Available online at http://ijim.srbiau.ac.ir

Int. J. Industrial Mathematics Vol. 2, No. 2 (2010) 89-96

Dominated Convergence for Fuzzy Random Variables

E. Deiri [a]*, B. G. Sadeghpour [b], G.H Yari [c]

(a, c) *Department of Statistic, Science and Research Branch, Islamic Azad University, Tehran, Iran.*

(b) *Department of Statistics, Faculty of Basic science, Babolsar, P. O. Box 47416-1467, Iran.*

Received 8 January 2010; revised 13 June 2010; accepted 15 June 2010.

Abstract

In this paper, the properties of fuzzy random variables with new meter and some extended results of monotone convergence theorem and dominated convergence theorem for fuzzy random variables are discussed. The main result is given by using Dp, q-distance defined on the set of fuzzy numbers.

Keywords : Fuzzy random variables; $L_P(F(R))$-Fuzzy integrable space; Dominated convergence theorem; Expectation of fuzzy random variables

---

1 Introduction

There are two types of common in the real-life world, randomness and fuzziness. Accordingly, there are two powerful theories, and possibility theory. For the development of possibility theory, Dubois and Prad (1988), Klir (1999) and Zadeh (1978) are referred. In many optimization problems such as Luhanjula (1996), Luhanjula and Gopta (1996), Yazenin (1987), and Liu (1999, 2001a, 2001b), randomness and fuzziness are often required for simultion. For other approaches to the combined treatment of randomnss and fuzziness, Puri and Ralescu (1985, 1986), Klement, Kruse and Meyer (1987), and Negoita and Ralescu (1987) are recommanded. The concept of fuzzy random variables was introduced by Kwakernaak (1978), who developed useful basic properties. Puri and Ralescu (1985,1986), used the concept of fuzzy random variables and expected value (these concept goes beyond those of Kwakernaak) for generalization of the result of random sets to fuzzy random sets and also they developed an important tool for representing imprecise data associated with the outcomes of a random experiment.
The paper is organized as follows:

*Corresponding author. Email address: Deiri53@gmail.com Tel:09113268212

In Section 2, the basic definitions and concepts are brought. In Section 3, main result of paper are discussed and finally, conclusion is drawn in Section 4.

2 Preliminaries

In this section, first we recall some notations of fuzzy sets, fuzzy numbers and fuzzy random variables.

Definition 2.1. *Let E be a universal set, then a fuzzy set $\widetilde{A}$ of E is defined by its membership function $\widetilde{A}: E \to [0,1]$, where $\widetilde{A}(x)$ is the membership grade of x in $\widetilde{A}$*

Definition 2.2. *$Co(\widetilde{A})$ is called the coreof $\widetilde{A}$, defined by $Co(\widetilde{A}) = \{x \in E : \widetilde{A}(x) = 1\}$.*

Definition 2.3. *$Supp(\widetilde{A})$ is called the support of $\widetilde{A}$, defined by $Supp(\widetilde{A}) = cl\{x \in E : \widetilde{A}(x) > 0\}$.*

Definition 2.4. *A_α is called the $\alpha-$level (cut) set of $\widetilde{A}$, defined by $A_\alpha = \{x \in E : \widetilde{A}(x) \geq \alpha\}$.According to the decomposition theorem of fuzzy set, we have: $\widetilde{A}(x) = \sup\{\alpha I_{\widetilde{A}_\alpha}(x) : \alpha \in [0,1]\}$,where $I_{\widetilde{A}_\alpha}$ is the indicator function of ordinary set $\widetilde{A}_\alpha$.*

Definition 2.5. *A fuzzy number is a fuzzy set of $\mathbb{R}$ such that the following conditions are satisfied:*

a) $\widetilde{A}$ is normal, that is, there exist x_0 such that $\widetilde{A}(x_0) = 1$,

b) $\widetilde{A}$ is convex, that is, $\forall x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, $\widetilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq min\left(\widetilde{A}(x_1), \widetilde{A}(x_2)\right)$,

c) $\widetilde{A}$ is upper semicontinuous with compact support.

