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A Method to Estimate the Solution of a Weakly Singular Non-linear Integro-differential Equations by Applying the Homotopy Methods

Sh. Sadigh Behzadi[a]*, A.Yildirim [b,c]

(a) *Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran..*

(b) *Department of Mathematics, Ege University, 35100 Bornova, I zmir, Turkey.*

(c) *University of South Florida, Department of Mathematics and Statistics, Tampa, FL 33620-5700.USA.*

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Abstract

In this paper, the weakly singular nonlinear integro-differential equation is solved by using the homotopy perturbation and homotopy analysis methods . The approximation solution of this equation is calculated in the form of a series which its components are computed easily . The existence and uniqueness of the solution and the convergence of the proposed method are proved. A numerical example is studied to demonstrate the accuracy of the presented method.

Keywords : Volterra integral equations, Integro-differential equations, Singular integral equations, Homotopy analysis method (HAM), Homotopy perturbation method (HPM).

1 Introduction

Since many physical problems are modeled by integro-differential equations, the numerical solutions of such integro-differential equations have been highly studied by many authors. In recent years some works have been done in order to find the numerical solution of singular integral and integro-differential equations, for example [2, 5, 6, 8, 9, 10, 12, 16,

*Corresponding author. Email address: shadan_behzadi@yahoo.com.

22, 23, 24]. In this study, we develop HPM and HAM to solve the weakly singular nonlinear Volterra integro-differential equations as follows:

$$y^{(k)}(x) = f(x) + \mu \int_a^x \frac{1}{\sqrt{x-t}} \, G(y(t)) \, dt, \; k \geq 1, a \leq t \leq x \leq b, \tag{1.1}$$

with initial conditions

$$y^{(r)}(a) = b_r, \quad r = 0, 1, 2, \ldots, k-1, \tag{1.2}$$

where a, b, μ, b_r are constant values, $f(x), G(y(t))$ are functions which have suitable derivatives on an interval $a \leq t \leq x \leq b$.

The paper is organized as follows. In section 2, the HPM and HAM are briefly presented. In section 3, these methods are presented for solving Eq.(1.1). Also, the existence and uniqueness of the solution and convergence of the proposed methods are proved. Finally, the numerical examples and computational complexity of the proposed methods are shown in section 4.

To obtain the approximate solution of Eq.(1.1), by integrating k times from Eq.(1.1) with respect to x and using the initial conditions we obtain,

$$y(x) = F(x) + L^{-1}\left(\int_a^x \frac{1}{\sqrt{x-t}} \, G(y(t)) \, dt\right), \tag{1.3}$$

where L^{-1} is the multiple integration operator as follows:

$$L^{-1}(.) = \int_a^x \int_a^x \ldots \int_a^x (.) dx dx \ldots dx \;, \quad (k \; times).$$

and,

$$F(x) = f(x) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r.$$

The following relations have been mentioned in [25]:

$$L^{-1}\left(\int_a^x \frac{1}{\sqrt{x-t}} \, G(y(t)) \, dt\right) = \frac{1}{k!} \int_a^x (x-t)^k \frac{1}{\sqrt{x-t}} \, G(y(t)) \, dt, \tag{1.4}$$

so, we can write,

$$y(x) = F(x) + \frac{\mu}{k!} \int_a^x (x-t)^k \frac{1}{\sqrt{x-t}} \, G(y(t)) \, dt. \tag{1.5}$$

In Eq.(1.5), we assume that $F(x)$ is bounded for all x in $C = [a, b]$ and

$$\left|\frac{\mu}{k!}(x-t)^{k-\frac{1}{2}}\right| \leq M'.$$

Also, we suppose the nonlinear term $G(y(t))$ is Lipschitz continuous with

$$|G(y) - G(y^*)| \leq L'|y - y^*|$$

We set,

$$\alpha = L'M'(b-a).$$

2 Introducing homotopy

2.1 Description of the HAM

Let

$$N[y] = 0,$$

where N is a nonlinear operator, $y(x)$ is an unknown function and x is an independent variable. If $y_0(x)$ denotes an initial guess of the exact solution $y(x)$, $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[r(x)] = 0$ when $r(x) = 0$, then by considering$q \in [0, 1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1 - q)L[\phi(x; q) - y_0(x)] - qhH(x)N[\phi(x; q)] = \hat{H}[\phi(x; q); y_0(x), H(x), h, q]. \quad (2.6)$$

