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A numerical approach on Hiemenz flow problem using radial basis functions

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Abstract

In this paper, we propose radial basis functions (RBF) to solve the two dimensional flow of fluid near a stagnation point named Hiemenz flow. The Navier-Stokes equations governing the flow can be reduced to an ordinary differential equation of third order using similarity transformation. Because of its wide applications the flow near a stagnation point has attracted many investigations during the past several decades. We satisfy boundary conditions such as infinity condition, by using Gaussian radial basis function through the both differential and integral operations. By choosing center points of RBF with shift on one point in uniform grid, we increase the convergence rate and decrease the collocation points.

Keywords : Hiemenz flow; Radial basis functions; Navier-Stokes equations; Collocation method.

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1 Introduction

Hiemenz [1], first examined the two dimensional flow of fluid near a stagnation point. He demonstrated that the Navier-Stokes equations governing the flow can be reduced to an ordinary differential equation of third order using similarity transformation. Because of the nonlinearities in the reduced differential equation, no analytical solution is available and the nonlinear equation is usually solved numerically subject to two-point boundary conditions, one of which is prescribed at infinity. The flow near a stagnation point has attracted many investigations during the past several decades because of its wide applications. Numerical solution of this problem by employing finite difference scheme is given by Howarth [2]. The results of axisymmetric three-dimensional stagnation point flow is applied in the prediction of skin friction as well as heat/ mass transfer near stagnation regions of bodies in supersonic or hypersonic speeds.

Also the effect of suction and blowing on Hiemenz flow is considered by Schlichting and Bussman (1943), Preston (1946), Ariel (1994) and Weidman and Mahalingam (1997) [3]. Some works has been done in porous medium for example I.A. Hassanien and T.H. Al-arabi [4], considered the unsteady mix convection flow near the stagnation point on a heated vertical surface embedded in a porous medium with thermal radiation and variable viscosity. The unsteadiness is caused by sudden motion of velocity and sudden variation in the surface temperature. This study has many application in several technical processes for example in electronic devices cooled by fans, heat exchangers, placed in low-velocity-environment and solar central receivers exposed to wind current. An analysis is presented to investigate the unsteady fluid dynamic characteristics of an axisymmetric stagnation flow on a circular cylinder performing a harmonic motion in its own plane by Gorla [5]. An analysis of magnetohydrodynamic (MHD) flow of incompressible fluid has been made by Sparrow to determine the reduction in stagnation point heat transfer when blowing and magnetic field act simultaneously. The results of this work has many important engineering applications in devices such as power generator, the cooling of reactors, the design of heat exchangers and MHD accelerators. That non-Newtonian fluids are finding increasing applications in industries has given impe-

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tus to many researchers. Rajeshwari and Rathna [6], were the first persons who considered non-Newtonian two dimensional Stagnation point flow and gave its solution for a viscoelastic second-order fluid. In this paper we want to consider the viscous two-dimensional stagnation point flow.

Many problems in science and engineering modelled as differential equations (DEs) [9, 10, 11, 12]. Solving equations by traditional numerical methods such as finite difference (FDM), finite element (FEM) needs generation of a regular mesh in the domain of the problem which is computationally expensive.

The meshless methods based on radial basis functions (RBF) has been considered as a powerful and prospective numerical method for the interpolation problems and solving differential equations of scattered data. A radial basis function is a positive real-valued function whose values depend only on the distance from some points, called centers. The center points are not necessarily structured, that is, they can have an arbitrary distribution. The arbitrary grid structure is one of the major differences between the RBF method and other global methods. Such a meshless grid structure yields high flexibility especially when the domain is irregular.

RBF was first studied by Roland Hardy, an Iowa State geodesist, in 1968, these methods allow for scattered data to easily be used in computations [13]. The concept of solving DEs by using RBF was first introduced by Kansa [14] who directly collocated the radial basis functions for the approximate solution of differential equations. Since then, it has received a great deal of attention from researchers. And subsequently, many further interesting developments and applications have been reported [15, 16, 17, 18].

Essentially, in a typical RBF collocation method, each variable and its derivatives are all expressed as weighted linear combinations of basis functions, where the sets of network weights are identical. These closed forms of representations are substituted into the governing equations as well as boundary conditions, and the point collocation technique is then employed to discretize the system. If all basis functions in networks are available in analytic forms, the RBF collocation methods can be regarded as truly meshless methods [19].