According to the definition $\alpha-$level (cut) set of a fuzzy number is a closed interval where we denote by $\widetilde{A}_\alpha = [A_\alpha^-, A_\alpha^+]$, i.e.

$$A_\alpha^- = \inf\left\{x \in \mathbb{R} : \widetilde{A}(x) \geq \alpha\right\}, \quad and \quad A_\alpha^+ = \sup\left\{x \in \mathbb{R} : \widetilde{A}(x) \geq \alpha\right\}.$$

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $F(\mathbb{R})$ denote the set of fuzzy defined on $\mathbb{R}$

Definition 2.6. *A Mapping $\widetilde{X} : \Omega \to F(\mathbb{R})$ is said to be a fuzzy random variable associated with $(\Omega, \mathcal{A})$ if and only if*

$$\{(\omega, x) : x \in X_\alpha(\omega)\} \in \mathcal{A} \times \mathcal{B},$$

where $\mathcal{B}$ denotes the σ-field of Borel set in $\mathbb{R}$.For a fuzzy random variable $\widetilde{X}$ and $\omega \in \Omega$, let $\widetilde{X}(\omega)$ be a fuzzy set with the membership function $\widetilde{X}(\omega)(x)$.

Definition 2.7. *$D : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to [0, \infty)$ by the equation*

$$D_{p,q}(\widetilde{A}, \widetilde{B}) = \begin{cases} \left(\int_0^1 |q(A_\alpha^+ - B_\alpha^+) + (1-q)(A_\alpha^- - B_\alpha^-)|^p \, d\alpha\right)^{1/p} & if \quad 1 \leq p < \infty \\ \\ \sup_{0 \leq \alpha \leq 1} |q(A_\alpha^+ - B_\alpha^+) + (1-q)(A_\alpha^- - B_\alpha^-)| & if \quad p = \infty. \end{cases}$$

The analytical properties of $D_{p,q}$ depend on the first parameter p, while the second parameter q of $D_{p,q}$ characterizes the subjective weight attributed to the sides of the fuzzy numbers.

3 Main Result

Theorem 3.1. *Let $\{\widetilde{X}_n, \widetilde{X} : n \geq 1\}$ be a sequence of fuzzy random variables of real value. If $\widetilde{X}_n \xrightarrow{a.s.D} \widetilde{X}$, then*

$$\widetilde{X}_n \xrightarrow{i.P.D} \widetilde{X},$$

where a.s.D i.e is almost surly based on $D_{p,q}$.

Proof: Since $\widetilde{X}_n \xrightarrow{\text{a.s.D}} \widetilde{X}$, therefore $P\{\omega \in \Omega : lim_{n\to\infty} D_{p,q}(\widetilde{X}_n, \widetilde{X}) = 0\} = 1$ On the other hand $\exists A \in F$ s.t $P(A) = 0, \forall \omega \in A^c$ then

$$lim_{n\to\infty} D_{p,q}(\widetilde{X}_n, \widetilde{X}) = 0$$

So, $\forall \epsilon > 0, \exists n' \in N$ s.t $\forall n' \leq n$, then $D_{p,q}(\widetilde{X}_n, \widetilde{X}) < \epsilon, \forall \omega \in \Omega$ Thus,

$$1 = P\{\bigcap_{\epsilon>0} \bigcup_{1\leq n'} \bigcap_{n\leq n'} [D_{p,q}(\widetilde{X}_n, \widetilde{X}) < \epsilon]\}$$
$$\leq P\{\bigcap_{\epsilon>0} \bigcup_{1\leq n'} [D_{p,q}(\widetilde{X}_n, \widetilde{X}) < \epsilon]\}$$

let $B_{n'} = \bigcap_{n'\leq n} [D_{p,q}(\widetilde{X}_n, \widetilde{X}) < \epsilon]$

$$= P\{\bigcap_{1\leq n'} B_{n'}\}$$

$$= lim_{n'\to\infty} P\{\bigcap_{n'\leq n} [D_{p,q}(\widetilde{X}_n, \widetilde{X}) < \epsilon]\}$$

$$\leq lim_{n'\to\infty} P\{[D_{p,q}(\widetilde{X}_{n'}, \widetilde{X}) < \epsilon]\}.$$

