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Duality in linear interval equations

M. Movahedian [a*], S. Salahshour [b], S. Haji Ghasemi [b], S. Khezerloo [c],
M. Khezerloo [b], S. M. Khorasany [b]

(a) *Department of Science, Quchan Branch, Islamic Azad University, Quchan, Iran.*
(b) *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.*
(c) *Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.*

Abstract
In this paper we find solution of linear interval equation in dual form based on the generalized procedure of interval extension which is called *interval extended zero* method. Moreover, proposed method has significant advantage, is that it substantially decreases the excess width effect.
Keywords : Interval equation, Interval extended zero method, Interval arithmetics

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1 Introduction

At the core of many engineering problems is the solution of sets of equations and inequalities and the optimization of cost function. Unfortunately, expect in special cases such as when a set of equations is linear in its unknowns or when a convex constraint, the results obtained by conventional numerical methods are only local and can be guaranteed.
By contrast, interval analysis makes it possible to obtain guaranteed approximations of the set of the all actual solutions of the problem being considered. So, to investigate interval analysis, we should propose interval arithmetic. Interval arithmetic is an arithmetic on sets of intervals, rather than sets of real numbers.
Moreover, consider $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$ be two crisp intervals and $@ \in \{+, -, *, /\}$, then

$$[x]@[y] = \{x@y \ \forall x \in [x], \ \forall y \in [y]\} \tag{1}$$

As the direct outcome of the basic definition(1), the following expressions were obtained:

$$[x] + [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \tag{2}$$

$$[x] - [y] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}] \tag{3}$$

*Corresponding author. Email address: movahedian_10@yahoo.com(M. Movahedian).

$$[x]*[y]=[\min(\underline{x}\underline{y},\overline{x}\overline{y},\underline{x}\overline{y},\overline{x}\underline{y}),\max(\underline{x}\underline{y},\overline{x}\overline{y},\underline{x}\overline{y},\overline{x}\underline{y})] \tag{4}$$

$$[x]/[y]=[\underline{x},\overline{x}]*[1/\overline{y},1/\underline{y}] \tag{5}$$

One of the important applications of interval analysis is finding solution of interval equation. However, the classical solution too often fails to exist[1, 2, 3, 4, 5, 6]. So, numerical methods are applied to find such equations.
The interval equation $ax = bx + c$ where a, b and c be intervals is called linear interval equation in dual form. Recently, Sevastjanov and Dymova[9] proposed a new method for solving linear interval and fuzzy equations in the form $ax = b$, where a, b are interval numbers. They shown that the resulting solution of interval linear equations based on the interval extended zero method may be naturally treated as a fuzzy number.
Since there is certain pluralism when choosing an appropriate method for solving interval equations in dual form, we propose (similar[9]) to turn back to the classical approach, but looking at the problem from other point of view.
We believe that the solution of the problem is that the equations

$$ax = bx + c, \quad ax - bx - c = 0$$

where a, b, c are interval numbers, are not equivalent ones. Moreover, the main problem is that the conventional interval extension of the usual equation, which leads to the interval equation such as $ax - bx - c = 0$, is not a correct procedure, since, in the left hand side we have interval and the right hand side we have real number zero.
So, to modified such shortcomings, we develop the method proposed in [8, 9] to solve interval equations in dual form.
The rest of the paper is set out as follows. In section 2, we review briefly *interval extended zero* method for solving interval equation in dual form. Finally, some numerical examples are given.

2 Interval extended zero method

Here, we briefly describe the interval extended zero method for solving interval equations in dual form.
Let us consider interval extensions of linear equation

$$ax = bx + c \tag{6}$$

and its algebraically equivalent forms:

$$x = \frac{c}{a-b} \tag{7}$$

$$ax - bx - c = 0 \tag{8}$$

for intervals a,b and c such that $0 \notin a - b$.
Let $[a] = [\underline{a}, \overline{a}]$, $[b] = [\underline{b}, \overline{b}]$ and $[c] = [\underline{c}, \overline{c}]$ be intervals. For the sake of simplicity, let us consider the case of $[a], [b], [c] > 0$. Moreover, for describing of our solution we assumed $\underline{a} > \max\{\overline{b}, \overline{c}\}$ and $\underline{b} > \overline{c}$. Notice that later assumption is applied only for simplicity and

another cases could be easily extended based on the such assumptions.
However, the interval extension of (8) is $[\underline{a},\overline{a}][\underline{x},\overline{x}] = [\underline{b},\overline{b}][\underline{x},\overline{x}] + [\underline{c},\overline{c}]$.
By applying conventional interval arithmetic rule(4), we get:

$$[\underline{ax},\overline{ax}] = [\underline{bx},\overline{b}\overline{x}] + [\underline{c},\overline{c}] = [\underline{bx}+\underline{c},\overline{b}\overline{x}+\overline{c}]$$