There are two basic approaches for obtaining new basis functions from RBF, namely direct approach (DRBF) based on a differential process (Kansa [14]) and indirect approach (IRBF) based on an integration process (Mai-Duy and Tran-Cong [13, 16, 20]). Both approaches were tested on the solution of second order DEs and the indirect approach was found to be superior to the direct approach (Mai-Duy and Tran-Cong [16]). In recent years, radial basis functions have been very effective tools to approximate the solutions of equations on a scattered or irregular grid. Boyd et al. [21] presented theory and numerical experiments for approximate the solutions on uniform grid of spacing h in which one point is shifted by an amount sh. They asserted, that the effects of a shifted grid are localized in the sense that the RBF approximation will be unchanged except within a few grid points of the shifted grid point.

This paper is arranged as follows: In section (2) problem formulation of Hiemenz flow is applied. In section (3) properties and interpolation of RBF is applied. In section (4) we apply new model of RBF through both the integration and differential process. In this section new method is applied to solve Hiemenz flow by choosing center points (η_i) in uniform grid. In subsection (4.1) present method is applied to solve Hiemenz flow by choosing center points (η_i) with shift on one point in uniform grid [21].

2 Problem formulation

Let us consider two-dimensional, Newtonian, viscous, incompressible, steady state flow of density ρ impinging on a plane situated at $x_2 = 0$ see Fig. 1. Governing equations are in tensor form. Continuity equation is given by

$$d_m V_m = 0, \tag{2.1}$$

momentum equations are

$$V_m d_m V_i = \frac{1}{\rho} d_i p + \nu d_{mm} V_i. \tag{2.2}$$

That $i, m = 1, 2$. If index m is equal to one it means that properties in direction x_1 is considered and if index m is equal to two it means that properties in direction x_2 is considered. d_1 is the first order derivation in x_1 direction and d_2 is the first order derivation in x_2 direction. The pressure is shown with p and d_{mm} is the second order derivation. ν is kinematic viscosity. The boundary conditions on wall are given by

$$V_1(x_1, x_2 = 0) = 0, \tag{2.3}$$
$$V_2(x_1, x_2 = 0) = 0,$$

where V_1 and V_2 are the velocity component in the Cartesian directions (x_1, x_2). Far away as $x_2 \to \infty$ we reach the invisible flow. The velocity in a potential flow is written in below form:

$$U(x_1) = V_1(x_1, x_2 \to \infty) = cx_1,$$
$$V(x_2) = V_2(x_1, x_2 \to \infty) = -cx_2,$$

where U and V are the potential flow velocity components and c is the dimensional constant. However, to allow for effect of viscous region at the stagnation point region on the outside invisid profile, we may write:

$$U(x_1) = V_1(x_1, x_2 \to \infty) = cx_1, \tag{2.4}$$
$$V_2(x_1, x_2 \to \infty) = -c(x_2 + \delta^*), \tag{2.5}$$

where δ^* is the thickness to shift the profile from the wall. we can try the following solution:

$$V_2(x_1, x_2) = f(x_2). \tag{2.6}$$

From Eq. (2.1) we have

$$V_1(x_1, x_2) = -x_1 f'(x_2), \tag{2.7}$$

from Eq. (2.2) in two dimensional we have

$$f'^2 - ff'' + \nu f''' = \frac{-d_1 p}{\rho x_1}, \tag{2.8}$$

$$ff' - \nu f'' = \frac{-d_2 p}{\rho}. \tag{2.9}$$

After integrating Eq. (2.9) and putting in Eq. (2.8) and applying Eqs. (2.3), (2.4), (2.6), (2.7) we have

$$f'^2 - ff'' + \nu f''' = c^2, \tag{2.10}$$
$$f(0) = f'(0) = 0,$$
$$f'(\infty) = -c.$$

