fuzzy initial conditions. In recent years, there has been a growing interest in the numerical treatment of pantograph equations. A special feature of this type is the existence of compactly supported solutions [16]. This phenomenon was studied in [15] and has direct applications to approximation theory and to wavelets [17]. Pantograph equations are characterized by the presence of a linear functional argument and play an important role in explaining many different phenomena. In particular they turn out to be fundamental when ODEs-based model fails. These equations arise in industrial applications [21, 31] and in studies based on biology, economy, control and electrodynamics [6, 12]. The structure of the paper is organized as follows:
In Section 2, some basic definitions which will be used later in the paper are provided. In Section 3, one method for solving fuzzy generalized pantograph equations with linear functional argument is introduced, then the proposed method is illustrated by solving two examples in Section 4,and the conclusion is considered in Section 5.

2 Preliminaries

A tilde is placed over a symbol to denote a fuzzy set, as in $\widetilde{\lambda}, \widetilde{f}(t)$, ...
An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)), \quad 0 \leq r \leq 1$, which satisfy the following requirements.

- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over [0, 1].
- $\overline{u}(r)$ is a bounded left continuous nonincreasing function over [0, 1].
- $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

Let $\mathbf{E}$ be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level intervals. This means that if $v \in \mathbf{E}$ then the r-level set

$$[v]^r = s|v(s) \geq r, \quad 0 < r \leq 1$$

is a closed bounded interval which is denoted by

$$[v]^r = [\underline{v}(r), \overline{v}(r)].$$

For arbitrary $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v})$ and $k \geq 0$,addition and multiplication by k are defined as follows:

- $(\underline{u+v}) = \underline{u}(r) + \underline{v}(r)$,
- $(\overline{u+v}) = \overline{u}(r) + \overline{v}(r)$,
- $(\underline{ku})(r) = k\underline{u}(r), \quad (\overline{ku})(r) = k\overline{u}(r)$.

Definition 2.1. *For arbitrary fuzzy quantities $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$,the quantity*

$$D(u, v) = [\int_0^1 (\underline{u}(r) - \underline{v}(r))^2 dr + \int_0^1 (\overline{u}(r) - \overline{v}(r))^2 dr]^{\frac{1}{2}}$$

is the distance between u and v.

3 New Method

In this section, we are going to solve the following problem (taken from [34], Eq. (3.1))

$$y^{(m)}(t) = \sum_{j=0}^{J} \sum_{k=0}^{m-1} P_{jk}(t) y^{(k)}(\alpha_j t + \beta_j) + f(t) \tag{3.1}$$

which is a generalization of the pantograph equations ,with the fuzzy initial conditions

$$\sum_{k=0}^{m-1} c_{ik} \tilde{y}^{(k)}(0) = \tilde{\lambda}_i, \quad i = 0, 1, 2, ..., m-1, \tag{3.2}$$

Here $P_{jk}(t)$ are analytical functions on some interval I. The interval I can be [0,T] for some $T > 0$. c_{ik}, α_j and β_j are real or complex constants and for $i = 0, 1, 2, ..., m-1$,$\tilde{\lambda}_i$ are fuzzy constants also , let

$$\forall j, k \quad P_{jk} \geq 0, \quad \forall i, k \quad c_{ik} \geq 0$$

Buckly-Feuring method of solution is to fuzzify the crisp solution to obtain a fuzzy function $\tilde{Y}(t)$, and then check to see if it satisfies the differential equation with fuzzy initial conditions. In this paper we proposed another method for solving m-order fuzzy differential pantograph equations with linear functional argument. This method is to find the solution in terms of the Taylor polynomial form, in the origin,

$$\tilde{y}_N(t) = \sum_{n=0}^{N} \tilde{y}_n t^n, \quad \tilde{y}_n = \frac{\tilde{y}^{(n)}(0)}{n!}, \tag{3.3}$$

where $\{t^n\}_{n=0}^{\infty}$ are positive basic functions whose all differentiations are positive. Now, the aim is to compute the fuzzy coefficients in (3.3) by setting the error to zero as follows,

