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Fusion frames in Hilbert modules over pro-C*-algebras

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Abstract

In this paper, we introduce fusion frames in Hilbert modules over pro-C*-algebras. Also, we give some useful results about these frames.

Keywords : Pro-C*-algebra; Hilbert modules; Bounded module maps; Fusion frames.

1 Introduction

Frames for Hilbert spaces were formally defined by Duffin and Schaeffer [5] in 1952 to study some deep problems in nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [6]. Frames have many nice properties which make them very useful in the characterization of function spaces, signal processing and many other fields. Many generalizations of frames were introduced, e.g. frames of subspaces [1] and g-frames [17]. Meanwhile, Frank and Larson presented a general approach to the frame theory in Hilbert C*-modules in [7]. Finally, A. and B. Khosravi [11] generalized the concept of fusion frames and g-frames to Hilbert C*-modules.

In this note, we generalize the theory of fusion frames to Hilbert modules over a pro-C*-algebra and give some useful results.

We refer the reader to [8],[15] for pro-C*-algebras and [13],[15] for Hilbert modules over pro-C*-algebras. We also refer the reader to [1],[3],[7],[12],[17] for more information about the theory of frames and its generalizations.

In section 2, we give some definitions and basic properties of pro-C*-algebras and we state some notations. In section 3, we recall the basic definitions and some notations about Hilbert pro-C*-modules. We also give some basic properties of such spaces which we will use in sequel. Finally, in section 4, we introduce fusion frames in Hilbert pro-C*-modules. We also generalize some of the results about fusion frames in Hilbert C*-modules to pro-C*-algebra case.

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2 Pro-C*-algebras

In the follwoing, we briefly recall some definitions and basic properties of pro-C*-algebras .

A pro-C*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C*-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 iff $p(a_\lambda) \to 0$ for any continuous C*-seminorm p on A and we have :

(a) $p(ab) \leq p(a)p(b)$

(b) $p(a^*a) = (p(a))^2$

for all C*-seminorm p on A and $a, b \in A$.

If the topology of a pro-C*-algebra is determined by only countably many C*-seminormes, then it is called a σ-C*-algebra.

Let A be a unital pro-C*-algebra with unit 1_A and let $a \in A$. Then, the spectrum $\mathrm{sp}(a)$ of $a \in A$ is the set $\{\lambda \in \mathbb{C} \ : \ \lambda 1_A - a$ is not invertible$\}$. If A is not unital, then the spectrum is taken with respect to its unitization $\tilde{A}$.

If A^+ denotes the set of all positive elements of A, then A^+ is a closed convex cone such that $A^+ \cap (-A^+) = 0$. We denote by $S(A)$, the set of all continuous C*-seminorms on A. For $p \in S(A)$, we put $\ker(p) = \{a \in A : p(a) = 0\}$; which is a closed ideal in A. For each $p \in S(A)$, $A_p = A/\ker(p)$ is a C*-algebra in the norm induced by p which defined as ;

$$\|a + \ker(p)\|_{A_p} = p(a) \quad , \quad p \in S(A) \ .$$

We have $A = \varprojlim_p A_p$ (see [15]) .

The canonical map from A onto A_p for $p \in S(A)$, will be denoted by π_p and the image of $a \in A$ under π_p will be denoted by a_p. Hence $l^2(A_p)$ is a Hilbert A_p-module (see [9]), with the norm, defined as :

$$\|(\pi_p(a_i))_{i\in\mathbb{N}}\|_p = [\, p(\textstyle\sum_{i\in\mathbb{N}} a_i a_i^*)\,]^{1/2} \quad , \quad p \in S(A) \ , \ (\pi_p(a_i))_{i\in\mathbb{N}} \in l^2(A_p) \ .$$

Example 2.1 Every C*-algebra is a pro-C*-algebra .

Example 2.2 A closed $*$-subalgebra of a pro-C*-algebra is a pro-C*-algebra.

Example 2.3([15]) Let X be a locally compact Hausdorff space and let $A = C(X)$ denotes all continuous complex-valued functions on X with the topology of uniform convergence on compact subsets of X. Then A is a pro-C*-algebra .

Example 2.4([15]) A product of C*-algebras with the product topology is a pro-C*-algebra .

Notation 2.1 $a \geq 0$ denotes $a \in A^+$ and $a \leq b$ denotes $a - b \geq 0$.

Proposition 2.1 ([8]) *Let A be a unital pro-C*-algebra with an identity 1_A. Then for any $p \in S(A)$, we have :*

(1) $p(a) = p(a^*)$ *for all* $a \in A$
(2) $p(1_A) = 1$
(3) *If* $a, b \in A^+$ *and* $a \leq b$ *, then* $p(a) \leq p(b)$
(4) $a \leq b$ *iff* $a_p \leq b_p$
(5) *If* $1_A \leq b$ *, then* b *is invertible and* $b^{-1} \leq 1_A$
(6) *If* $a, b \in A^+$ *are invertible and* $0 \leq a \leq b$ *, then* $0 \leq b^{-1} \leq a^{-1}$
(7) *If* $a, b, c \in A$ *and* $a \leq b$ *, then* $c^*ac \leq c^*bc$
(8) *If* $a, b \in A^+$ *and* $a^2 \leq b^2$ *, then* $0 \leq a \leq b$

Proposition 2.2 *If* $\sum_{i=1}^{\infty} a_i$ *is a convergent series in a pro-C*-algebra* A *and* $a_i \geq 0$ *for* $i \in \mathbb{N}$, *then it converges unconditionally.*

Proof. For $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} a_i$. Then for any $\varepsilon \geq 0$ and $p \in S(A)$, there is a positive integer N_p such that for $m, n \geq N_p$;

$$p(\textstyle\sum_{i=m}^{n} a_i) \leq \varepsilon \ .$$

For a permutation σ of $\mathbb{N}$, we define , $S'_n = \sum_{i=1}^{n} a_{\sigma(i)}$. Let $k \in \mathbb{N}$ such that

$$\{1, 2, ..., N_p\} \subseteq \{\sigma(1), \sigma(2), ..., \sigma(k)\} \ .$$

Then $S'_n - S_n$ for $n \geq k$, do not have any a_i for $1 \leq i \leq N_p$. Hence for $n \geq k$,

$$p(S'_n - S_n) \leq \varepsilon \ .$$

Thus for $S = \sum_{i=1}^{\infty} a_i$ and $n \geq k$, we have ,

$$p(S'_n - S) \leq p(S'_n - S_n) + p(S_n - S) \leq 2\varepsilon \ .$$

This means that $\lim_{n\to\infty} S'_n = S$. □

3 Hilbert pro-C*-modules

In this section, we recall some of the basic definitions and properties of Hilbert modules over pro-C*-algebras from [15].