Theorem 3.2. *Let $\{\widetilde{X}_n, \widetilde{X} : n \geq 1\}$ be a sequence of fuzzy random variables and integrable. If $\widetilde{X}_n \xrightarrow{i.P.D} \widetilde{X}$, then*

$$E[\widetilde{X}_n] \xrightarrow{D} [\widetilde{X}],$$

for , $p = 1$.

Proof: By the assumption we have $D_{1,q}(\widetilde{X}_n, \widetilde{X}) \xrightarrow{\text{P}} 0$. Now we show that $D_{1,q}(E[\widetilde{X}_n], E[\widetilde{X}]) \xrightarrow{\text{P}} 0$, as, $n \to \infty$.

$$D_{1,q}(E[\widetilde{X}_n], E[\widetilde{X}]) = \int_0^1 |q(\mu^+_{n\alpha} - \mu^+_{\alpha}) + (1-q)(\mu^-_{n\alpha} - \mu^-_{\alpha})| d\alpha$$

$$\leq E[\int_0^1 |q(X^+_{n\alpha} - X^+_{\alpha}) + (1-q)(X^-_{n\alpha} - X^-_{\alpha})| d\alpha]$$

$$= E[D_{1,q}(\widetilde{X}_n, \widetilde{X})]$$

$$= E[D_{1,q}(\widetilde{X_n}, \widetilde{X}) I_{\{\omega\in\Omega | D_{1,q}(\widetilde{X}_n,\widetilde{X})\geq\epsilon\}}] + E[D_{1,q}(\widetilde{X}_n, \widetilde{X}) I_{\{\omega\in\Omega | D_{1,q}(\widetilde{X}_n,\widetilde{X})<\epsilon\}}]$$

let $I_n = E[D_{1,q}(\widetilde{X}_n, \widetilde{X}) I_{\{\omega\in\Omega | D_{1,q}(\widetilde{X}_n,\widetilde{X})\geq\epsilon\}}]$

$$\leq I_n + \epsilon P\{\omega \in \Omega | D_{1,q}(\widetilde{X}_n, \widetilde{X}) < \epsilon\}$$

$$\leq I_n + \epsilon$$

To complete the proof we show that $I_n \to 0, n \to \infty$.

$$I_n = E[D_{1,q}(\widetilde{X}_n, \widetilde{X}) I_{\{\omega\in\Omega|D_{1,q}(\widetilde{X}_n,\widetilde{X})\geq\epsilon\}}]$$

$$= \int_{\{\omega\in\Omega|D_{1,q}(\widetilde{X}_n,\widetilde{X})\geq\epsilon\}} D_{1,q}(\widetilde{X}_n, \widetilde{X}) dP(\omega)$$

$$= \int_{A_n} D_{1,q}(\widetilde{X}_n, \widetilde{X}) dP(\omega).$$

Since $\widetilde{X}_n, \widetilde{X}$ are integrable, we have $\left\|\widetilde{X}_n\right\| < \infty$ $a.s.D$, and $D_{1,q}(\widetilde{X}_n, \widetilde{X}) < \infty$ a.s.. Indeed $P(A_n) \to 0$, $n \to \infty$, since $\widetilde{X}_n \xrightarrow{\text{i.P.D}} \widetilde{X}$. Then,

$I_n = \int_{A_n} D_{1,q}(\widetilde{X}_n, \widetilde{X}) dP(\omega) \leq sup_{\omega\in\Omega} D_{1,q}(\widetilde{X}_n, \widetilde{X}) P(A_n) \to 0$, as $n \to \infty$.