$$\begin{aligned} Error \quad &= D(\tilde{y}^{(m)}(t) - \textstyle\sum_{j=0}^{J} \sum_{k=0}^{m-1} P_{jk}(t) \tilde{y}^{(k)}(\alpha_j t + \beta_j), \tilde{f}(t)) \\ &+ D(\textstyle\sum_{k=0}^{m-1} c_{0k} \tilde{y}^{(k)}(0), \tilde{\lambda}_0) + D(\textstyle\sum_{k=0}^{m-1} c_{1k} \tilde{y}^{(k)}(0), \tilde{\lambda}_1) \\ &+ ... + D(\textstyle\sum_{k=0}^{m-1} c_{m-1k} \tilde{y}^{(k)}(0), \tilde{\lambda}_{m-1}). \end{aligned} \tag{3.4}$$

To illustrate this approach, let $\tilde{y}_N(t)$ be the fuzzy solution of (3.1) such that

$$\underline{y_N}(t, r) = \underline{\sum_{n=0}^{N} y_n t^n} = \sum_{n=0}^{N} \underline{y_n t^n} = \sum_{n=0}^{N} \underline{y_n} t^n, \tag{3.5}$$

$$\overline{y}_N(t, r) = \overline{\sum_{n=0}^{N} y_n t^n} = \sum_{n=0}^{N} \overline{y_n t^n} = \sum_{n=0}^{N} \overline{y_n} t^n, \quad 0 < r \leq 1 \tag{3.6}$$

We substitute (3.5) and (3.6) in (3.4), then we have two initial value problems as following

$$\begin{cases} \underline{y^{(m)}(t,r) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t)y^{(k)}(\alpha_j t+\beta_j,r) = f(t,r)} \\ \underline{\sum_{k=o}^{m-1} c_{ik}y^{(k)}(0,r) = \underline{\lambda_i}(r)} \qquad i=0,1,2,...,m-1 \end{cases} \tag{3.7}$$

$$\begin{cases} \overline{y^{(m)}(t,r) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t)y^{(k)}(\alpha_j t+\beta_j,r) = \overline{f(t,r)}} \\ \underline{\sum_{k=o}^{m-1} c_{ik}y^{(k)}(0,r) = \overline{\lambda_i}(r)} \qquad i=0,1,2,...,m-1 \end{cases} \tag{3.8}$$

now , since $\forall j,k,\ P_{jk} \geq 0$,we have

$$\begin{cases} \underline{y}^{(m)}(t,r) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t)\underline{y}^{(k)}(\alpha_j t+\beta_j,r) = \underline{f}(t,r) \\ \sum_{k=o}^{m-1} c_{ik}\underline{y}^{(k)}(0,r) = \underline{\lambda_i}(r) \qquad i=0,1,2,...,m-1 \end{cases} \tag{3.9}$$

$$\begin{cases} \overline{y}^{(m)}(t,r) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t)\overline{y}^{(k)}(\alpha_j t+\beta_j,r) = \overline{f}(t,r) \\ \sum_{k=o}^{m-1} c_{ik}\overline{y}^{(k)}(0,r) = \overline{\lambda}_i(r) \qquad i=0,1,2,...,m-1 \end{cases} \tag{3.10}$$

we solve the initial value problem (3.9) as following :

let us convert expressions defined in (3.9), (3.5) to the matrix forms. Let us first assume that the functions $\underline{y}(t,r)$ and its derivative $\underline{y}^{(k)}(t,r)$ can be expanded to Taylor series about $t=0$ in the form

$$\underline{y}^{(k)}(t,r) = \sum_{n=0}^{\infty} \underline{y_n}^{(k)} t^n, \tag{3.11}$$

where for k = 0, $\underline{y}^{(0)}(t,r) = \underline{y}(t,r)$ and $\underline{y_n}^{(0)} = \underline{y_n}$.
Now, let us differentiate expression (3.11) with respect to t and then put $n \to n+1$

$$\underline{y}^{(k+1)}(t,r) = \sum_{n=1}^{\infty} n\underline{y_n}^{(k)} t^{n-1} = \sum_{n=0}^{\infty}(n+1)\underline{y_{n+1}}^{(k)} t^n. \tag{3.12}$$

