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The Variational Iteration Method for a Class of Tenth-Order Boundary Value Differential Equations

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Abstract

In this paper, the variational iteration method, as a well-known method for solving functional equations, has been employed to solve a class of tenth-order boundary value problems, which governs on scientific and engineering experimentations. Some special cases of the mentioned equations are solved as example to illustrate ability and reliability of the method. The results reveal that the method is very effective and convenient.

Keywords : Variational iteration method; Tenth-order boundary-value problems

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1 Introduction

The solution of nonlinear problems by analytic techniques is often rather difficult [1]. Higher-order initial-boundary value problems arise in many engineering applications [2, 3, 4, 5, 6]. If an infinite horizontal layer of fluid is heated from below, with the supposition that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modelled by tenth-order boundary value problem. When instability sets in as overstability, it is modelled by twelfth-order boundary value problem [7]. The literature on the numerical solutions of tenth-order boundary value problems and associated eigenvalue problems, is seldom. Twizell et al. [8] developed numerical methods for eighth, tenth and twelfth order eigenvalue problems arising in thermal instability. Siddiqi and Twizell [9] presented the solutions of tenth-order boundary value problems using tenth degree spline, where some unexpected results, for the solution and higher order derivatives, were obtained near the boundaries of the interval.

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In this paper, a class of tenth-order boundary-value problems is considered. Three examples, solved in [10], are dealt with again to obtain accurate results in the entire domain via the variational iteration method. Consider the following class of tenth-order boundary value problem

$$u^{(10)}(x) + \phi(x)u(x) = \psi(x), \qquad a \leq x \leq b, \tag{1.1}$$

with boundary conditions

$$\begin{array}{lllll} u(a) = A_0, & u^{(1)}(a) = A_1, & u^{(2)}(a) = A_2, & u^{(3)}(a) = A_3, & u^{(4)}(a) = A_4, \\ u(b) = B_0, & u^{(1)}(b) = B_1, & u^{(2)}(b) = B_2, & u^{(3)}(b) = B_3, & u^{(4)}(b) = B_4, \end{array}$$

where $u = u(x)$ and $\phi(x)$ and $\psi(x)$ are continuous functions defined on $[a, b]$ and the constants A_i and B_i are finite real numbers.

2 Variational iteration method

To illustrate the basic concept of variational iteration method [11, 12, 13, 14, 15, 16], we consider the following general nonlinear system

$$L[u(x)] + N[u(x)] = \psi(x),$$

where L is a linear operator, N is a nonlinear operator and $\psi(x)$ is a given continuous function. According to the variational iteration method, we can construct a correction functional in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[Lu_n(s) + N\tilde{u}_n(s) - \psi(s)]ds,$$

where $u_0(x)$ is an initial approximation with possible unknowns, λ is a Lagrange multiplier which can be identified optimally via variational theory, the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$. It is shown this method is very effective and easy for linear problem, its exact solution can be obtained by only one iteration, because λ can be exactly identified. It should be specially pointed out that the variational iteration method is a powerful method for engineering applications [17, 18, 19, 20, 21, 22].

For tenth-order boundary value problems mentioned above, according to the variational iteration method, the non-linear terms have to be considered as restricted variation. So we drive a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[u_n^{(10)}(s) + \phi(s)\tilde{u}_n(s) - \psi(s)]ds,$$

and the stationary condition of the above correction functional can be expressed as follows:

$$\begin{array}{ll} \lambda^{(10)}(s) = 0, & \\ 1 - \lambda^{(9)}(s)|_{s=x} = 0, & \\ \lambda^{(i)}(s)|_{s=x} = 0, & i = 1, 2, \ldots, 8 \\ \lambda(s)|_{s=x} = 0. & \end{array}$$

The Lagrange multiplier, therefore, can be identified as follows:

$$\lambda = \frac{1}{9!}(s-x)^9,$$

and hence, we obtain the following iteration formula

$$u_{n+1}(x) = u_n(x) + \frac{1}{9!}\int_0^x (s-x)^9[u_n^{(10)}(s) + \phi(s)u_n(s) - \psi(s)]ds. \tag{2.2}$$

3 Applications

In this section, we present three examples to show efficiency and high accuracy of the present method.