Applying theorems (3.1) and (3.2) we have the following theorems.

Theorem 3.3. *Let $\{\widetilde{X}_n, \widetilde{X} : n \geq 1\}$ be a sequence of fuzzy random variables and be independent of random variable Y such that $Y \in L_1, \widetilde{X} \in L_1(F), \widetilde{X}_n \in L_1(F)$. If $\widetilde{X}_n \xrightarrow{a.s.D} \widetilde{X}$, then*

$$E[\widetilde{X}_n Y] \xrightarrow{D} E[\widetilde{X}]E[Y].$$

Proof: By theorems (3.1) and (3.2), $E[\widetilde{X}_n] \xrightarrow{D} E[\widetilde{X}]$. since $\widetilde{X}_n \xrightarrow{a.s.D} \widetilde{X}$. Therefore $E[\widetilde{X}_n]E[Y] \xrightarrow{D} E[\widetilde{X}]E[Y]$. Now, to complete the proof we show that $E[\widetilde{X}_n]E[Y] = E[\widetilde{X}_n Y]$. For all fuzzy random variables we have,

$$\widetilde{X}_n(\omega) = \{\ [X^-_{n\alpha}(\omega), X^+_{n\alpha}(\omega)]\ |\ 0 \leq \alpha \leq 1\}, \quad \forall \omega \in \Omega.$$

Thus, $\sigma(\widetilde{X}_n) = \sigma(\{\widetilde{X}_{n\alpha}; 0 \leq \alpha \leq 1\})$. We also know that a fuzzy random variable $\widetilde{X}_n$ and a real-valued random variable Y are independent if and only if $\sigma(\widetilde{X}_n)$ and $\sigma(Y)$ are independent for $n \geq 1$, i.e, for any $A \in \sigma(\widetilde{X}_n),\ B \in \sigma(Y),\ P(A \cap B) = P(A)P(B)$. Therefore it is enough to show that

$$E[\widetilde{X}_{n\alpha}]E[Y] = E[\widetilde{X}_{n\alpha} Y], \quad \forall \alpha \in [0, 1].$$

Now here we consider three cases for a random variable.

(1) Let $Y = I_A$ for $A \in \sigma(Y)$. Since I_A is a random set thus $I_A \widetilde{X}_{n\alpha}$ is a random set. By Aumann integral we have

$$E[\widetilde{X}_{n\alpha}] = \{E(Z) |\ Z(\omega) \in \widetilde{X}_{n\alpha}(\omega)\},$$

and

$$E[\widetilde{X}_{n\alpha} I_A] = \{E(Z I_A) |\ Z(\omega) \in \widetilde{X}_{n\alpha}(\omega)\} = \{E(Z)P(A) |\ Z(\omega) \in \widetilde{X}_{n\alpha}(\omega)\}$$
$$= P(A)\{E(Z)) |\ Z(\omega) \in \widetilde{X}_{n\alpha}(\omega)\} = E[Y]E[\widetilde{X}_{n\alpha}].$$

Now if $Y=\sum_{i=1}^{n} a_i I_{A_i}$, then we have

$$\begin{aligned}E[Y\widetilde{X}_{n\alpha}] &= E[\sum_{i=1}^{n} a_i I_{A_i}\widetilde{X}_{n\alpha}] = \sum_{i=1}^{n} a_i E[I_{A_i}\widetilde{X}_{n\alpha}] \\ &= \sum_{i=1}^{n} a_i P(A_i)E[\widetilde{X}_{n\alpha}] = \sum_{i=1}^{n} a_i E[I_{A_i}]E[\widetilde{X}_{n\alpha}] \\ &= E[\sum_{i=1}^{n} a_i I_{A_i}]E[\widetilde{X}_{n\alpha}] = E[Y]E[\widetilde{X}_{n\alpha}].\end{aligned}$$