It is clear, from (3.11), that

$$\underline{y}^{(k+1)}(t,r)=\sum_{n=0}^{\infty}\underline{y_n}^{(k+1)}t^n. \tag{3.13}$$

Using relations (3.12) and (3.13), we have the recurrence relation between Taylor coefficients of $\underline{y}^{(k)}(t,r)$ and $\underline{y}^{(k+1)}(t,r)$

$$\underline{y_n}^{(k+1)}=(n+1)\underline{y_{n+1}}^{(k)},\quad n,k=0,1,2,.... \tag{3.14}$$

If we take n = 0, 1, ... , N and assume $\underline{y_n}^{(k)}=0$ for $n>N$, then we can transform system (3.14) into the matrix form

$$\underline{\mathbf{Y}}^{(k+1)}=\mathbf{M}\underline{\mathbf{Y}}^{(k)},\quad k=0,1,2,.... \tag{3.15}$$

where

$$\underline{\mathbf{Y}}^{(k)}=\begin{pmatrix}\underline{y_0}^{(k)}\\ \underline{y_1}^{(k)}\\ \vdots\\ \underline{y_N}^{(k)}\end{pmatrix},\quad \mathbf{M}=\begin{pmatrix}0&1&0&\dots&0\\0&0&2&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&N\\0&0&0&\dots&0\end{pmatrix}$$

For k = 0, 1, 2, . . ., it follows from relation (3.15) that

$$\underline{\mathbf{Y}}^{(k)}=\mathbf{M}^k\mathbf{Y}, \tag{3.16}$$

where clearly

$$\underline{\mathbf{Y}}^{(0)}=\underline{\mathbf{Y}}=[\underline{y_0}(r)\ \ \underline{y_1}(r)\ \ \dots\ \ \underline{y_N}(r)]^T.$$

On the other hand, solution expressed by (3.5) and its derivatives can be written in the matrix forms

$$\underline{y}(t,r)=\mathbf{T}\underline{\mathbf{Y}}$$

and

$$\underline{y}^{(k)}(t,r)=\mathbf{T}\underline{\mathbf{Y}}^{(k)}$$

or using relation (3.16)

$$\underline{y}(t,r)=\mathbf{T}\mathbf{M}^{\mathbf{k}}\underline{\mathbf{Y}}, \tag{3.17}$$

where

$$\mathbf{T}=[1\ \ t\ \ t^2\ \ \dots\ \ t^N].$$

To obtain the matrix form of the part

$$\sum_{j=0}^{J}\sum_{k=0}^{m-1}P_{jk}(t)\underline{y}^{(k)}(\alpha_j t+\beta_j,r) \tag{3.18}$$

which is defined in Eq. (3.9), we first write the function $P_{jk}(t)$ in the form

$$P_{jk}(t)=\sum_{i=0}^{N}P_{jk}^{(i)}t^i,\quad P_{jk}^{(i)}=\frac{P_{jk}^{(i)}(0)}{i!}$$

and then, substitute into (3.18). It is seen from relation (3.11) and binomial expansion that

$$\underline{y}^{(k)}(\alpha_j t+\beta_j,r)=\sum_{n=0}^{N}\underline{y_n}^{(k)}(\alpha_j t+\beta_j)^n=\sum_{n=0}^{N}\sum_{v=0}^{n}\binom{n}{v}\alpha_j^{n-v}\beta_j^v t^{n-v}\underline{y_n}^{(k)}.$$