Definition 3.1 A pre-Hilbert module over pro-C*-algebra A is a complex vector space E which is also a left A-module compatible with the complex algebra structure, equipped with an A-valued inner product $\langle .,. \rangle : E \times E \to A$ which is $\mathbb{C}$-and A-linear in its first variable and satisfies the following conditions:

(i) $\langle x, y \rangle^* = \langle y, x \rangle$
(ii) $\langle x, x \rangle \geq 0$
(iii) $\langle x, x \rangle = 0$ iff $x = 0$

for every $x, y \in E$. We say that E is a Hilbert A-module (or Hilbert pro-C*-module over A) if E is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_E(x) = \sqrt{p(\langle x, x \rangle)} \qquad x \in E \ , \ p \in S(A) \ .$$

Let E be a pre-Hilbert A-module. By Lemma 2.1 of [19] for every $p \in S(A)$ and for all $x, y \in E$, the following Cauchy-Bunyakovskii inequality holds

$$p(\langle x, y\rangle)^2 \leq p(\langle x, x\rangle)p(\langle y, y\rangle) .$$

Consequently, for each $p \in S(A)$, we have :

$$\bar{p}_E(ax) \leq p(a)\bar{p}_E(x) \quad a \in A \ , \ x \in E.$$

If E is a Hilbert A-module and $p \in S(A)$, then $\ker(\bar{p}_E) = \{x \in E \ : \ p(\langle x, x\rangle) = 0\}$ is a closed submodule of E and $E_p = E/\ker(\bar{p}_E)$ is a Hilbert A_p-module with scalar product

$$a_p.(x + \ker(\bar{p}_E)) = ax + \ker(\bar{p}_E) \qquad a \in A \ , \ x \in E$$

and inner product

$$\langle \ x + \ker(\bar{p}_E) \ , \ y + \ker(\bar{p}_E) \ \rangle = \langle x, y\rangle_p \quad x, y \in E \ .$$

By Proposition 4.4 of [15], we have $E \cong \varprojlim_p E_p$.

Example 3.1 If A is a pro-C*-algebra, then it is a Hilbert A-module with respect to the inner product defined by :

$$\langle a, b\rangle = ab^* \qquad a, b \in A \ .$$

Example 3.2 (See[15], Remark 4.8) Let $l^2(A)$ be the set of all sequences $(a_n)_{n\in\mathbb{N}}$ of elements of a pro-C*-algebra A such that the series $\sum_{i=1}^{\infty} a_i {a_i}^*$ is convergent in A. Then $l^2(A)$ is a Hilbert module over A with respect to the pointwise operations and inner product defined by :

$$\langle \ (a_i)_{i\in\mathbb{N}} \ , \ (b_i)_{i\in\mathbb{N}} \ \rangle = \sum_{i=1}^{\infty} a_i {b_i}^* \ .$$

Example 3.3 If $\{M_i\}_{i\in J}$ is a finite family of Hilbert A-modules. Then the direct sum $\bigoplus_{i\in J} M_i$ is a Hilbert A-module with pointwise operations and A-valued inner product $\langle x, y\rangle = \sum_{i\in J}\langle x_i, y_i\rangle$, where $x = (x_i)_{i\in J}$ and $y = (y_i)_{i\in J}$ are in $\bigoplus_{i\in J} M_i$.

Example 3.4 Let E_i for $i \in \mathbb{N}$, be a Hilbert A-module with the topology induced by the family of continuous seminorms $\{\bar{p}_i\}_{p\in S(A)}$ defined as :

$$\bar{p}_i(x) = \sqrt{p(\langle x, x\rangle)} \quad , \quad x \in E_i \ .$$

Direct sum of $\{E_i\}_{i\in\mathbb{N}}$ is defined as follows :

$$\bigoplus_{i\in\mathbb{N}} E_i = \{(x_i)_{i\in\mathbb{N}} \ : \ x_i \in E_i \ , \ \sum_{i=1}^{\infty}\langle x_i, x_i\rangle \ is\ convergent\ in\ A\} \ .$$

It has been shown (see [12], Example 3.2.3) that the direct sum $\bigoplus_{i\in\mathbb{N}} E_i$ is a Hilbert A-module with A-valued inner product $\langle x, y\rangle = \sum_{i=1}^{\infty}\langle x_i, y_i\rangle$, where $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{N}}$ are in $\bigoplus_{i\in\mathbb{N}} E_i$, pointwise operations and a topology determined by the family of seminorms

$$\bar{p}(x) = \sqrt{p(\langle x, x\rangle)} \quad , \quad x \in \bigoplus_{i\in\mathbb{N}} E_i \ , \ p \in S(A) \ .$$

The direct sum of a countable copies of a Hilbert module E is denoted by $l^2(E)$.