Obviously, the equality is hold iff $\underline{ax} = \underline{bx} + \underline{c}$ and $\overline{ax} = \overline{b}\overline{x} + \overline{c}$. So we have:

$$\underline{x} = \frac{\underline{c}}{\underline{a}-\underline{b}},\ \ \overline{x} = \frac{\overline{c}}{\overline{a}-\overline{b}} \tag{9}$$

and as a consequence of rule(5), interval extension of Eq.(8) lead to obtain

$$\underline{x} = \frac{\underline{c}}{\overline{a}-\underline{b}},\ \ \overline{x} = \frac{\overline{c}}{\underline{a}-\overline{b}} \tag{10}$$

To illustrate the solution (4) and (5), consider three examples as following:

Example 2.1 *Let a=[5,8],b=[2,4] and c=[1,2]. Then from Eq.(9) we get* $\underline{x} = \overline{x} = \frac{1}{3}$ *and from Eq.(10)* $\underline{x} = \frac{1}{6}$ *and* $\overline{x} = 2$.

Example 2.2 *Let a=[7,8],b=[5,6] and c=[3,3.5]. Then from Eq.(9) we get* $\underline{x} = \frac{3}{2}$, $\overline{x} = \frac{3.5}{2}$ *and from Eq.(10),* $\underline{x} = 1$,$\overline{x} = 3.5$

Example 2.3 *Let a=[6,9], b=[5,5] and c=[1,2]. Then from Eq.(9) we get* $\underline{x} = 1$, $\overline{x} = 0.5$ *and from Eq.(10)* $\underline{x} = \frac{1}{4}$, $\overline{x} = 2$

We can see that interval extension of Eq.() may result in the inverted interval [x], i.e., $\overline{x} < \underline{x}$, while the extension of Eq.(8) provides correct interval($\underline{x} < \overline{x}$).
The standard interval extension of Eq.(8) is $[\underline{ax},\overline{ax}] - [\underline{bx},\overline{b}\overline{x}] - [\underline{c},\overline{c}] = 0$, by applying the interval arithmetic we get finally:

$$\underline{x} = \frac{\overline{c}}{\underline{a}-\overline{b}},\quad \overline{x} = \frac{\underline{c}}{\overline{a}-\underline{b}}.$$

It is easy to see that in any case $\underline{x} > \overline{x}$, i.e., we get an inverted interval.
Also, we define the degenerate (usual zero) as the result of the operation $a - a$, where a is any real valued number or variable. Therefore, in a similar way, we can define the interval zero as the result of the operation $[a] - [a]$, where $[a]$ is an interval. So we get $[\underline{a},\overline{a}] - [\underline{a},\overline{a}] = [\underline{a}-\overline{a},\overline{a}-\underline{a}] = [-(\overline{a}-\underline{a}),\overline{a}-\underline{a}]$.
Thus, in any case the result of the interval subtraction $[a] - [a]$ is an interval symmetrical with respect to 0.
So, by such computations, the interval extension of Eq.(8) should be as following:

$$[\underline{a},\overline{a}][\underline{x},\overline{x}] - [\underline{b},\overline{b}][\underline{x},\overline{x}] - [\underline{c},\overline{c}] = [-y,y] \tag{11}$$

where the right hand side of Eq.(11) is an interval centered around zero.
So, by using Eq.(11) we get:

$$\begin{cases} \underline{ax} - \overline{b}\overline{x} - \overline{c} = -y \\ \overline{ax} - \underline{bx} - \underline{c} = y \end{cases} \tag{12}$$

Summing the expressions in the left and right hand sides of Eq.(12) we get:

$$(\underline{a} - \underline{b})\underline{x} + (\overline{a} - \overline{b})\overline{x} - (\underline{c} + \overline{c}) = 0 \tag{13}$$

It is impossible to get a single real valued solution of (13) as it is an under-determined equation. Similar [9], Eq.(13) is applied as the so called constraint satisfaction problem[10]. In addition, by using another constraint its interval solution may be derived.
To this end, such constraint which is applied on the variables $\underline{x}$ and $\overline{x}$ is the solution of Eq.(13) by setting $\underline{x} = \overline{x}$. In this degenerate case the solution of Eq.(13) as:

$$x_m = \frac{\underline{c} + \overline{c}}{(\underline{a} - \underline{b}) + (\overline{a} - \overline{b})} \tag{14}$$