The proper characteristic scale for length is $\sqrt{\nu/c}$ and for velocity is $\sqrt{\nu c}$. By use of characteristic scales, the non-dimensional form of Eq. (2.10) is defined as below

$$\psi''' + \psi\psi'' - \psi'^2 + 1 = 0, \tag{2.11}$$
$$\psi(0) = 0 \, , \psi'(0) = 0,$$
$$\psi'(\infty) = 1,$$

where η and $\psi(\eta)$ are

$$\eta = \sqrt{c/\nu} \, x_2, \tag{2.12}$$

$$\psi(\eta) = \frac{-f(x_2)}{\sqrt{\nu c}}. \tag{2.13}$$

By use of above equations we have

$$f(x_2) = -\sqrt{\nu c} \, \psi(\eta),$$
$$f'(x_2) = -c \, \psi'(\eta).$$

Subsequently, by substituting in Eqs. (2.6) and (2.7) we have

$$V_1 = cx_1\psi'(\eta), \tag{2.13}$$
$$V_2 = -\sqrt{\nu c} \, \psi(\eta).$$

Consequently, ψ' is proportional to V_1 and ψ is proportional to negative of V_2
For boundary layer flow, the wall skin friction τ_w is given by:

$$\tau_w = \mu \frac{\partial V_1}{\partial x_2}|_{x_2=0}, \tag{2.14}$$

where μ is the viscosity coefficient. By use of Eq. (2.4), the skin friction coefficient c_f can be defined as:

$$c_f = \frac{\tau_w}{\rho U^2}. \tag{2.15}$$

Table 1: Some well–known functions that generate RBFs ($r = \|x - x_i\| = r_i$), $\epsilon > 0$

| Name of functions	Definition
Inverse quadrics (IQ)	$1/(1 + \epsilon^2 r^2)$
Gaussian (GA)	$2/\sqrt{\pi}e^{-\epsilon^2 r^2}$
Hyperbolic secant (sech)	$sech(\epsilon r)$

Substituting Eq. (2.12) and Eq. (2.14) into Eq. (2.15) ,we have

$$c_f Re_{x_1}^{1/2} = \psi''(0),$$

where $Re_{x_1} = \frac{x_1 U}{\nu}$ is local Reynolds number. Thus, $\psi''(0)$ is proportional to wall skin friction. Because of their relation to physical quantities, we discuss the ψ, ψ' and $\psi''(0)$ in our results.
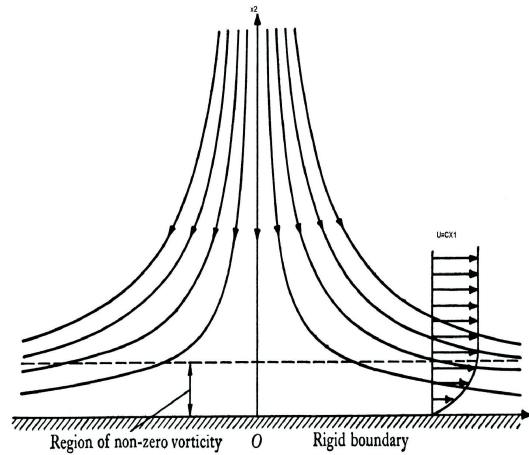
Figure 1: Hiemenz flow of density ρ impinging on a plane situated at $x_2 = 0$
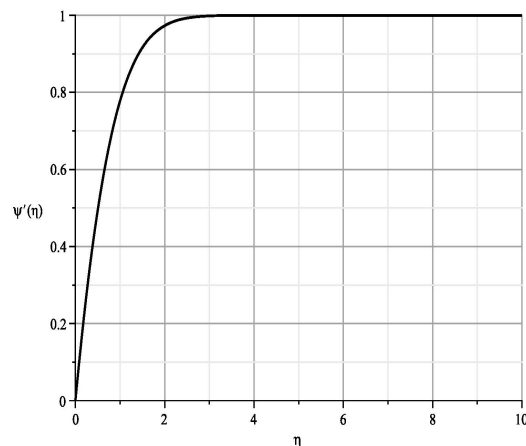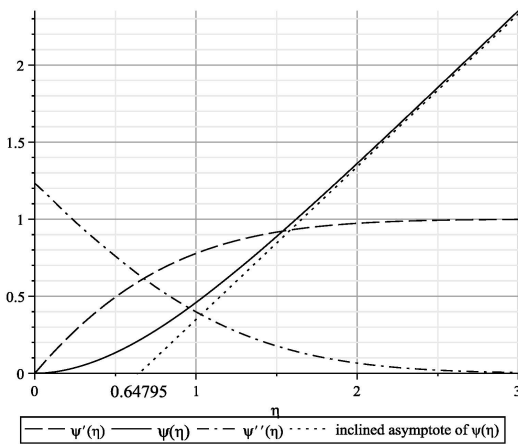