We recall that an element a in A (x in E) is bounded, if

$$\|a\|_\infty = \sup\{p(a) \ ; \ p \in S(A)\} < \infty \ ,$$

$$(\|x\|_\infty = \sup\{\bar{p}_E(x) \ ; \ p \in S(A)\} < \infty) \ .$$

The set of all bounded elements in A (in E) will be denoted by $b(A)$ ($b(E)$). We know that $b(A)$ is a C*-algebra in the C*-norm $\|.\|_\infty$ and $b(E)$ is a Hilbert $b(A)$-module.([15], Prop. 1.11 and [19], Theorem 2.1)

Let $M \subset E$ be a closed submodule of a Hilbert A-module E and let

$$M^\perp = \{y \in E \ : \ \langle x, y\rangle = 0 \quad for\ all\ x \in M\} \ .$$

Note that the inner product in a Hilbert modules is separately continuous, hence $M^\perp$ is a closed submodule of the Hilbert A-module E. Also, a closed submodule M in a Hilbert A-module E is called orthogonally complementable if $E = M \oplus M^\perp$. A closed submodule M in a Hilbert A-module E is called topologically complementable if there exists a closed submodule N in E such that $M \oplus N = E$, $N \cap M = \{0\}$.

Let A be a pro-C*-algebra and let E and F be two Hilbert A-modules. An A-module map $T : E \to F$ is said to bounded if for each $p \in S(A)$, there is $C_p > 0$ such that :

$$\bar{p}_F(Tx) \leq C_p.\bar{p}_E(x) \qquad (x \in E) \ ,$$

where $\bar{p}_E$, respectively $\bar{p}_F$, are continuous seminorms on E, respectively F. A bounded A-module map from E to F is called an operator from E to F. We denote the set of all operators from E to F by $Hom_A(E, F)$, and we set $Hom_A(E, E) = End_A(E)$.

Let $T \in Hom_A(E, F)$. We say T is adjointable if there exists an operator $T^* \in Hom_A(F, E)$ such that :

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in E$, $y \in F$.
We denote by $Hom^*_A(E,F)$, the set of all adjointable operators from E to F and $End^*_A(E) = Hom^*_A(E,E)$.

By a little modification in the proof of Lemma 3.2 of [19], we have the following result :

Proposition 3.1 *Let $T : E \to F$ and $T^* : F \to E$ be two maps such that the equality*

$$\langle x, T^*y\rangle = \langle Tx, y\rangle$$

holds for all $x \in E, y \in F$. Then $T \in Hom^*_A(E,F)$.

It is easy to see that for any $p \in S(A)$, the map defined by

$$\hat{p}_{E,F}(T) = \sup\{\ \bar{p}_F(Tx) \ : \ x \in E\ ,\ \bar{p}_E(x) \leq 1\}\ ,\ T \in Hom_A(E,F)\ ,$$

is a seminorm on $Hom_A(E,F)$. Moreover $Hom_A(E,F)$ with the topology determined by the family of seminorms $\{\hat{p}_{E,F}\}_{p\in S(A)}$ is a complete locally convex space ([10], Prop. 3.1) . Moreover using Lemma 2.2 of [19], for each $y \in F$ and $p \in S(A)$, we can write

$$\begin{aligned} \bar{p}_E(T^*(y)) &= \sup\{p\langle T^*(y), x\rangle : \bar{p}_E(x) \leq 1\} \\ &= \sup\{p\langle T(x), y\rangle : \bar{p}_E(x) \leq 1\} \\ &\leq \sup\{\bar{p}_F T(x) : \bar{p}_E(x) \leq 1\}\bar{p}_F(y) \\ &= \hat{p}(T)\bar{p}_F(y)\ . \end{aligned}$$

Thus for each $p \in S(A)$, we have $\hat{p}_{F,E}(T^*) \leq \hat{p}_{E,F}(T)$ and since $T^{**} = T$, by replacing T with T^*, for each $p \in S(A)$, we obtain:

$$\hat{p}_{F,E}(T^*) = \hat{p}_{E,F}(T). \tag{3.1}$$

By Proposition 4.7 of [15], we have the canonical isomorphism

$$Hom_A(E,F) \cong \varprojlim_p Hom_{A_p}(E_p, F_p).$$

Consequently, $End^*_A(E)$ is a pro-C*-algebra for any Hilbert A-module E and its topology is obtained by $\{\hat{p}_E\}_{p\in S(A)}$ ([19]). By Prop. 3.2 of [19], T is a positive element of $End^*_A(E)$ if and only if $\langle Tx, x\rangle \geq 0$ for any $x \in E$.

Lemma 3.1 *Let X be a Hilbert module over C^*-algebra B, $S \in End^*_B(X)$ and $S \geq 0$, i.e. this element is positive in C^*-algebra $End^*_B(X)$. Then for each $x \in X$,*

$$\langle Sx, x\rangle \leq \|S\|\langle x, x\rangle\ .$$

Proof. Since S is a positive element in $End^*_B(X)$, we have, $S \leq \|S\|I$, such that I is the identity element in $End^*_B(X)$. Hence $S - \|S\|I \geq 0$, and then

$$\langle(\|S\|I - S)x\ ,\ x\rangle \geq 0 \quad , \quad \forall x \in X\ .$$

Therefore, we have :

$$\langle Sx, x\rangle \leq \|S\|\langle x, x\rangle,$$

for all $x \in X$. □

Remark 3.1 Note that if $T \in End^*_B(X)$, then T^*T is a positive element in $End^*_B(X)$. Thus, we can write :

$$\langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*T\|\langle x, x\rangle = \|T\|^2\langle x, x\rangle,$$

for all $x \in X$.

Definition 3.2 Let E and F be two Hilbert modules over pro-C*-algebra A. Then the operator $T : E \to F$ is called uniformly bounded, if there exists $C > 0$ such that for each $p \in S(A)$,

$$\bar{p}_F(Tx) \leq C\bar{p}_E(x) \quad , \quad \forall x \in E. \tag{3.2}$$

The number C in (2) is called an upper bound for T and we set:

$$\|T\|_\infty = \inf\{C\ :\ \text{C is an upper bound for T}\}.$$

Clearly, in this case we have:

$$\hat{p}(T) \leq \|T\|_\infty \quad , \quad \forall p \in S(A).$$

Proposition 3.2 *Let E be a Hilbert module over pro-C^*-algebra A and T be an invertible element in $End^*_A(E)$ such that both are uniformly bounded. Then for each $x \in E$,*

$$\|T^{-1}\|^{-2}_\infty\langle x, x\rangle \leq \langle Tx, Tx\rangle \leq \|T\|^2_\infty\langle x, x\rangle.$$