Example 3.1.([10]) Consider Eq. (1.1) with $[a,b] = [-1,1]$, $\phi(x) = -(x^2-2x)$ and $\psi(x) = 10\cos(x) - (x-1)^3\sin(x)$, and the boundary conditions

$$\begin{array}{ll} A_0 = 2\sin(1), & A_1 = (-2\cos(1) - \sin(1)), \\ A_2 = (2\cos(1) - 2\sin(1)), & A_3 = (2\cos(1) + 3\sin(1)), \\ A_4 = (-4\cos(1) + 2\sin(1)), & \\ B_0 = 0, & B_1 = \sin(1), \\ B_2 = 2\cos(1), & B_3 = -3\sin(1), \\ B_4 = -4\cos(1). & \end{array} \tag{3.3}$$

According to (2.2) we have the following iteration formula

$$u_{n+1}(x) = u_n(x) + \frac{1}{9!}\int_0^x (s-x)^9[u_n^{(10)}(s) - (s^2-2s)u_n(s) - 10\cos(s) + (s-1)^3\sin(s)]ds. \tag{3.4}$$

Now, we begin with an arbitrary initial approximation:

$$u_0(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + ix^8 + jx^9$$

where a, b, c, d, e, f, g, h, i and j are constants to be determined. By the variational iteration formula (3.4), we have

$$\begin{aligned} u_1(x) &= u_0(x) + \frac{1}{9!}\int_0^x (s-x)^9[u_0^{(10)}(s) - (s^2-2s)u_0(s) - 10\cos(s) + (s-1)^3\sin(s)]ds \\ &= (1300+a) + (b-269)x + (c-353)x^2 + (d+\frac{167}{6})x^3 + (e+\frac{41}{3})x^4 \\ &\quad +(f-\frac{89}{120})x^5 + (g-\frac{59}{360})x^6 + (h+\frac{1}{144})x^7 + (i+\frac{1}{1440})x^8 \\ &\quad +(j-\frac{1}{72576})x^9 - \frac{a}{19958400}x^{11} + (\frac{a}{239500800} - \frac{b}{119750400})x^{12} \\ &\quad +(\frac{b}{1037836800} - \frac{c}{518918400})x^{13} + (\frac{c}{3632428800} - \frac{d}{1816214400})x^{14} \\ &\quad +(\frac{d}{10897286400} - \frac{e}{5448643200})x^{15} + (\frac{e}{29059430400} - \frac{f}{14529715200})x^{16} \\ &\quad +(\frac{f}{70572902400} - \frac{g}{35286451200})x^{17} + (\frac{g}{158789030400} - \frac{h}{79394515200})x^{18} \end{aligned}$$

$$+(\frac{h}{335221286400}-\frac{i}{167610643200})x^{19}+(\frac{i}{670442572800}-\frac{j}{335221286400})x^{20}$$
$$+\frac{j}{1279935820800}x^{21}-1300\cos(x)-3\sin(x)x^2-60\cos(x)x+\sin(x)x^3$$
$$+30\cos(x)x^2-327\sin(x)x+329\sin(x).$$

Incorporating the boundary conditions, Eqs. (3.3), into $u_1(x)$, yields a system with 10 equations and 10 variables which solving this system simultaneously, we have $a=0$, $b=-1$, $c=1$, $d=\frac{1}{6}$, $e=\frac{-1}{6}$, $f=\frac{-1}{120}$, $g=\frac{1}{120}$, $h=\frac{1}{5040}$, $i=\frac{-1}{5040}$ and $j=\frac{-1}{362880}$. Thus we obtain the following first order approximate solution

$$u_1(x)=-x+x^2+1/6\,x^3-1/6\,x^4-\frac{1}{120}x^5+\frac{1}{120}x^6+\frac{1}{5040}x^7-\frac{1}{5040}x^8-\frac{1}{362880}x^9.$$

As the same, we can find $u_2(x)$ as follows:

$$\begin{aligned} u_2(x) &= u_1(x)+\frac{1}{9!}\int_0^x (s-x)^9[u_1^{(10)}(s)-(s^2-2s)u_1(s)-10\cos(s)+(s-1)^3\sin(s)]ds \\ &= -x+x^2+\frac{1}{6}x^3-\frac{1}{6}x^4-\frac{1}{120}x^5+\frac{1}{120}x^6+\frac{1}{5040}x^7-\frac{1}{5040}x^8-\frac{1}{362880}x^9 \\ &\quad +\frac{1}{362880}x^{10}+\frac{1}{39916800}x^{11}-\frac{1}{39916800}x^{12}-\frac{1}{6227020800}x^{13} \\ &\quad +\frac{1}{6227020800}x^{14}+\frac{1}{1307674368000}x^{15}-\frac{1}{1307674368000}x^{16} \\ &\quad -\frac{1}{355687428096000}x^{17}+\frac{1}{355687428096000}x^{18}+\frac{1}{121645100408832000}x^{19} \\ &\quad -\frac{1}{121645100408832000}x^{20}+O\left(x^{21}\right). \end{aligned}$$

This gives the solution in a closed form by $(x-1)\sin(x)$.