Figure 2: Graph of $\psi'(\eta)$ by using GA-RBF on uniform grid with $N = 35$ and $\epsilon = 1$

Table 2: Comparison of the some values of ψ, ψ', ψ'', for the present method(GA), on uniform grid with $N = 35$, $\epsilon = 1$, and numerical values given by Howarth [2]

η	$\psi - numerical$	$\psi - GA$	$\psi' - numerical$	$\psi' - GA$	$\psi'' - numerical$	$\psi'' - GA$
0	0	0	0	0	1.2326	1.229742
0.2	0.0233	0.023296	0.2266	0.226445	1.0345	1.034585
0.6	0.1867	0.186623	0.5663	0.566178	0.6752	0.675274
1	0.4592	0.459116	0.7779	0.777802	0.3980	0.398094
1.4	0.7967	0.796521	0.8968	0.896770	0.2110	0.211050
1.8	1.1689	1.168712	0.9568	0.956808	0.1000	0.099983
2	1.3620	1.361826	0.9732	0.973194	0.0658	0.065834
2.4	1.7553	1.755097	0.9905	0.990527	0.0260	0.026014
2.8	2.1530	2.152829	0.9970	0.997018	0.0090	0.009033
3	2.3526	2.352384	0.9984	0.998393	0.0051	0.005059

Table 3: Comparison of the some values of ψ, ψ', ψ'', for the present method (shifted point with $\sigma = 0.0555$) with $N = 24$, $\epsilon = 0.5$, and numerical values given by Howarth [2]

η	$\psi - numerical$	$\psi - GA$	$\psi' - numerical$	$\psi' - GA$	$\psi'' - numerical$	$\psi'' - GA$
0	0	0	0	0	1.2326	1.232588
0.2	0.0233	0.023355	0.2266	0.226800	1.0345	1.032368
0.6	0.1867	0.186715	0.5663	0.566324	0.6752	0.675697
1	0.4592	0.459236	0.7779	0.777840	0.3980	0.398446
1.4	0.7967	0.796657	0.8968	0.896778	0.2110	0.211040
1.8	1.1689	1.168855	0.9568	0.956811	0.1000	0.099876
2	1.3620	1.361968	0.9732	0.973193	0.0658	0.065870
2.4	1.7553	1.755238	0.9905	0.990515	0.0260	0.026008
2.8	2.1530	2.152965	0.9970	0.997001	0.0090	0.008995
3	2.3526	2.352516	0.9984	0.998379	0.0051	0.005140

Figure 3: Graph of $\psi(\eta)$, $\psi'(\eta)$, $\psi''(\eta)$ and inclined asymptote of $\psi(\eta)$ by using GA-RBF on uniform grid with $N = 35$ and $\epsilon = 1$

3   Properties of RBF

Let $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$ be the non-negative half-line and let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $\phi(0) \geq 0$. A radial basis function on $\mathbb{R}^d$ is a function of the form

$$\phi(\|\vec{X} - \vec{X}_i\|)$$

where $\vec{X}$, $\vec{X}_i \in \mathbb{R}^d$ and $\|.\|$ denotes the Euclidean distance between $\vec{X}$, $\vec{X}_i$. If one chooses N points $\{\vec{X}_i\}_{i=1}^N$ in $\mathbb{R}$ then by custom

$$s(\vec{X}) = \sum_{i=1}^N \lambda_i \phi(\|\vec{X} - \vec{X}_i\|); \quad \lambda_i \in \mathbb{R}$$

is called a radial basis function as well [22].

3.1   RBF interpolation

One dimensional function $u(x)$ to be interpolated or approximated can be represented by an RBF as:

$$u(x) \approx s(x) = \sum_{i=0}^N \lambda_i \phi_i(x) = \Phi^T(x)\Lambda \qquad (3.16)$$

where

$$\phi_i(x) = \phi(\|x - x_i\|),$$
$$\Phi^T(x) = [\phi_0(x), \phi_1(x), ..., \phi_N(x)],$$
$$\Lambda = [\lambda_0, \lambda_1, ..., \lambda_N]^T,$$

x is the input and $\{\lambda_i\}_{i=0}^N$ are the set of coefficients to be determined. By choosing $N + 1$ interpolate nodes $\{x_i\}_{i=0}^N$ in Eq. (3.16), we can approximate the function