Proof. Recall that for each $p \in S(A)$, the space $End^*_{A_p}(E_p)$ is a C*-algebra and T_p belong to this space with the norm defined by:

$$\|T_p\|_p = \hat{p}_E(T).$$

Therefore by Remark 3.1, for each $p \in S(A)$ and $x \in E$,

$$\begin{aligned}\langle Tx, Tx\rangle_p &= \langle (Tx)_p, (Tx)_p\rangle \\ &= \langle T_p(x_p), T_p(x_p)\rangle \\ &\leq \|T_p\|_p^2 \langle x_p, x_p\rangle \\ &= \hat{p}_E(T)^2 \langle x, x\rangle_p \\ &\leq \|T\|_\infty^2 \langle x, x\rangle_p.\end{aligned}$$

By Remark 2.2 of [8], we have:

$$\langle Tx, Tx\rangle \leq \|T\|_\infty^2 \langle x, x\rangle \quad , \quad \forall x \in E. \quad (3.3)$$

On the other hand, by replacing T^{-1} and y instead of T and x in (3), we obtain:

$$\langle T^{-1}y, T^{-1}y\rangle \leq \|T^{-1}\|_\infty^2 \langle y, y\rangle.$$

Let $x \in E$ such that $Tx = y$. Then, we can conclude:

$$\langle x, x\rangle \leq \|T^{-1}\|_\infty^2 \langle Tx, Tx\rangle.$$

because T is an invertible operator, it can be concluded that: $\|T^{-1}\|_\infty > 0$ and hence:

$$\|T^{-1}\|_\infty^{-2} \langle x, x\rangle \leq \langle Tx, Tx\rangle \ \forall x \in E.$$

Let N and M be closed submodules in a Hilbert module E such that $E = M \oplus N$. We denote by P_M, the projection onto M along N.

Proposition 3.3 *Let M be an orthogonally complemented submodule of a Hilbert A-module E. Then $P_M \in End_A^*(E)$.*

Proof. Let $x, y \in E$. Then, there exist unique elements $a, b \in M$ and $a', b' \in M^\perp$ such that, $x = a + a'$, $y = b + b'$. Therefore

$$\langle P_M(x) , y\rangle = \langle a , b + b'\rangle = \langle a , b\rangle .$$

On the other hand,

$$\langle x , P_M(y)\rangle = \langle a + a' , b\rangle = \langle a , b\rangle .$$

By Lemma 3.2 of [19], we have $P_M = {P_M}^*$. Using Prop 3.1, we conclude $P_M \in End_A^*(E)$. □

Proposition 3.4 *Let M be an orthogonally complemented submodule of a Hilbert A-module E and let $T \in End_A^*(E)$ be an ivertible operator such that $T^*TM \subseteq M$. Then we have:*

$$T(M^\perp) = (TM)^\perp \quad , \quad P_{TM} = TP_M T^{-1} .$$

Proof. Let $u \in M$ and $v \in M^\perp$. Since $T^*Tu \in M$, then we have $\langle Tu, Tv\rangle = \langle T^*Tu, v\rangle = 0$. Thus $T(M^\perp) \subseteq (TM)^\perp$. On the other hand if $y \in (TM)^\perp$, then there exists $x \in E$ such that $y = Tx$. Let $x = m + n$ for some $m \in M$ and $n \in M^\perp$, then we have

$$\langle y, Tm\rangle = \langle Tx, Tm\rangle = \langle Tm + Tn, Tm\rangle = \langle Tm, Tm\rangle + \langle Tn, Tm\rangle = 0.$$

Since $\langle Tn, Tm\rangle = 0$, we have $\langle Tm, Tm\rangle = 0$ and then $Tm = 0$. Thus $y = Tn$, and we have $(TM)^\perp \subseteq T(M^\perp)$. Let $x \in E$. Since $E = M + M^\perp$, so we have $x = u + v$, $u \in M$, $v \in M^\perp$. Hence, $Tx = Tu + Tv$. On the other hand, we have, $TM^\perp = (TM)^\perp$. Thus $Tu \in TM$ and $Tv \in (TM)^\perp$. Therefore :

$$P_{TM}(Tx) = Tu \quad , \quad TP_M(x) = Tu .$$

This completes the proof . □

Remark 3.2 Let A be a σ-C*-algebra and T a continuous invertible operator on a Hillbert A-module E. Then, $T(M)$ is a closed submodule of E for any closed submodule $M \subseteq E$.(See [4],[16])

4 Fusion frames in Hilbert pro-C*-modules

In this section, we will assume that A is a unital pro-C*-algebra, X is a Hilbert A-module and I is a finite or countable index set. Furthermore, if M is an orthogonally complemented submodule of X, we let P_M denote the orthogonal projection of X onto M. By Prop. 3.3, we have, $P_M \in End_A^*(X)$.

Definition 4.1 A sequence $\{x_i \ : \ i \in I\}$ of elements in a Hilbert A-module X is said to be a *frame* if there are real constants $C, D > 0$ such that:

$$C\langle x, x\rangle \leq \sum_{i\in I} \langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle \quad (4.4)$$

for any $x \in X$. The constants C and D are called lower and upper frame bounds for the frame, respectively. If $C = D = \lambda$, the frame is called a λ-*tight frame*. If $C = D = 1$, the frame is called a *Parseval frame*. We say that $\{x_i : i \in I\}$ is a *frame sequence* if it is a frame for the closure of A-linear hull of $\{x_i : i \in I\}$. If, in (4) we only require to have the upper bound, then $\{x_i : i \in I\}$

is called a *Bessel sequence* with Bessel bound D.