Thus, the term $t^i\underline{y}^{(k)}(\alpha_j t+\beta_j,r)$ is obtained and its matrix representations become

$$t^i\underline{y}^{(k)}(\alpha_j t+\beta_j,r)=\sum_{n=0}^{N}\sum_{v=0}^{n}\binom{n}{v}\alpha_j^{n-v}\beta_j^v t^{n-v+i}\underline{y_n}^{(k)}=\mathbf{TI}_i\mathbf{A}_j\underline{\mathbf{Y}}^{(k)}$$

or from (3.16)

$$t^i\underline{y}^{(k)}(\alpha_j t+\beta_j,r)=\mathbf{TI}_i\mathbf{A}_j\mathbf{M}^k\underline{\mathbf{Y}},\quad i=0,1,...,N, \tag{3.19}$$

where

$$\mathbf{I}_0=\begin{pmatrix}1&0&\dots&0&0\\0&1&\dots&0&0\\0&0&\dots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\dots&0&1\end{pmatrix},\quad \mathbf{I}_1=\begin{pmatrix}0&0&\dots&0&0\\1&0&\dots&0&0\\0&1&\dots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\dots&1&0\end{pmatrix},$$

$$\mathbf{I}_N=\begin{pmatrix}0&0&\dots&0&0\\0&0&\dots&0&0\\0&0&\dots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\1&0&\dots&0&0\end{pmatrix}$$

For $\beta_j\neq 0$,

$$\mathbf{A}_j=\begin{pmatrix}\binom{0}{0}(\alpha_j)^0(\beta_j)^0&\binom{1}{1}(\alpha_j)^0(\beta_j)^1&\dots&\binom{N}{N}(\alpha_j)^0(\beta_j)^N\\0&\binom{1}{0}(\alpha_j)^1(\beta_j)^0&\dots&\binom{N}{N-1}(\alpha_j)^1(\beta_j)^{N-1}\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&\binom{N}{0}(\alpha_j)^N(\beta_j)^0\end{pmatrix}$$

and for $\beta_j=0$,

$$\mathbf{A}_j=\begin{pmatrix}(\alpha_j)^0&0&\dots&0\\0&(\alpha_j)^1&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&(\alpha_j)^N\end{pmatrix}$$

Utilizing expression (3.19), we obtain the matrix form of the part (3.18) as

$$\sum_{j=0}^{J}\sum_{k=0}^{m-1}\sum_{i=0}^{N}P_{jk}^{(i)}\mathbf{TI}_i\mathbf{A}_j\mathbf{M}^k\underline{\mathbf{Y}}. \tag{3.20}$$

We now assume that the function $f(t)$ can be expanded as

$$f(t) = \sum_{n=0}^{N} f_n t^n, \quad f_n = \frac{f^{(n)}(0)}{n!}$$

or written in the matrix form

$$f(t) = \mathbf{TF}. \tag{3.21}$$

where

$$\mathbf{F} = [\ f_0\ \ f_1\ \ .\ .\ .\ \ f_N\ ]^T.$$

Next, by means of relation (3.17), we can obtain the corresponding matrix form for the initial conditions (3.9) as

$$\sum_{k=0}^{m-1} c_{ik}\mathbf{T}(0)\mathbf{M}^k\underline{\mathbf{Y}} = \underline{\lambda_i}(r) \quad i = 0, 1, ..., m-1, \tag{3.22}$$

where

$$\mathbf{T}(0) = [\ 1\ \ 0\ \ 0\ \ .\ .\ .\ \ 0\ ].$$

We are now ready to construct the fundamental matrix equation corresponding to Eq. (3.9). For this purpose, substituting matrix relations (3.17), (3.20) and (3.21) into Eq. (3.9) and then simplifying, we obtain the fundamental matrix equation

$$\{\mathbf{M}^m - \sum_{j=0}^{J}\sum_{k=0}^{m-1}\sum_{i=0}^{N} P_{jk}^{(i)}\mathbf{I}_i\mathbf{A}_j\mathbf{M}^k\}\underline{\mathbf{Y}} = \mathbf{F} \tag{3.23}$$

which corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $\underline{y_0}(r), \underline{y_1}(r), ..., \underline{y_N}(r)$. Briefly, we can write Eq. (3.23) in the form