(2) Let Y is a non-negative random variable (i.e $Y \geq 0$, a.s.). Therefore there exist a simple random variable $0 \leq Y_m = \sum_{i=1}^{m} a_i I_{A_i}$ such that $0 \leq Y_m \nearrow Y$. By monotone convergence theorem for classical case we have $E[Y_m] \nearrow E[Y]$, so $Y_m\widetilde{X}_{n\alpha} \to Y\widetilde{X}_{n\alpha}$. Thus

$$E[Y\widetilde{X}_{n\alpha}] = Lim_{m\to\infty}E[Y_m\widetilde{X}_{n\alpha}] = Lim_{m\to\infty}E[Y_m]E[\widetilde{X}_{n\alpha}] = E[Y]E[\widetilde{X}_{n\alpha}].$$

(3) Let Y is a arbitrary random variable, then there exist non-negative random variable Y_1 and Y_2 such that $Y = Y_2 - Y_1$, therefore

$$\begin{aligned}E[Y\widetilde{X}_{n\alpha}] &= E[(Y_2 - Y_1)\widetilde{X}_{n\alpha}] = E[Y_2\widetilde{X}_{n\alpha}] - E[Y_1\widetilde{X}_{n\alpha}] \\ &= E[Y_2]E[\widetilde{X}_{n\alpha}] - E[Y_1]E[\widetilde{X}_{n\alpha}] = (E[Y_2] - E[Y_1])E[\widetilde{X}_{n\alpha}] = E[Y]E[\widetilde{X}_{n\alpha}].\end{aligned}$$

Theorem 3.4. *Let $\{\widetilde{X}_n, \widetilde{X} : \ n \geq 1\}$ be a sequence of fuzzy random variables and integrable, the following conditions are equivalent.*
(i) $D_{1,q}(\widetilde{X}_n, \widetilde{X}) \xrightarrow{P} 0, \quad E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})] \to E[D_{1,q}(\widetilde{X}, I_{\{0\}})]$
(ii) $E[D_{1,q}(\widetilde{X}_n, \widetilde{X})] \to 0$
(iii) $D_{1,q}(\widetilde{X}_n, \widetilde{X}) \xrightarrow{P} 0, \quad D_{1,q}(\widetilde{X}_n, I_{\{0\}})$ *is uniformly integrable.*

Proof: $(i) \Longrightarrow (ii)$: Let $Y_n = D_{1,q}(\widetilde{X}_n, \widetilde{X})$. Therefore Y_n is a nonnegative real-value random variable and $E[Y_n] < \infty$ and $Y_n \xrightarrow{P} 0$, thus $E[Y_n] \to 0$, i.e $D_{1,q}(\widetilde{X}_n, \widetilde{X}) \to 0$.
$(ii) \Longrightarrow (iii)$: For all $\epsilon > 0$ we have

$$P(\omega \in \Omega |\ D_{1,q}(\widetilde{X}_n, \widetilde{X}) \geq \epsilon) \leq \frac{E[D_{1,q}(\widetilde{X}_n, \widetilde{X})]}{\epsilon} \to 0,$$

as $n \to \infty$. Since $\widetilde{X}_n$, is integrable then

$$P\{\omega \in \Omega |\ D_{1,q}(\widetilde{X}_n, I_{\{0\}}) = \infty\} = 0,$$

and

$$P\{\omega \in \Omega |\ D_{1,q}(\widetilde{X}_n, I_{\{0\}}) \leq a\} \to 1, \quad as,\ n \to \infty \tag{3.1}$$

or

$$P\{\omega \in \Omega |\ D_{1,q}(\widetilde{X}_n, I_{\{0\}}) > a\} \to 0, \quad as,\ n \to \infty. \tag{3.2}$$

Therefore $E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})] = I_a + II_a$, where

$$I_a = E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})I_{\{\omega\in\Omega|\ D_{1,q}(\widetilde{X}_n, I_{\{0\}})>a\}}],$$

$$II_a = E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})I_{\{\omega\in\Omega|\ D_{1,q}(\widetilde{X}_n,I_{\{0\}})\le a\}}].$$