Definition 4.2 Let $\{v_i : i \in I\}$ be a family of weights in A, i.e., each v_i is a positive invertible element from the center of A, and let $\{M_i : i \in I\}$ be a family of orthogonally complemented submodules of X. Then $\{(M_i, v_i) : i \in I\}$ is a *fusion frame* if there exist real constants $C, D > 0$ such that:

$$C\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle \leq D\langle x, x\rangle \tag{4.5}$$

for any $x \in X$. We call C and D the lower and upper bounds of the fusion frame. If $C = D = \lambda$, the family $\{(M_i, v_i) : i \in I\}$ is called a λ-*tight* fusion frame and if $C = D = 1$, it is called a *Parseval* fusion frame. If in (5), we only have the upper bound, then $\{(M_i, v_i) : i \in I\}$ is called a Bessel fusion sequence with Bessel bound D. Let X be a Hilbert A-module and $\{v_i : i \in I\}$ be a family of weights in A. Let for each $i \in I$, M_i be an orthogonally complemented submodule of X and let $\{x_{ij} : j \in J_i\}$ be a frame for M_i with bounds C_i and D_i. Suppose that $0 < C = \inf_i C_i \leq D = \sup_i D_i < \infty$. Then, the following conditions are equivalent.

(a) $\{v_i x_{ij} : i \in I, j \in J_i\}$ is a frame for X.
(b) $\{(M_i, v_i) : i \in I\}$ is a fusion frame for X.

Proof. Since for each $i \in I$, $\{x_{ij} : j \in J_i\}$ is a frame for M_i with frame bounds C_i and D_i, we have :

$$C_i\langle x, x\rangle \leq \sum_{j\in J_i} \langle x, x_{ij}\rangle\langle x_{ij}, x\rangle \leq D_i\langle x, x\rangle \quad \text{for all } x \in M_i \ .$$

Therefore, for all $x \in X$, we obtain :

$$\begin{aligned} C\sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle &\leq \sum_{i\in I} C_i v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle \\ &= \sum_{i\in I} C_i \langle v_i P_{M_i}(x), v_i P_{M_i}(x)\rangle \\ &\leq \sum_{i\in I}\sum_{j\in J_i} \langle v_i P_{M_i}(x), x_{ij}\rangle\langle x_{ij}, v_i P_{M_i}(x)\rangle \\ &= \sum_{i\in I} v_i^2 \sum_{j\in J_i} \langle P_{M_i}(x), x_{ij}\rangle\langle x_{ij}, P_{M_i}(x)\rangle \\ &\leq \sum_{i\in I} v_i^2 D_i \langle P_{M_i}(x), P_{M_i}(x)\rangle \\ &\leq D\sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle \ . \end{aligned}$$

So, we can briefly write :

$$C\sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle \leq \sum_{i\in I}\sum_{j\in J_i} \langle x, v_i x_{ij}\rangle\langle v_i x_{ij}, x\rangle \leq D\sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle \ .$$

This shows that if $\{v_i x_{ij} : i \in I, j \in J_i\}$ is a frame for X with frame bounds A and B, then, $\{(M_i, v_i) : i \in I\}$ form a fusion frame for X with frame bounds $\frac{A}{D}$ and $\frac{B}{C}$. Moreover, if $\{(M_i, v_i) : i \in I\}$ is a fusion frame for X with frame bounds A and B, the above calculation implies that $\{v_i x_{ij} : i \in I, j \in J_i\}$ is a frame for X with frame bounds AC and BD. Thus (a) $\Leftrightarrow$ (b) . □

Definition 4.3 Let X be a Hilbert A-module. A squence $\{x_i : i \in I\}$ in X is called *complete*, if the A-linear hull of $\{x_i : i \in I\}$ is dense in X. Also, a family of closed submodules $\{M_i : i \in I\}$ of X is called *complete* if the A-linear hull of $\cup_{i\in I} M_i$ is dense in X.

Lemma 4.1 *Let $\{(M_i, v_i) : i \in I\}$ be a fusion frame for Hilbert A-module X such that the closed A-linear hull of $\{M_i : i \in I\}$ is orthogonally complemented. Then $\{M_i : i \in I\}$ is complete .*

Proof. Let M be the closed A-linear hull of $\{M_i : i \in I\}$. Then $X = M \oplus M^{\perp}$. Assume that $\{M_i : i \in I\}$ is not complete. Then there exists some $x \in X$, $x \neq 0$ with $x \perp M$. Hence for each $i \in I$, $x \perp M_i$. It follows $\sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle = 0$, hence $\{(M_i, v_i) : i \in I\}$ is not a fusion frame for X . □

The following lemma follows immediately from the definitions.

Lemma 4.2 *Let $\{M_i : i \in I\}$ be a family of closed submodules of a Hilbert A-module X such that the closed A-linear hull of $\{M_i : i \in I\}$ is orthogonally complemented and for each $i \in I$, let $\{x_{ij} : j \in J_i\}$ be a frame for M_i. Then $\{M_i : i \in I\}$ is complete if and only if $\{x_{ij} : i \in I, j \in J_i\}$ is complete .*

Lemma 4.3 *Let X be a Hilbert A-module and $\{(M_i, v_i) : i \in I\}$ be a fusion frame for X with frame bounds C and D. Let M be an orthogonally complemented submodule of X. Then $\{(M_i \cap M, v_i) : i \in I\}$ is a fusion frame for M with frame bounds C and D.*

Proof. For all $i \in I$ and $x \in M$, we have:

$$P_{M_i \cap M}(x) = P_{M_i}(P_M(x)) = P_{M_i}(x) \ .$$

Hence, for all $x \in M$, we can write:

$$\sum_{i\in I} v_i^2 \langle P_{M_i\cap M}(x), P_{M_i\cap M}(x)\rangle = \sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle \ .$$

Now, the result follows. □

Lemma 4.4 *Let $\{(M_i, v_i) : i \in I\}$ be a Bessel fusion sequence for a Hilbert A-module X with Bessel bound D. Then, for each $x = (x_i)_{i\in I}$ in the Hilbert A-module $M = \bigoplus_{i\in I} M_i$, the series $\sum_{i\in I} v_i x_i$ converges unconditionally and for each $p \in S(A)$, we have:*

$$\bar{p}_X(\sum_{i\in I} v_i x_i) \le \sqrt{D}\bar{p}_M(x) \ .$$

Proof. Let $x = (x_i)_{i\in I}$ be an element of $M = \bigoplus_{i\in I} M_i$ and J be a finite subset of I. Let $y = \sum_{i\in J} v_i x_i$. Since the projection P_{M_i} is self adjoint, we have

$$\langle y, y\rangle = \langle y, \sum_{i\in J} v_i x_i\rangle = \sum_{i\in J} v_i \langle y, x_i\rangle = \sum_{i\in J} \langle v_i P_{M_i}(y), x_i\rangle \ .$$