$$\mathbf{W}\underline{\mathbf{Y}} = \mathbf{F} \ \ or \ \ [\mathbf{W}; \mathbf{F}],$$

where

$$\mathbf{W} = [w_{nh}], \quad n, h = 0, 1, ..., N.$$

Also, the matrix form (3.22) for conditions (3.22) can be written as

$$\mathbf{U}_i\underline{\mathbf{Y}} = \underline{\lambda_i}(r) \ \ or \ \ [\mathbf{U}_i; \underline{\lambda_i}(r)], \quad i = 0, 1, ..., m-1,$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m-1} c_{ik}\mathbf{T}(0)\mathbf{M}^k = [\ u_{i0}\ \ u_{i1}\ \ .\ .\ .\ \ u_{iN}\ ].$$

To obtain the solution of Eq. (3.9), by replacing the m rows matrices $[\mathbf{U}_i; \underline{\lambda_i}(r)]$ by the last m rows of the matrix $[\mathbf{W}; \mathbf{F}]$, we have the augmented matrix

$$[\widehat{\mathbf{W}}, \widehat{\mathbf{F}}] = \begin{pmatrix} w_{00} & w_{01} & \dots & w_{0N} & ; & f_0 \\ w_{10} & w_{11} & \dots & w_{1N} & ; & f_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \dots & w_{N-m,N} & ; & f_{N-m} \\ u_{00} & u_{01} & \dots & u_{0N} & ; & \underline{\lambda_0}(r) \\ u_{10} & u_{11} & \dots & u_{1N} & ; & \underline{\lambda_1}(r) \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \dots & u_{m-1,N} & ; & \underline{\lambda_{m-1}}(r) \end{pmatrix}$$

If det $\widehat{\mathbf{W}} \neq 0$, then we can write

$$\underline{\mathbf{Y}} = (\widehat{\mathbf{W}})^{-1}\widehat{\mathbf{F}}.$$

Thus the coefficients $\underline{y_n}(r), \quad n = 0, 1, ..., N$ are uniquely determined by this equation.

In a similar way, we can solve the initial value problem (3.10). we determin the coefficients $\overline{y_n}(r), \quad n = 0, 1, ..., N$.

4 Illustrative examples

In this section, two numerical examples are given to illustrate the properties of the method.

Example 4.1. *(Evans and Raslan, [20]). Consider the fuzzy pantograph equation of second order*

$$y''(t) = \frac{3}{4}y(t) + y(\frac{t}{2}) - t^2 + 2, \quad \tilde{y}(0) = (r, -r + 2), \quad \tilde{y'}(0) = (r, 3 - 2r), \quad 0 \leq r \leq 1$$

we have two initial value problems as following :

$$\begin{cases} \underline{y}''(t) = \frac{3}{4}\underline{y}(t) + \underline{y}(\frac{t}{2}) - t^2 + 2, \quad r \in [0, 1] \\ \underline{y}(0) = r, \quad \underline{y}'(0) = r. \end{cases} \tag{4.24}$$

and

$$\begin{cases} \overline{y}''(t) = \frac{3}{4}\overline{y}(t) + \overline{y}(\frac{t}{2}) - t^2 + 2, \quad r \in [0, 1] \\ \overline{y}(0) = 2 - r, \quad \overline{y}'(0) = 3 - 2r. \end{cases} \tag{4.25}$$

we solve the initial value problem (4.24) as following : The fundamental matrix equation of this problem is

$$(\mathbf{M}^2 - \frac{3}{4}\mathbf{I}_0 - \mathbf{A}_1)\underline{\mathbf{Y}} = \underline{\mathbf{F}}.$$

Here I_0 is unit matrix and for $N = 4$ others

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{16} \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ r \\ r \end{pmatrix}$$