By using (3.1) and (3.2), we have

$$Lim_{n\to\infty}E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})] = Lim_{a\to\infty}I_a + Lim_{a\to\infty}II_a$$
$$= Lim_{a\to\infty}I_a + Lim_{n\to\infty}E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})]$$

Therefore

$$0 = Lim_{a\to\infty}I_a = Lim_{a\to\infty}E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})I_{\{\omega\in\Omega|\ D_{1,q}(\widetilde{X}_n,I_{\{0\}})>a\}}]$$

$(iii) \Longrightarrow (i)$:

$$\begin{aligned}|E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})] - E[D_{1,q}(\widetilde{X}, I_{\{0\}})]| &\le E[|D_{1,q}(\widetilde{X}_n, I_{\{0\}}) - D_{1,q}(\widetilde{X}, I_{\{0\}})|]\\ &= E[|\int_0^1 [|qX^+_{n\alpha} + (1-q)X^-_{n\alpha}| - |qX^+_\alpha + (1-q)X^-_\alpha|]d\alpha|]\\ &\le E[\int_0^1 [|qX^+_{n\alpha} + (1-q)X^-_{n\alpha} - qX^+_\alpha - (1-q)X^-_\alpha|]d\alpha]\\ &= E[D_{1,q}(\widetilde{X}_n, \widetilde{X})] \to 0, \quad as,\ n\to\infty.\end{aligned}$$

Because, $\widetilde{X}_n \xrightarrow{i.P.D} \widetilde{X}$ or $D_{1,q}(\widetilde{X}_n, \widetilde{X}) \xrightarrow{P} 0$. Then by using the theorem (3.3) the proof is completed.

Theorem 3.5. *(Dominated convergence theorem for fuzzy random variables) Let* $\{\widetilde{X}_n, \widetilde{X} :\ n \ge 1\}$ *be a sequence of fuzzy random variables and* Y *be a real-valued random variable and* $Y \in L_1$. *If* $\widetilde{X}_n \xrightarrow{a.s.D} \widetilde{X}$ *and* $D_{1,q}(\widetilde{X}_n, I_{\{0\}}) \le |Y|$, *a.s. for* $n \ge 1$ *then*

$$\widetilde{X}_n, \widetilde{X} \in L_1(F) \quad and \quad E[\widetilde{X}_n] \xrightarrow{D} E[\widetilde{X}].$$

Proof: From theorem (3.1) and (3.2) to complete the proof it is enough to show that $\widetilde{X}_n, \widetilde{X} \in L_1(F)$. Since $\widetilde{X}_n \xrightarrow{a.s.D} \widetilde{X}$, then

$$D_{1,q}(\widetilde{X}_n, \widetilde{X}) \xrightarrow{a.s.} 0,$$

and there exist an event $E \in F$, such that $P(E) = 0$, and for all $\omega \in \Omega$,

$$Lim_{n\to\infty}D_{1,q}(\widetilde{X}_n, \widetilde{X}) = 0,$$

in the other hand

$$\forall\epsilon > 0,\ \exists n' \in N\ s.t\ \forall n \ge n' \implies D_{1,q}(\widetilde{X}_n, \widetilde{X}) < \epsilon.$$

Since $D_{1,q}(\widetilde{X}_n, I_{\{0\}}) \le |Y|$, a.s., and $Y \in L_1$, then

$$E[D_{1,q}(\widetilde{X}_n, I_{\{0\}})] \le E[|Y|] < \infty,$$

and $\widetilde{X}_n \in L_1(F)$. So for the enough large n

$$\begin{aligned}D_{1,q}(\widetilde{X}, I_{\{0\}}) &\le D_{1,q}(\widetilde{X}_n, \widetilde{X}) + D_{1,q}(\widetilde{X}_n, I_{\{0\}})\\ &\le \epsilon + D_{1,q}(\widetilde{X}_n, I_{\{0\}}) < \infty \quad \text{a.s.}.\end{aligned}$$

Therefore $\widetilde{X} \in L_1(F)$.