Using Cauchy-Bunyakovskii inequality, we have

$$p(\langle y, y\rangle)^2 = p(\sum_{i\in J} \langle v_i P_{M_i}(y), x_i\rangle)^2 = p(\langle \{v_i P_{M_i}(y)\}_{i\in J}, \{x_i\}_{i\in J}\rangle)^2$$
$$\le p(\langle \{v_i P_{M_i}(y)\}_{i\in J}, \{v_i P_{M_i}(y)\}_{i\in J}\rangle) p(\langle \{x_i\}_{i\in J}, \{x_i\}_{i\in j}\rangle)$$
$$= p(\sum_{i\in J} v_i^2 \langle P_{M_i}(y), P_{M_i}(y)\rangle) p(\sum_{i\in J} \langle x_i, x_i\rangle)$$

for all $p \in S(A)$. Since $\{(M_i, v_i) : i \in I\}$ is a Bessel fusion sequence with Bessel bound D, we can write

$$\sum_{i\in J} v_i^2 \langle P_{M_i}(y), P_{M_i}(y)\rangle \le \sum_{i\in I} v_i^2 \langle P_{M_i}(y), P_{M_i}(y)\rangle \le D\langle y, y\rangle \ .$$

Hence, for all $p \in S(A)$, we have

$$p(\langle y, y\rangle)^2 \le D.p(\langle y, y\rangle) p(\sum_{i\in J} \langle x_i, x_i\rangle) \ .$$

Using $\bar{p}_X(y) = \sqrt{p(\langle y, y\rangle)}$, we obtain

$$\bar{p}_X(\sum_{i\in J} v_i x_i) = \bar{p}_X(y) \le \sqrt{D}.p(\sum_{i\in J} \langle x_i, x_i\rangle)^{1/2} \tag{4.6}$$

$p \in S(A)$.

Since $x = (x_i)_{i\in I}$ is an element of $\bigoplus_{i\in I} M_i$, the series $\sum_{i\in I} \langle x_i, x_i\rangle$ is convergent in A and by Prop. 2.2, converges unconditionally. So, by (6), the series $\sum_{i\in I} v_i x_i$ converges unconditionally and we have

$$\bar{p}_X(\sum_{i\in I} v_i x_i) \le \sqrt{D} p(\sum_{i\in I} \langle x_i, x_i\rangle)^{1/2} = \sqrt{D}\bar{p}_M(x)$$

for all $p \in S(A)$. □ Let $\{(M_i, v_i) : i \in I\}$ be a fusion frame for a Hilbert A-module X with frame bounds C and D. Then, the corresponding frame transform $\theta : X \to l^2(X)$ defined by $\theta(x) = (v_i P_{M_i}(x))_{i\in I}$ for $x \in X$, is an isomorphic imbedding with closed range, and its adjoint operator $\theta^* : l^2(X) \to X$ is bounded and defined by $\theta^*(y) = \sum_{i\in I} v_i P_{M_i}(y_i)$ for $y = (y_i)_{i\in I}$ in $l^2(X)$. **Proof.** Since $\{(M_i, v_i) : i \in I\}$ is a fusion frame, for each $x \in X$, we have:

$$C\langle x, x\rangle \le \sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle \le D\langle x, x\rangle \ .$$

Thus, the frame transform is well-defined and by Prop. 2.1, for each $p \in S(A)$ and $x \in X$, we can write

$$Cp\langle x, x\rangle \le p\langle \theta(x), \theta(x)\rangle \le Dp\langle x, x\rangle \ .$$

Hence, for each $p \in S(A)$ and $x \in X$, we obtain

$$\sqrt{C}\bar{p}_X(x) \le \bar{p}_{l^2(X)}(\theta(x)) \le \sqrt{D}\bar{p}_X(x), \tag{4.7}$$

where $\bar{p}_X$ and $\bar{p}_{l^2(X)}$, are continuous seminorms on X and $l^2(X)$, respectively. Thus, θ is bounded by the second inequality of (7) and is injective by the first inequality and Lemma 2.2 of [19]. Now, we show that θ has closed range. Let $\{\theta(x_n)\}$ be a sequence in Hilbert A-module $l^2(X)$ such that converges to an element y. Thus, $\{\theta(x_n)\}$ is a Cauchy sequence and by definition, for each $\varepsilon \ge 0$ and $p \in S(A)$, there exists positive number N such that for $m, n \ge N$, we have

$$\bar{p}_{l^2(X)}(\theta(x_n) - \theta(x_m)) < \sqrt{C}\varepsilon \ .$$

Using the first inequality of (4.4), we can write

$$\sqrt{C}\bar{p}_X(x_n - x_m) \leq \bar{p}_{l^2(X)}(\theta(x_n) - \theta(x_m)) < \sqrt{C}\varepsilon$$

for $m, n \geq N$. Hence, $\{x_n\}$ is a Cauchy sequence in Hilbert A-module X and therefore converges to an element $x \in X$. By the second inequality of (7), for each $p \in S(A)$, we have

$$\bar{p}_{l^2(X)}(\theta(x_n - x)) \leq \sqrt{D}\bar{p}_X(x_n - x) \to 0 \ .$$

Thus, the sequence $\{\theta(x_n)\}$ converges to $\theta(x)$ in Hilbert A-module $l^2(X)$. This means that $\theta(x) = y$. Therefore $\theta : X \to l^2(X)$ is an isomorphic imbedding with closed range. Now, for each $y = (y_i)_{i\in I}$ in $l^2(X)$ define:

$$\theta^* : l^2(X) \to X \quad , \quad \theta^*(y) = \textstyle\sum_{i\in I} v_i P_{M_i}(y_i)$$