It is easy to see that x_m is the upper bound for $\underline{x}$ and the lower bound for $\overline{x}$. Also, the lower bound for $\underline{x}$ and the upper bound for $\overline{x}$ should be defined too.
So, we define the natural lower bound for $\underline{x}$ and the upper bound for $\overline{x}$ as follows: $\underline{x} = \frac{\underline{c}}{\overline{a}-\underline{b}}$, $\overline{x} = \frac{\overline{c}}{\underline{a}-\overline{b}}$. Thus, we have $[\underline{x}] = \left[\frac{\underline{c}}{\overline{a}-\underline{b}}, x_m\right]$ and $[\overline{x}] = \left[x_m, \frac{\overline{c}}{\underline{a}-\overline{b}}\right]$.
It is clear that the right bound of $\underline{x}$ and the left bound of $\overline{x}$, i.e., x_m can not b changed as they present the degenerate(real value) solution of (13). So, from (13), we get:

$$\underline{x} = \frac{c + c + (b - a)}{a - b}, \quad \overline{x} \in \left[x_m, \frac{\overline{c}}{\underline{a} - \overline{b}}\right] \tag{15}$$

$$\overline{x} = \frac{c + c + (b - a)}{a - b}, \quad \underline{x} \in \left[\frac{\underline{c}}{\overline{a} - \underline{b}}, x_m\right] \tag{16}$$

Obviously, when $\overline{x}$ is maximal in the interval $\left[x_m, \frac{\overline{c}}{\underline{a}-\overline{b}}\right]$, i.e., $\overline{x} = \frac{\overline{c}}{\underline{a}-\overline{b}}$ we get the minimal value of $\underline{x}$, i.e., $\underline{x}_{\min} = \frac{\underline{c}}{\underline{a}-\underline{b}} + \frac{\overline{c}(\underline{a}-\overline{a})}{(\underline{a}-\underline{b})(\underline{a}-\overline{b})}$.
Similarly, we get the maximal value of $\overline{x}$, i.e., $\overline{x}_{\max} = \frac{\overline{c}}{\overline{a}-\overline{b}} + \frac{\underline{c}(\overline{a}-\underline{a})}{(\overrightarrow{a}-\overline{b})(\overline{a}-\underline{b})}$, when $\underline{x} = \frac{\underline{c}}{\overline{a}-\underline{b}}$.
Generally, it is possible that $\underline{x}_{\min} < \frac{\underline{c}}{\overline{a}-\underline{b}}$ and $\overline{x}_{\max} > \frac{\overline{c}}{\underline{a}-\overline{b}}$.
So, the minimal lower bound o $\underline{x}$ and the maximal upper bound of $\overline{x}$ can be presented as following:

$$\underline{x}^L_{\min} = \min\left(\frac{\underline{c}}{\overline{a} - \underline{b}}, \frac{\underline{c}}{\underline{a} - \underline{b}} + \frac{\overline{c}(\underline{a} - \overline{a})}{(\underline{a} - \underline{b})(\underline{a} - \overline{b})}\right) \tag{17}$$

$$\overline{x}^U_{\max} = \max\left(\frac{\overline{c}}{\underline{a} - \overline{b}}, \frac{\overline{c}}{\overline{a} - \overline{b}} + \frac{\underline{c}(\overline{a} - \underline{a})}{(\overline{a} - \overline{b})(\overline{a} - \underline{b})}\right) \tag{18}$$

Therefore, we get the following interval solution of Eq.(8):

$$[\underline{x}] = \left[\underline{x}^L_{\min}, \frac{\underline{c} + \overline{c}}{(\underline{a} - \underline{b}) + (\overline{a} - \overline{b})}\right], \quad [\overline{x}] = \left[\frac{\underline{c} + \overline{c}}{(\underline{a} - \underline{b}) + (\overline{a} - \overline{b})}, \overline{x}^U_{\max}\right] \tag{19}$$

It is seen that Eq.(19) define all possible solutions of Eq.(8). The maximal interval solution width $w_{\max} = \overline{x}^U_{\max} - \underline{x}^L_{\min}$ corresponds to the maximal value of y:

$$y_{\max} = \max\left\{\frac{\overline{c}\underline{a}}{\underline{a} - \overline{b}} - \frac{\underline{c}\underline{a}}{\overline{a} - \underline{b}}\ ,\ \frac{\overline{c}\overline{a}}{\underline{a} - \overline{b}} - \frac{\underline{c}\overline{a}}{\overline{a} - \underline{b}}\right\} \tag{20}$$

and $y_{\min}$ is obtained by substituting the degenerate solution $\underline{x} = \overline{x} = x_m$ in Eq.(13) such that

$$y_{\min} = \min\left\{(\overline{b} - \underline{a})x_m + \overline{c}, (\overline{a} - \underline{b})x_m - \underline{c}\right\} \tag{21}$$

3 Conclusion

in this paper we proposed a new method to solve linear interval equation in dual form. To this end, we turn back to the classical approach but looking at the problem from the other point of view. An important of advantage of a new method is that substantially decreases the excess width effect. Moreover, our approach guaranteed that such new method always gives interval solution not inverted interval solution while, classical interval method has no such property every where.

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