4 Conclusion

In this paper we have developed some important theorems of probability theory by using a distance defined on the set of fuzzy numbers and their α cuts. The results is agreed to crisp case where α= 1.
And finally, Special thanks to reviewers for their worthwhile and useful offers.

References

[2] A. Acosta, Inequaliy for B-value Random Vectors with Application to the Strong Law of Large Numbers. Ann. Probab. 9 (1981) 157-163.

[1] B.D. Choi, and S.H. Sung. On Chung Strong Law of Large Numbers in General Space. Bull. Austral. Math. Soc. 37 (1988) 93-100.

[3] D. Dubois, and H.Prad, Possibility Theory. New York: Plenum Press, (1988).

[4] Y. Feng, Convergence Theory for Fuzzy Random Variables and Fuzzy Martingales. Fuzzy Set and Systems 103 (1999) 435-441.

[5] Y. Feng, Note on some of Inequality Fuzzy Random Variables, Fuzzy Set and Systems. 143 (2004) 479-485.

[6] S.Y. Joo, and Y.K. Kim, and J, S, Kwon, Strong Convergence for Weighted Sums of Fuzzy Random Sets, Information Science. 176 (2006) 1086-1099.

[7] E.P Kelement, and M.L. Puri, and D.A. Ralescu, Limit Theory for Fuzzy Random Variables, Press, R.Soc .Lond. A 407 (1986) 171-182.

[8] G.J. Klir, On Fuzzy-Set Interpretation of Possibility Theory, Fuzzy Set and System 108 (1999) 263-273.

[9] V. Kratchmer, Limit Theorems for Fuzzy Random Variables. Fuzzy Sets and Systems. 126 (2002) 253-263.

[10] R. Kruse, The Strong Low of Large Numbers for Fuzzy Random Variables, Information Sciences 28 (1982) 233-241.

[11] R. Kruse, and K.D. Meyer, Statistics with Vague Data. Dordrecht: D.Reidel Publishing Company (1987).

[12] H. Kwakernaak, Fuzzy Random Variables I, Information Sciences 15 (1978) 1-29.

[13] H. Kwakernaak, Fuzzy Random Variables II, Information Sciences 17 (1978) 153-178.

[14] B. Liu, Uncertain Programming.Viley, New York,(1999).

[15] B. Liu, Fuzzy Random Chance-Constrained Programming, IEEE Transations on Fuzzy systems 9 (2001a) 713-720.

[16] B. Liu, Fuzzy Random Dependent- Chance Programming, IEEE Transations on Fuzzy Systems 9 (2001b) 721-726.

[17] M.K. Luhanjula, Fuzziness and Randomness in an Optimization Framework, Fuzzy Set and Systems 77 (1996) 291-297.

[18] M.L. Negoita, and D.A. Ralescu, Simulation, Knowledg-Based omputing, and Fuzzy Statistics.New York: Van Nostrand Reinhold company,(1987).

[19] M.L. Puti, and D.A. Ralescu, The concept of Normality for Fuzzy Random Variables, Annals of Probabilty 13 (1985) 1371-1379.

[20] M.L. Puri, and D.A. Ralescu Convergence Theorem for Fuzzy Martingales, J.Math. Anal.Appl.160 (1991) 197-122.

[21]] M.L. Puti, and D.A. Ralescu, (1986) Fuzzy Random Variables, J, Math.Anal Appl.114 (1986) 409-422.

[22] R.L. Taylor, and L. Seymour, and Y. Chen, Weak Laws of Large Numbers for Fuzzy Random Sets. Nonlinear Anal. 47 (2001) 1245-1256.

[23] A.V. Yazenin, Fuzzy and Stochastic Programming, Fuzzy Set and Systems 22 (1987) 171-180.

[23] L.A. Zadeh, Fuzzy Sets As a Basis for a Theory of Possibility, Fuzzy Sets and Systems 1 (1978) 3-28.