By Proposition 2.2, the series $\sum_{i\in I}\langle y_i, y_i\rangle$ converges unconditionally. Moreover, we have

$$\textstyle\sum_{i\in I}\langle P_{M_i}(y_i), P_{M_i}(y_i)\rangle \leq \sum_{i\in I}\langle y_i, y_i\rangle \ .$$

Hence, $\{P_{M_i}(y_i)\}_{i\in I}$ is in $\bigoplus_{i\in I} M_i$. Thus, by Lemma 4.4, $\sum_{i\in I} v_i P_{M_i}(y_i)$ converges unconditionally and θ^* is well-defined. On the other hand, for each $x \in X$ and $y = (y_i)_{i\in I}$ in $l^2(X)$, we have

$$\langle x, \theta^*(y)\rangle = \langle x, \textstyle\sum_{i\in I} v_i P_{M_i}(y_i)\rangle = \sum_{i\in I}\langle v_i P_{M_i}(x), y_i\rangle = \langle\theta(x), y\rangle.$$

This shows that θ^* is bounded (Prop. 3.1). □

Proposition 4.1 *Let $\{(M_i, v_i) : i \in I\}$ be a Parseval fusion frame for a Hilbert A-module X. Then, the corresponding frame transform θ preserves the inner product.*

Proof. For each $x, y \in X$, we have the polarization identity as follows

$$\langle x, y\rangle = \tfrac{1}{4}\textstyle\sum_{k=0}^{3} i^k\langle x + i^k y, x + i^k y\rangle \ .$$

On the other hand, by the definition of frame transform θ, for each $x \in X$, we have

$$\begin{aligned}\langle\theta(x), \theta(x)\rangle &= \langle(v_i P_{M_i}(x))_{i\in I}, (v_i P_{M_i}(x))_{i\in I}\rangle \\ &= \textstyle\sum_{i\in I}\langle v_i P_{M_i}(x), v_i P_{M_i}(x)\rangle \\ &= \textstyle\sum_{i\in I} v_i^2\langle P_{M_i}(x), P_{M_i}(x)\rangle \\ &= \langle x, x\rangle \ .\end{aligned}$$

Using the polarization identity, for each $x, y \in X$, we can write

$$\begin{aligned}\langle\theta(x), \theta(y)\rangle &= \tfrac{1}{4}\textstyle\sum_{k=0}^{3} i^k\langle\theta(x) + i^k\theta(y), \theta(x) + i^k\theta(y)\rangle \\ &= \tfrac{1}{4}\textstyle\sum_{k=0}^{3} i^k\langle\theta(x + i^k y), \theta(x + i^k y)\rangle \\ &= \tfrac{1}{4}\textstyle\sum_{k=0}^{3} i^k\langle x + i^k y, x + i^k y\rangle \\ &= \langle x, y\rangle \ .\end{aligned}$$

This shows that θ preserves the inner product. □

Remark 4.1 By Proposition 4.1, if θ is a frame transform for a Parseval fusion frame of a Hilbert A-module X then for each $x \in X$, we have $\langle\theta(x), \theta(x)\rangle = \langle x, x\rangle$. Hence for each $p \in S(A)$, we obtain:

$$\bar{p}_X(x) = \bar{p}_{l^2(X)}(\theta(x)) \quad , \quad x \in X \ .$$

Then, by (1) and for each $p \in S(A)$, we have $\hat{p}_{X,l^2(X)}(\theta) = \hat{p}_{l^2(X),X}(\theta^*) = 1$.

Definition 4.4 Let $M = \{(M_i, v_i) : i \in I\}$ be a fusion frame for a Hilbert A-module X. Then the *fusion frame operator* S for M is defined by

$$S(x) = \theta^*\theta(x) = \textstyle\sum_{i\in I} v_i^2 P_{M_i}(x) \quad , \quad x \in X.$$

Let $\{(M_i, v_i) : i \in I\}$ be a fusion frame for a Hilbert A-module X with fusion frame operator S and fusion frame bounds C and D. Then, S is a positive, self-adjoint and invertible operator on X and we have the reconstruction formula

$$x = \textstyle\sum_{i\in I} v_i^2 S^{-1} P_{M_i}(x) \quad , \quad \text{for all } x \in X \ .$$

Proof. It is easy to see that for each $x, y \in X$, we have $\langle S(x), y\rangle = \langle x, S(y)\rangle$. Hence, by Proposition 3.1, we obtain $S \in End_A^*(X)$ and $S^* = S$. On the other hand, for any $x \in X$, we can write

$$\langle S(x), x\rangle = \textstyle\sum_{i\in I} v_i^2\langle P_{M_i}(x), x\rangle = \sum_{i\in I} v_i^2\langle P_{M_i}(x), P_{M_i}(x)\rangle \ .$$

Thus, S is a positive operator. Since $\{(M_i, v_i) : i \in I\}$ is a fusion frame with bounds C and D, for each $x \in X$, we have

$$C\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle = \langle S(x), x\rangle \leq D\langle x, x\rangle \ .$$

Thus, $CI \leq S \leq DI$ that I is the identity element in pro-C*-algebra $End_A^*(X)$. By (6) and (7) of Proposition 2.1, S is invertible and $D^{-1}I \leq S^{-1} \leq C^{-1}I$. We also have

$$x = S^{-1}S(x) = \sum_{i\in I} v_i^2 S^{-1} P_{M_i}(x) \quad , \quad x \in X.$$

This completes the proof. □

Remark 4.2 Since S and S^{-1} are positive elements from pro-C*-algebra $End_A^*(X)$, by (2) and (3) of Proposition 2.1, we have

$$C \leq \hat{p}_X(S) \leq D \quad , \quad D^{-1} \leq \hat{p}_X(S^{-1}) \leq C^{-1} \ .$$