It should be emphasized that we have great freedom to choose the initial guess $y_0(x)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x)$.
Enforcing the homotopy Eq.(2.6) to be zero, i.e.,

$$\hat{H}[\phi(x; q); y_0(x), H(x), h, q] = 0, \quad (2.7)$$

we get the zero-order deformation equation as follows:

$$(1 - q)L[\phi(x; q) - y_0(x)] = qhH(x)N[\phi(x; q)]. \quad (2.8)$$

When $q = 0$, the zero-order deformation Eq.(2.8) becomes

$$\phi(x; 0) = y_0(x), \quad (2.9)$$

and when $q = 1$, since $h \neq 0$ and $H(x) \neq 0$, the Eq.(2.8) is equivalent to

$$\phi(x; 1) = y(x). \quad (2.10)$$

Thus, according to Eq.(2.9) and Eq.(2.10), as the embedding parameter q increases from 0 to 1, $\phi(x; q)$ varies continuously from the initial approximation $y_0(x)$ to the exact solution $y(x)$. Such a kind of continuous variation is called deformation in homotopy [1, 2, 11, 17, 18, 19, 20, 21].

Due to Taylor's theorem, $\phi(x; q)$ can be expanded in a power series of q as follows

$$\phi(x; q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)q^m, \quad (2.11)$$

where,

$$y_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x; q)}{\partial q^m} \,|_{q=0} .$$

Let the initial guess $y_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x)$ be properly chosen so that the power series Eq.(2.11) of $\phi(x;q)$ converges at $q = 1$, then, we under these assumptions the solution series is obtained:

$$y(x) = \phi(x;1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \tag{2.12}$$

From Eq.(2.11) and Eq.(2.8) can be written as follows

$$L[\sum_{m=1}^{\infty} y_m(x)\ q^m] - q\ L[\sum_{m=1}^{\infty} y_m(x) q^m] = q\ h\ H(x) N[\phi(x,q)]. \tag{2.13}$$

By differentiating Eq.(2.13) m times with respect to q, we obtain

$$m!\ L[y_m(x) - y_{m-1}(x)] = h\ H(x)\ m\ \tfrac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}\ |_{q=0}\ .$$

Therefore,

$$\begin{array}{l} L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)\Re_m(y_{m-1}(x)), \\ y_m(0) = 0, \end{array} \tag{2.14}$$

where,

$$\Re_m(y_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{2.15}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases}$$

Note that the high-order deformation Eq.(2.14) is governing the linear operator L, and the term $\Re_m(y_{m-1}(x))$ can be expressed simply by Eq.(2.15) for any nonlinear operator N.

To obtain the approximation solution of Eq.(1.1) based on the HAM let

$$N[y] = y(x) - L^{-1}(f(x)) - \textstyle\sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r - \mu L^{-1}(\int_a^x \frac{1}{\sqrt{x-t}}\ G(y(t))\ dt) = 0.$$

We obtain the term $\sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r$ from the initial conditions.

We have,

$$\begin{aligned} \Re_m(y_{m-1}(x)) &= y_{m-1}(x) - \mu L^{-1}\left(\textstyle\int_a^x \frac{1}{\sqrt{x-t}}\ G(y_{m-1}(t))\ dt\right) \\ &\quad -(1-\chi_m)\left(L^{-1}(f(x)) + \textstyle\sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r\right), \quad m \geq 1. \end{aligned} \tag{2.16}$$

Substituting Eq.(2.16) into Eq.(2.14),

$$\begin{aligned} L[y_m(x) - \chi_m y_{m-1}(x)] &= hH(x)\left[y_{m-1}(x) - \mu L^{-1}\left(\textstyle\int_a^x \frac{1}{\sqrt{x-t}}\ G(y_{m-1}(t))\ dt\right)\right. \\ &\quad \left. -(1-\chi_m)\left(L^{-1}(f(x)(x)) + \textstyle\sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r\right)\right]. \end{aligned} \tag{2.17}$$

We take an initial guess $y_0(x) = F(x) = L^{-1}(f(x)) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r$, an auxiliary linear operator $Ly = y$, a nonzero auxiliary parameter $h = -1$, and an auxiliary function $H(x) = 1$. This is substituted into Eq.(2.17) to give the recurrence relation

$$\begin{aligned} y_0(x) &= L^{-1}(f(x)) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r, \\ y_m(x) &= \int_a^x \frac{\mu}{k!}(x-t)^{k-\frac{1}{2}} G(y_{m-1}(t))\, dt),\ m \geq 1. \end{aligned} \tag{2.18}$$

Relation Eq.(2.18) will enable us to determine the components $y_m(x)$ recursively for $m \geq 0$.