After the ordinary operations and following the method in Section 3, the augmented matrix for the problem is gained as

$$[\widehat{\mathbf{W}},\widehat{\mathbf{F}}]=\begin{pmatrix} -\frac{7}{4} & 0 & 2 & 0 & 0 & ; & 2 \\ 0 & -\frac{5}{4} & 0 & 6 & 0 & ; & 0 \\ 0 & 0 & -1 & 0 & 12 & ; & -1 \\ 1 & 0 & 0 & 0 & 0 & ; & r \\ 0 & 1 & 0 & 0 & 0 & ; & r \end{pmatrix}$$

where the last two rows indicate the augmented matrix of the conditions $[\mathbf{U}_i;\underline{\lambda_i}(r)]$*.Solving this system, we get*

$$\underline{y_0}(r)=r,\quad \underline{y_1}(r)=r,\quad \underline{y_2}(r)=1+\frac{7}{8}r,\quad \underline{y_3}(r)=\frac{5}{24}r,\quad \underline{y_4}(r)=\frac{7}{96}r$$

if $r=0.8$*, then*

$$\underline{y_0}=\frac{4}{5},\quad \underline{y_1}=\frac{4}{5},\quad \underline{y_2}=\frac{17}{10},\quad \underline{y_3}=\frac{1}{6},\quad \underline{y_4}=\frac{7}{120}$$

In a similar way, we can solve the initial value problem (4.25) then , we have

$$\overline{y}_0(r)=2-r,\ \ \overline{y}_1(r)=3-2r,\quad \overline{y}_2(r)=\frac{11}{4}-\frac{7}{8}r,\quad \overline{y}_3(r)=\frac{5}{8}-\frac{5}{12}r,\quad \overline{y}_4(r)=\frac{7}{48}-\frac{7}{96}r$$

if $r=0.8$*, then*

$$\overline{y}_0=\frac{6}{5},\quad \overline{y}_1=\frac{7}{5},\quad \overline{y}_2=\frac{41}{20},\quad \overline{y}_3=\frac{7}{24},\quad \overline{y}_4=\frac{7}{80}$$

we have

$$y_0(r)=[r,2-r],\quad y_1(r)=[r,3-2r],\quad y_2(r)=[1+\frac{7}{8}r,\frac{11}{4}-\frac{7}{8}r],$$

$$y_3(r)=[\frac{5}{24}r,\frac{5}{8}-\frac{5}{12}r],\quad y_4(r)=[\frac{7}{96}r,\frac{7}{48}-\frac{7}{96}r],$$

also, $\forall r\in[0,1]$ *:*

$$y(t,r)=[r,2-r]+[r,3-2r]t+[1+\frac{7}{8}r,\frac{11}{4}-\frac{7}{8}r]t^2+[\frac{5}{24}r,\frac{5}{8}-\frac{5}{12}r]t^3+[\frac{7}{96}r,\frac{7}{48}-\frac{7}{96}r]t^4$$

If $r=0.8$*, then*

$$y(t)=[\frac{4}{5},\frac{6}{5}]+[\frac{4}{5},\frac{7}{5}]t+[\frac{17}{10},\frac{41}{20}]t^2+[\frac{1}{6},\frac{7}{24}]t^3+[\frac{7}{120},\frac{7}{80}]t^4$$

Example 4.2. *Considering the fuzzy pantograph equation of third order*

$$y'''(t)=ty''(t)-y'(t)-y(\frac{t}{2})+t\cos(2t)+\cos(\frac{t}{2}),$$

$$\tilde{y}(0)=(r,2-r),\quad \tilde{y}'=(r,3-2r),\quad \tilde{y}''=(r,5-4r),\quad 0\le r\le 1$$

we have two initial value problems as following:

$$\begin{cases} \underline{y}'''(t) = t\underline{y}''(t) + \underline{y}'(t) + \underline{y}(\frac{t}{2}) + t\cos(2t) + \cos(\frac{t}{2}), \quad r \in [0,1] \\ \underline{y}(0) = r, \quad \underline{y}'(0) = r, \quad \underline{y}''(0) = r. \end{cases} \tag{4.26}$$