Proposition 4.2 *Let A be a σ-C^*-algebra and let $\{(M_i, v_i) : i \in I\}$ be a fusion frame for a Hilbert A-module X with fusion frame operator S and fusion frame bounds C and D. If T is an invertible element of $End_A^*(X)$ such that both are uniformly bounded and for each $i \in I$, $T^*T(M_i) \subseteq M_i$, then $\{(TM_i, v_i) : i \in I\}$ is a fusion frame for X with fusion frame operator TST^{-1}.*

Proof. By definition, M_i for each $i \in I$, is a closed submodule of X, so each TM_i is also a closed submodule of X (Remark 3.2). Since T is surjective, for each $x \in X$, there exist a unique $y \in X$ such that $x = Ty$. On the other hand, for each $i \in I$, M_i is an orthogonally complemented submodule, hence there exist unique elements $u \in M_i$ and $v \in M_i^{\perp}$ such that $y = u + v$. Thus $x = Tu + Tv$ and since $TM_i \cap TM_i^{\perp} = \{0\}$, for each $i \in I$, we have

$$X = T(M_i) \oplus T(M_i^{\perp}) \ .$$

On the other hand, by Proposition 3.4, we have $T(M_i^{\perp}) = (TM_i)^{\perp}$, for each $i \in I$, which implies that TM_i is orthogonally complemented. Using Propositions 3.2 and 3.4, for each $x \in X$ and $p \in S(A)$, we have

$$\begin{aligned}\sum_{i\in I} v_i^2 \langle P_{TM_i}(x), P_{TM_i}(x)\rangle &= \sum_{i\in I} v_i^2 \langle TP_{M_i}T^{-1}(x), TP_{M_i}T^{-1}(x)\rangle \\ &\leq \|T\|_\infty^2 \sum_{i\in I} v_i^2 \langle P_{M_i}(T^{-1}x), P_{M_i}(T^{-1}x)\rangle \\ &\leq D\|T\|_\infty^2 \langle T^{-1}x, T^{-1}x\rangle \\ &\leq D\|T\|_\infty^2 \|T^{-1}\|_\infty^2 \langle x, x\rangle \ .\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i\in I} v_i^2 \langle P_{TM_i}(x), P_{TM_i}(x)\rangle &\geq \|T^{-1}\|_\infty^{-2} \sum_{i\in I} v_i^2 \langle P_{M_i}(T^{-1}x), P_{M_i}(T^{-1}x)\rangle \\ &\geq C\|T^{-1}\|_\infty^{-2} \langle T^{-1}x, T^{-1}x\rangle \\ &\geq C\|T^{-1}\|_\infty^{-2} \|T\|_\infty^{-2} \langle x, x\rangle \ .\end{aligned}$$

This shows that $\{(TM_i, v_i) : i \in I\}$ is a fusion frame for X. Furthermore, if S' is the corresponding fusion frame operator, then by using Proposition 3.4, for any $x \in X$, we have

$$S(T^{-1}x) = \sum_{i\in I} v_i^2 P_{M_i}(T^{-1}x) = T^{-1} \sum_{i\in I} v_i^2 P_{TM_i}(x) = T^{-1}S' \ .$$

Thus, $S' = TST^{-1}$. □

References

[1] P. G. Casazza, G. Kutyniok, it Frames of subspaces, in Wavelets, Frames, and Operator Theory, in: Contemp. Math., vol. 345, Amer. Math. Soc. (2004) 87-113.

[2] P. G. Casazza, G. Kutyniok, S. Li, *Fusion frames and distributed processing*, to appear in Appl. Comput. Harmon. Anal. (2007); doi: 10.1016/j.acha.2007.10.001./

[3] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[4] J. B. Conway, *A course in Functional Analysis,* Springer-Verlag, 1978.

[5] I. Daubechies, A. Grossman, Y. Meyer, *Painless nonorthogonal expansions,* J. Math. Phys. 27 (1986) 1271-1283.

[6] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series,* Trans. Amer. Math. Soc. 72 (1952) 341-366.

[7] M. Frank, D. R. Larson, *Frames in Hilbert C*-modules and C*-algebras,* J. Operator Theory 48 (2002) 273-314.

[8] A. Inoue, *Locally C*-algebras,* Mem. Fac. Sci. Kyushu Univ. Ser. A 25 (1971) 197-235.

[9] M. Joita, *The stabilization theorem for Hilbert modules over locally C*-algebras,* The 3rd International Conference on Topological Algebra and Applications, (ICTAA3), Oulu, Finland July (2001) 2-6.

[10] M. Joita, *On bounded module maps between Hilbert modules over locally C*-algebras,* Acta Math. Univ. Comenianae. Vol. LXXIV 1 (2005) 71-78.

[11] A. Khosravi, B. Khosravi, *Fusion frames and g-frames in Hilbert C*-modules,* Int. J. Wavelets Multiresolution, Inf. Process 6 (2008) 433-466.

[12] A. Khosravi, M. S. Asgari, *Frames and bases in Hilbert modules over locally C*-algebras,* International Journal of Pure and Applied Mathematics14 (2004) 169-187.

[13] E. C. Lance, *Hilbert C*-modules,* A toolkit for operator algebraists, London Math. Soc. Lecture Note Series 210. Cambridge Univ. Press, Cambridge, 1995.

[14] A. Mallios, *Hermitian K-theory over Topological ∗-Algebras,* J. Math. Anal. Appl. 106 (1985) 454-539.

[15] N. C. Phillips, *Inverse limits of C*-algebras,* J. Operator Theory 19 (1988) 159-195.

[16] W. Rudin, *Functional Analysis,* 2nd. ed., Mc Graw-Hill, New York, 1991.

[17] W. Sun, *G-frames and g-Riesz bases,* J. Math. Anal. Appl. 322 (2006) 437-452.

[18] N. E. Wegge-Olsen, K-Theory and C*-Algebras, A Friendly Approach, Oxford University Press, Oxford, England, 1993.

[19] Yu. I. Zhuraev, F. Sharipov, *Hilbert modules over locally C*-algebra,* Preprint, posted on Arxiv Org. Math. OA/0011053 V3 (2001).

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