2.2 Description of the HPM

To explain HPM [3, 4, 7, 13, 14, 15], we consider the following general nonlinear differential equation:

$$Ly + Ny = f(y), \tag{2.19}$$

with initial conditions

$$y_0(x) = f(x).$$

According to HPM, we construct a homotopy which satisfies the following relation

$$H(y,p) = Ly - Lv_0 + p\ Lv_0 + p\ [Ny - f(y)] = 0, \tag{2.20}$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Eq.(2.20) is expressed as

$$y(x) = y_0(x) + p\ y_1(x) + p^2\ y_2(x) + \cdots \tag{2.21}$$

Hence the approximate solution of Eq.(2.19) can be expressed as a series of the power of p, i.e.

$$y = \lim_{p \to 1} y = y_0 + y_1 + y_2 + \cdots$$

we have the recursive relation as follows:

$$\begin{aligned} & y_0(x) = F(x), \\ & \vdots \\ & y_m(x,t) = \sum_{k=1}^{m-1} \frac{\mu}{k!}(x-t)^{k-\frac{1}{2}}\ G(u_{m-k-1}(t))\ dt, \quad m \geq 1. \end{aligned} \tag{2.22}$$

3 Existence solution and convergence of iterative methods

Theorem 3.1. *The weakly singular nonlinear Volterra integro-differential equation in Eq.(1.1), has a unique solution whenever* $0 < \alpha < 1$.

Proof: Let y and y^* be two different solutions of Eq.(1.5) then

$$\begin{aligned} |y-y^*| &= \left|\int_a^x \frac{\mu(x-t)^k}{\sqrt{x-t}\ k!}[G(y)-G(y^*)]\ dt\right| \\ &\le \int_a^x \left|\frac{\mu}{k!}(x-t)^{k-\frac{1}{2}}\right|\ |G(y)-G(y^*)|\ dt \\ &\le (b-a)\ L^{'}M^{'}\ |y-y^*|, \end{aligned}$$

Then we obtain $(1-\alpha)|y-y^*| \le 0$. Since $0<\alpha<1$, so $|y-y^*|=0$. Therefore, $y=y^*$ and this completes the proof.

Theorem 3.2. *If the series solution $y(x)=\sum_{m=0}^{\infty} y_m(x)$ obtained from Eq.(2.18) by using HAM is convergent then it converges to the exact solution of the Eq.(1.1).*

Proof: We assume:

$$\widehat{G}(y(t)) = \sum_{m=0}^{\infty} G(y_m(x)), \tag{3.23}$$

where,

$$\lim_{m\to\infty} y_m(x) = 0.$$

We can write

$$\sum_{m=1}^{n}[y_m(x)-\chi_m y_{m-1}(x)] = y_1+(y_2-y_1)+\cdots+(y_n-y_{n-1}) = y_n(x). \tag{3.24}$$

Hence, from Eq.(3.23),

$$\lim_{n\to\infty} y_n(x) = 0. \tag{3.25}$$

So, using Eq.(3.25)and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[y_m(x)-\chi_m y_{m-1}(x)] = L[\sum_{m=1}^{\infty}[y_m(x)-\chi_m y_{m-1}(x)]] = 0.$$

Therefore, from Eq.(2.13) we can obtain that

$$\sum_{m=1}^{\infty} L[y_m(x)-\chi_m y_{m-1}(x)] = hH(x)\sum_{m=1}^{\infty}\Re_m(y_{m-1}(x)) = 0.$$

Since $h\neq 0$ and $H(x)\neq 0$, we have

$$\sum_{m=1}^{\infty}\Re_m(y_{m-1}(x)) = 0. \tag{3.26}$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation Eq.(3.26) and simplifying it , we have

$$\begin{aligned} \sum_{m=1}^{\infty}\Re_m(y_{m-1}(x)) &= \sum_{m=1}^{\infty}\left[y_{m-1}-\mu L^{-1}\left(\int_a^x \frac{1}{\sqrt{x-t}}\ G(y_{m-1}(t))\ dt\right)-(1-\chi_m)F(x)\right] \\ &= y(x)-F(x)-\mu L^{-1}\left(\int_a^x \frac{1}{\sqrt{x-t}}\ [\sum_{m=1}^{\infty} G(y_{m-1}(t))]\right). \end{aligned} \tag{3.27}$$

From Eq.(3.26) and Eq.(3.27), we have

$$y(x) = F(x) + \mu L^{-1}(\int_a^x \frac{1}{\sqrt{x-t}}\ \widehat{G}(y(t))\ dt),$$

therefore, $y(x)$ must be the exact solution of Eq.(1.1).