and

$$\begin{cases} \overline{y}'''(t) = t\overline{y}''(t) + \overline{y}'(t) + \overline{y}(\frac{t}{2}) + t\cos(2t) + \cos(\frac{t}{2}), \quad r \in [0,1] \\ \overline{y}(0) = 2 - r, \quad \overline{y}'(0) = 3 - 2r, \quad \overline{y}''(0) = 5 - 4r. \end{cases} \tag{4.27}$$

we solve the initial value problem (4.26) as following : The fundamental matrix equation of this problem is

$$(\mathbf{M}^3 - \mathbf{I}_1\mathbf{A}_0\mathbf{M}^2 - \mathbf{A}_2 - \mathbf{M})\underline{\mathbf{Y}} = \underline{\mathbf{F}}.$$

where $\mathbf{A}_0$ *and* $\mathbf{A}_2$ *are defined in relation (3.19) for* $\alpha_0 = 2, \beta_0 = 0$ *and* $\alpha_2 = 1/2, \beta_2 = 0$, *respectively.*

If we take $N = 6$ *and follow the Taylor series method in Section 3, the augmented matrix becomes*

$$[\widehat{\mathbf{W}}, \widehat{\mathbf{F}}] = \begin{pmatrix} -1 & -1 & 0 & 6 & 0 & 0 & 0 & ; & 1 \\ 0 & \frac{-1}{2} & -4 & 0 & 24 & 0 & 0 & ; & 1 \\ 0 & 0 & \frac{-1}{4} & -15 & 0 & 60 & 0 & ; & -\frac{1}{8} \\ 0 & 0 & 0 & \frac{-1}{8} & -52 & 0 & 120 & ; & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & r \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & r \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & ; & r \end{pmatrix}$$

we obtain the Taylor coefficient vector

$$\underline{\mathbf{Y}} = [\ r \quad r \quad 0.5r \quad 0.1667{+}0.3334r \quad 0.0416{+}0.1042r \quad 0.0395{+}0.0854r \quad 0.0016{+}0.0455r\ ].$$

we have

$$\begin{aligned} \underline{y}(t,r) &= r + rt + 0.5rt^2 + (0.1667 + 0.3334r)t^3 \\ &\quad + (0.0416 + 0.1042r)t^4 + (0.0395 + 0.0854r)t^5 + (0.0016 + 0.0455r)t^6 \end{aligned}$$

In a similar way, we can solve the initial value problem (4.27) then , we have

$$\overline{\mathbf{Y}} = [\ 2{-}r \quad 3{-}2r \quad 2.5{-}2r \quad 1{-}0.5r \quad 0.5208{-}0.375r \quad 0.2584{-}0.1334r \quad 0.2101{-}0.1630r\ ]$$

also

$$\begin{aligned} \overline{y}(t,r) &= (2 - r) + (3 - 2r)t + (2.55 - 2r)t^2 + (1 - 0.5r)t^3 \\ &\quad + (0.5208 - 0.375r)t^4 + (0.2584 - 0.1334r)t^5 + (0.2101 - 0.1630r)t^6 \end{aligned}$$

The following remark shows when above-mentioned case has a fuzzy approximated solution.

Remark 4.1. The sufficient conditions for $(\underline{y}(t,r), \overline{y}(t,r))$ to define the parametric form of a fuzzy number are as follows:

- $\sum_{n=0}^{N} \underline{y_n} t^n$ is a bounded left continuous nondecreasing function over $t \in T$.
- $\sum_{n=0}^{N} \overline{y_n} t^n$ is a bounded left continuous nonincreasing function over $t \in T$.
- $\sum_{n=0}^{N} \underline{y_n} t^n \leq \sum_{n=0}^{N} \overline{y_n} t^n, 0 \leq r \leq 1$.

5 Conclusions

In this paper a numerical method similar to the collocation method, based on a Taylor series ,with a positive basis $\{t^n\}_{n=0}^{\infty}$ for solving the fuzzy pantograph equations was discussed.fuzzy approximate solutions were obtained by solving an extended system of generalized pantograph equations with linear functional argument.

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