$u(x)$ by

$$u_j = \sum_{i=0}^{N} \lambda_i \phi_i(x_j), \quad (j = 0, 1, 2, ..., N). \quad (3.17)$$

To brief discussion on coefficient matrix we define:

$$A\Lambda = U, \quad (3.18)$$

where

$$U = [u_0, u_1, ..., u_N]^T, \quad (3.19)$$

$$A = [\Phi^T(x_0), \Phi^T(x_1), ..., \Phi^T(x_N)]^T$$

$$= \begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_N(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_N(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_N) & \phi_1(x_N) & \dots & \phi_N(x_N) \end{pmatrix}. \quad (3.20)$$

Note that $\phi_i(x_j) = \phi(\|x_i - x_j\|)$ therefor we have $\phi_i(x_j) = \phi_j(x_i)$ consequently $A = A^T$.

All the infinitely smooth RBF choices are listed in Table 1 will give the coefficient matrices A in (3.20). They are symmetric and nonsingular [23], i.e. there is a unique interpolant of the form (3.16), no matter how the distinct data points are scattered in any number of space dimensions. In the cases of inverse quadratic (IQ), sech and Gaussian (GA) the matrix A is positive definite [23].

The shape parameter ϵ which is appeared in Table 1 affects both the accuracy of the approximation and the conditioning of the interpolation matrix [24]. In general, for a fixed number of N, smaller shape parameters produce the more accurate approximations, but also are associated with a poorly conditioned A. The condition number also grows with N for fixed values of the shape parameter ϵ. Small ϵ means peaked radial functions, whereas big ϵ it means flat ones. Many researchers [25, 26] have attempted to develop algorithms for selecting optimal values of the shape parameter. The optimal choice of the shape parameter is still an open question. In practice it is most often selected by brute force. Recently, Fornberg et al. [27] developed a Contour–Padé algorithm which is capable of stably computing the RBF approximation for all $\epsilon > 0$, [24].

For some key theorems regarding RBF interpolation and the convergence of its interpolation, see [23, 28, 29, 30].

4 Solving the problem by using RBF method

In this problem, we use Gaussian (GA): $2/\sqrt{\pi} \, e^{-\epsilon^2 r^2}$ which is positive definite function and can get high accurate solution [23], where $r = \|\eta - \eta_i\|$ and ϵ is a positive constants that control the widths of the basis functions, called shape parameters.
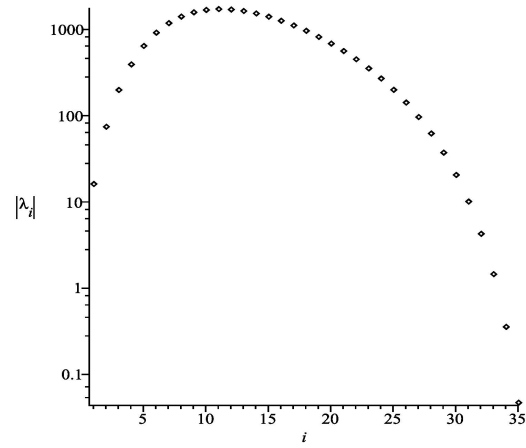

Figure 4: Graph of coefficients λ_i by using GA-RBF on uniform grid with $N = 35$ and $\epsilon = 1$
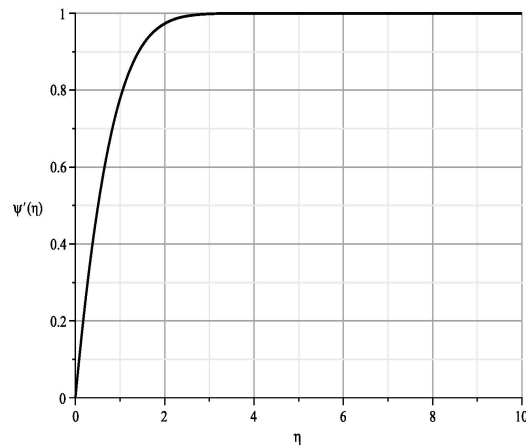

Figure 5: Graph of $\psi'(\eta)$ by using GA-RBF on uniform grid with one shifted point with $N = 24$ and $\epsilon = 0.5$