Theorem 3.3. *The series solution $y(x) = \sum_{m=0}^{\infty} y_m(x)$ obtained from Eq.(2.22) by using HPM is convergent then it converges to the exact solution of the Eq.(1.1).*

Proof: We set,

$$\begin{aligned} \phi_n(x) &= \sum_{i=1}^{n} y_i(x), \\ \phi_{n+1}(x) &= \sum_{i=1}^{n+1} y_i(x) \end{aligned}$$

so, we have

$$\begin{aligned} \mid \phi_{n+1}(x) - \phi_n(x) \mid &= \mid \phi_n + y_n - \phi_n \mid \\ &= \mid y_n \mid \\ &\leq \sum_{k=1}^{m-1} \mid \frac{\mu}{k!}(x-t)^{k-\frac{1}{2}} \mid\mid G(y_{m-k-1}(t)) \mid \ dt. \end{aligned}$$

then,

$$\sum_{n=0}^{\infty} \parallel \phi_{n+1}(x) - \phi_n(x) \parallel \leq (m-1)\alpha \mid F(x) \mid \sum_{n=0}^{\infty} \alpha^n,$$

Since $0 < \alpha < 1$, therefore,

$$\lim_{n\to\infty} y_n(x) = y(x).$$

4 Numerical example

In this section, we compute a numerical example which is solved by the HAM and HPM . The programs have been provided with mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm:
Step 1. $n \leftarrow 0$.
Step 2. Calculate the recursive relation using Eq.(2.18) for HAM and Eq.(2.22) for HPM.
Step 3. If $\mid y_{n+1} - y_n \mid < \varepsilon$ then go to step 4 else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $y(x) = \sum_{i=0}^{n} y_i$ as the approximate of the exact solution.

Lemma 4.1. *The computational complexity of the above algorithm for HAM is $O(n)$ and for HPM is $O(n^2)$.*

Proof: The number of computations including division, production, sum and subtraction.

HAM:
In step 2,
$y_1 : 5.$
.
.
$y_{n+1} : 5, \quad n \geq 0.$

In step 4, the total number of the computations is equal to
$\sum_{i=1}^{n} y_i(x,t) = 5n = O(n).$

HPM:
In step 2,
$y_1 : 5.$
.
.
$y_{n+1} : 5, \quad n \geq 0.$

In step 4, the total number of the computations is equal to
$\sum_{i=1}^{n} y_i(x,t) = 4(n-1)n = O(5n^2).$

Example 4.1. *Let us now study the nonlinear singular integro-differential equation as follows*

$$y''(x) = e^x - 3.34067 \times 10^{-16} x^{\frac{3}{2}} - 0.2\sqrt{x - 0.3} - 1.3333x + \int_{0.3}^{x} \frac{[y(t)]^2}{\sqrt{x-t}} dt,$$

with initial conditions $y(0) = 1$, $y'(0) = 1$. *The exact solution is* $y(x) = e^x$, $\alpha = 0.98$

Table 1
Numerical results for Example 4.1

x	Errors (HPM, n=6))	Errors (HAM,n=3)
0.30	0.050281	0.030281
0.35	0.054184	0.032267
0.40	0.058754	0.036754
0.45	0.062683	0.038867
0.50	0.065375	0.043578
0.55	0.067284	0.045638
0.60	0.069881	0.047245
0.65	0.072674	0.051257
0.70	0.075843	0.053897
0.75	0.077698	0.056245
0.80	0.079675	0.057895

Table 1 shows that, the approximation solution of the weakly singular nonlinear Volterra integro-differential equation is convergent with 3 iterations by using the HAM.

5 Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which converge rapidly to the exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the weakly singular nonlinear integro-differential equation . For this purpose in examples, have shown that the HAM converges more rapidly than the HPM . Also, the number of computations in HAM is less than the number of computations in HPM.

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