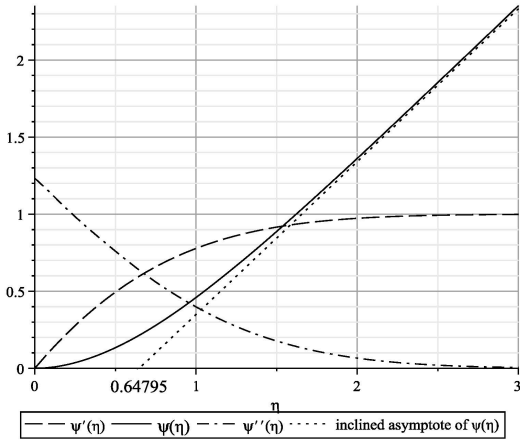Now we approximate $\psi''(\eta)$ and $\psi'''(\eta)$ as

$$\psi''(\eta) \simeq \psi''_N(\eta) = \sum_{i=0}^{N} \lambda_i \phi_i(\eta), \quad (4.21)$$

$$\psi'''(\eta) \simeq \psi'''_N(\eta) = \sum_{i=0}^{N} \lambda_i \phi'_i(\eta), \quad (4.22)$$

by using integral operation $f'(\eta)$ is obtained as

$$\int_{\eta}^{\infty} \psi''_N(t)dt = \sum_{i=0}^{N} \lambda_i \int_{\eta}^{\infty} \phi_i(t) \, dt,$$

$$\psi'_N(\infty) - \psi'_N(\eta) = \sum_{i=0}^{N} \lambda_i \int_{\eta}^{\infty} \phi_i(t) \, dt,$$

$$\psi'(\eta) \simeq \psi'_N(\eta) = \quad (4.23)$$

$$1 - \sum_{i=0}^{N} \lambda_i \int_{\eta}^{\infty} \phi_i(t) \, dt.$$

Now for obtaining $f(\eta)$ simply we choose integral operation $I(g(\eta)) = \int_0^{\eta} g(z)dz$ and by using Eq. (4.21)

Figure 6: Graph of $\psi(\eta)$, $\psi'(\eta)$, $\psi''(\eta)$ and inclined asymptote of $\psi(\eta)$ by using GA-RBF on uniform grid with one shifted point with $N = 24$ and $\epsilon = 0.5$

we have

$$\int_0^\eta \psi_N''(t)dt = \sum_{i=0}^N \lambda_i \int_0^\eta \phi_i(t) \ dt,$$

$$\psi_N'(\eta) - \psi_N'(0) = \sum_{i=0}^N \lambda_i \int_0^\eta \phi_i(t) \ dt,$$

$$\int_0^\eta \psi_N'(t)dt = \sum_{i=0}^N \lambda_i \int_0^\eta \int_0^t \phi_i(v) \ dv \ dt,$$

$$\psi(\eta) \simeq \psi_N(\eta) = \sum_{i=0}^N \lambda_i \int_0^\eta \int_0^t \phi_i(v) \ dv \ dt,$$

where $\psi(0) = \psi'(0) = 0$. Fortunately, integrals on right hand sides over the finite interval between 0 and η can be reduced to one dimensional integrals by using the formula of iterated integrals (Abramowitz and Stegun [33]) as

$$\psi(\eta) \simeq \psi_N(\eta) = \qquad (4.24)$$

$$\eta^2 \sum_{i=0}^N \lambda_i \int_0^1 t \ \phi_i(\eta(1-t))dt.$$

By substituting Equations Eqs. (4.21), (4.22), (4.23) and (4.24) in Eq. (2.11), Residual can be defined

$$Res(\eta) = \psi_N'''(\eta) + \psi_N(\eta)\psi_N''(\eta) - \psi_N'^2(\eta) + 1.$$

Now by using $N + 1$ interpolate nodes $\{\eta_j\}_{j=0}^N$, same as centers, the set of equations can be solved and consequently, the coefficients $\{\lambda_i\}_{i=0}^N$ will be obtained

$$Res(\eta_j) = 0 \quad , \quad j = 0, 1, ..., N.$$

In this method all of the boundary conditions such as infinity condition ($\psi'(\infty) = 1$) is satisfy. Also the problem is solved in semi-interval domain by using collocation points $\eta_j = jh$. Here, we set $h = 10/N$.

For the numerical solution $\psi''(0)$ is important, thus $\psi''(\eta)$ is approximated by RBF. We compare the present method by using GA-RBF by $\epsilon = 1$ and $N = 35$ with numerical solution obtained by Howarth [2] in Table 2. Graph of $\psi'(\eta)$ by using GA-RBF on uniform grid is shown by Fig. 2. Graphs of $\psi(\eta)$, $\psi'(\eta)$, $\psi''(\eta)$ and inclined asymptote of $\psi(\eta)$ by using present method on uniform grid is shown by Fig. 3. In Fig. 4 the coefficients of GA-RBF on uniform grid is shown.

To achieve more accurate approximations, we should choose smaller shape parameter and big number of N, but also both of them are converted the problem to ill-condition system. Therefore, we apply a scheme that for a fixed number of N, can choose a smaller shape parameter and achieve accurate approximations.
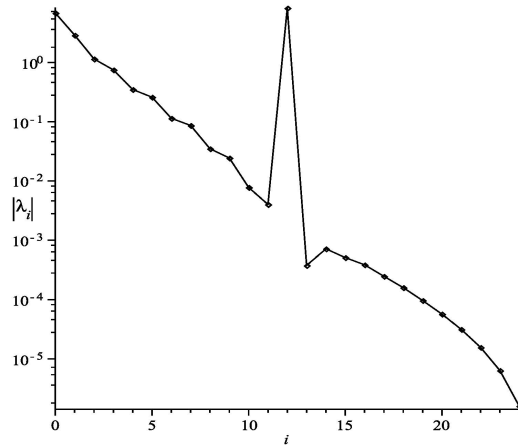
Figure 7: Graph of coefficients λ_i by using GA-RBF on uniform grid with one shifted point with $N = 24$ and $\epsilon = 0.5$
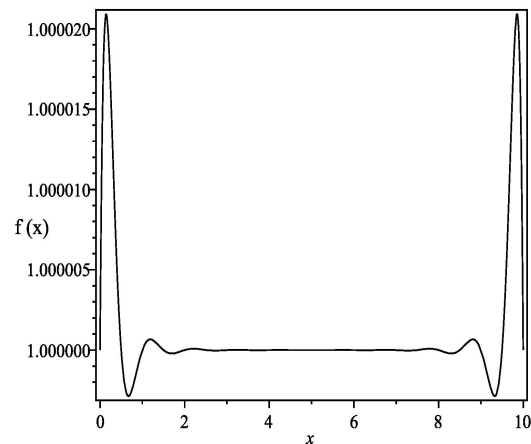
Figure 8: Graph of approximated of $f(x) = 1$ by using GA-RBF with $N = 20$ and $\epsilon = 0.5$ on uniform grid
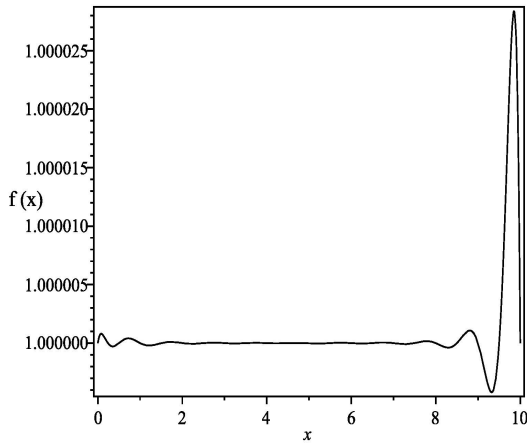
Figure 9: Graph of approximated of $f(x) = 1$ by using GA-RBF with $N = 20$ and $\epsilon = 0.5$ on uniform grid with one shifted point

4.1 **RBF with shift on one point in uniform grid**

In the first step of our work, we rearrange $\psi''(\eta)$ by RBF approximate for $x \geq 0$ as:

$$\psi''(\eta) \simeq \psi''_N(\eta, \sigma) = \sum_{i=0}^{N} \lambda'_i \phi(\eta - \eta_i),$$

$$\begin{cases} \eta_i = ih \quad, & i \neq [\frac{N}{2}], \\ \eta_i = \sigma \quad, & i = [\frac{N}{2}], \end{cases}$$

where h is the average grid spacing and $0 < \sigma < h$ is shifted parameter. Now we can show $\psi(\eta)$ as

$$\psi''(\eta) \simeq \psi''_N(\eta, \sigma)$$
$$=$$
$$\sum_{i=0}^{m-1} \lambda'_i \phi_i(\eta) + \sum_{i=m+1}^{N} \lambda'_i \phi_i(\eta) + \lambda'_m \phi_m(\eta),$$

$$(4.25)$$

where $m = [N/2]$ and $\phi_m(\eta) = \phi(\eta - \sigma)$. We apply the function (4.25) by $\epsilon = 0.5$ and $N = 24$ for solving Eq. (2.11) same as pervious section. Solving this problem by using points in uniform grid by $\epsilon = 0.5$ and $N = 24$ is converted to ill-condition system and isn't computational affordable. But using RBF with shift on one point in uniform grid is eliminated this problem. The coefficients of GA-RBF in this case, is shown by Fig. 7. The graphs illustrate that the series expansion of $\psi''(x)$ has a good convergence rate.

In the other hand, by omitting one point in middle of chosen points and adding another point near origin, we try to decrease the error near origin and access good value for $\psi''(0)$. In the simple example, we interpolate $f(x) = 1$ by using GA-RBF in two cases and show that by using nodes with shift on one point in uniform grid, the error near origin is decreased. Figs. 8 and 9 recognize this fact.

Buhmann, Wendland and Fasshauer [28, 31, 32] discuss theorems that show that RBF converge on irregular grids. Recently Boyd et al. [21] presented the theory about Sensitivity of RBF interpolation on an otherwise uniform grid with a shifted on one point.

We compare the present method by using GA-RBF with shift on one point in uniform grid, with numerical solution which is obtained by Howarth [2] in Table 3. Graph of $\psi'(\eta)$ by using present method with shift on one point in uniform grid, is shown by Fig. 5. Graphs of $\psi(\eta)$, $\psi'(\eta)$, $\psi''(\eta)$ and inclined asymptote of $\psi(\eta)$ by using present method with shift on one point in uniform grid, is shown by Fig. 6.

5   Results and Conclusions

The $x_1 - Dir$ velocity, V_1, has the same shape $\psi'(\eta)$ at each location x_1 while the magnitude increases linearly. Hence we can define a viscous diffusion length (δ). However, the 0.99% of the the maximum velocity is reached at about $\eta = 2.4$ (Tables 2 and 3) and the corresponding value of x_2, which is δ, from Eq. (2.12) is

$$\delta = 2.4 \sqrt{\frac{\nu}{c}}.$$

Thus, the δ is proportional to $\sqrt{\nu}$. From the solution we have (it is shown in Figs. 3 and 6)

$$\lim_{x_2 \to \infty} \eta(x_2) = 0.64795 + \psi(\eta). \qquad (5.26)$$

Substituting Eq. (2.13), Eq. (2.5) and Eq. (2.12) into Eq. (5.26), We have

$$\delta^* = 0.64795 \sqrt{\nu/c}.$$

Accurate numerical integration using a shooting algorithm yields the initial value $\psi''(0) = 1.232588$ [34] which we achieve it by present method on uniform grid with shift on one point (Table 3). The coefficients of present method by using GA-RBF on uniform grid with $N = 35$, and GA-RBF with shift on one point in uniform grid with $N = 24$, are shown by Figs. 4 and 7. The results are shown that for big N, $\lambda_n \longrightarrow 0$ and consequently, it leads to convergence of the method. In Fig. 7 Value of λ_{12} have a jump, because of using shift on $\eta_{12} = \sigma$ (Eq. (4.25)). Comparison between two set of coefficients show that the method with shift on one point in uniform grid is more efficient and reliable than using uniform grid.

Hiemenz flow is the two dimensional flow of fluid near a stagnation point. Because of the nonlinearities in the reduced differential equation, no analytical solution is available. The flow near a stagnation point has attracted many investigations during the past several decades because of its wide applications. The solution of this equation is obtained by using traditional

numerical methods such as finite difference, finite element and boundary element methods need generation of regular mesh points or initial guess or domain truncation of the problem which is computationally expensive and Inefficient for problems prescribed at infinity. Radial basis functions have been very effective tools to approximate the solutions of equations on meshless points without using initial guess. This method is very easy to apply and has good accuracy. In this paper we obtain new method based on RBF by using two sets of center points, the first set points on uniform grid, the second set points with shift on one point in uniform grid. Both of them have good accuracy, but using the shift on one point in uniform grid method is a powerful procedure to approximate functions which are important on origin. Additionally, high convergence rates and good accuracy are obtained by the proposed method using relatively low numbers of data points.

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