Example 3.2.([10]) in this example consider Eq. (1.1)with$[a,b]=[-1,1]$, $\phi(x)=-x$ and $\psi(x)=-(55+17^x+x^2-x^3)e^x$, with boundary conditions

$$\begin{array}{llll} A_0=0, & A_2=2/e, & A_4=-4/e, & A_6=-18/e, \\ B_0=0, & B_2=-6e, & B_4=-20e, & B_6=-42e. \end{array} \tag{3.5}$$

According to (2.2) we have the following iteration formula

$$u_{n+1}(x)=u_n(x)+\frac{1}{9!}\int_0^x (s-x)^9[u_n^{(10)}(s)-su_n(s)+(55+17s+s^2-s^3)e^s]ds. \tag{3.6}$$

Now, we begin with an arbitrary initial approximation:

$$u_0(x)=a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7+ix^8+jx^9,$$

where a, b, c, d, e, f, g, h, i and j are constants to be determined. By the variational iteration formula (3.6), we have

$$\begin{aligned} u_1(x) &= u_0(x) + \frac{1}{9!}\int_0^x (s-x)^9[u_0^{(10)}(s) - su_0(s) + (55+17s+s^2-s^3)e^s]ds \\ &= 1309 + a + (980+b)\,x + \left(c + \frac{713}{2}\right)x^2 + \left(\frac{251}{3} + d\right)x^3 + \left(\frac{341}{24} + e\right)x^4 \\ &+ \left(f + \frac{28}{15}\right)x^5 + \left(g + \frac{29}{144}\right)x^6 + \left(\frac{7}{360} + h\right)x^7 + \left(\frac{11}{5760} + i\right)x^8 \\ &+ \left(\frac{19}{90720} + j\right)x^9 + \frac{1}{39916800}\,x^{11}a + \frac{1}{239500800}\,x^{12}b + \frac{1}{1037836800}\,x^{13}c \\ &+ \frac{1}{3632428800}\,x^{14}d + \frac{1}{10897286400}\,x^{15}e + \frac{1}{29059430400}\,x^{16}f \\ &+ \frac{1}{70572902400}\,x^{17}g + \frac{1}{158789030400}\,x^{18}h + \frac{1}{335221286400}\,x^{19}i \\ &+ \frac{1}{670442572800}\,x^{20}j - 1309\,e^x + e^x x^3 - 31\,e^x x^2 + 329\,e^x x. \end{aligned}$$

Incorporating the boundary conditions, Eqs. (3.5), into $u_1(x)$, yields a system with 10 equations and 10 variables which solving this system simultaneously, we have $a = 1,\ b = 1,\ c = \frac{-1}{2},\ d = \frac{-5}{6},\ e = \frac{-11}{24},\ f = \frac{-19}{120},\ g = \frac{-29}{720},\ h = \frac{-41}{5040},\ i = \frac{-11}{8064}$ and $j = \frac{-71}{362880}$. Thus we obtain the following first order approximate solution

$$u_1(x) = 1 + x - 1/2\,x^2 - 5/6\,x^3 - \frac{11}{24}\,x^4 - \frac{19}{120}\,x^5 - \frac{29}{720}\,x^6 - \frac{41}{5040}\,x^7 - \frac{11}{8064}\,x^8 - \frac{71}{362880}\,x^9.$$

As the same, we can find $u_2(x)$ as follows

$$\begin{aligned} u_2(x) &= u_1(x) + \frac{1}{9!}\int_0^x (s-x)^9[u_1^{(10)}(s) - su_1(s) + (55+17s+s^2-s^3)e^s]ds \\ &= 1 + x - \frac{1}{2}x^2 - \frac{5}{6}x^3 - \frac{11}{24}x^4 - \frac{19}{120}x^5 - \frac{29}{720}x^6 - \frac{41}{5040}x^7 - \frac{11}{8064}x^8 \\ &- \frac{71}{362880}x^9 - \frac{89}{3628800}x^{10} - \frac{109}{39916800}x^{11} - \frac{131}{479001600}x^{12} \\ &- \frac{31}{1245404160}x^{13} - \frac{181}{87178291200}x^{14} - \frac{19}{118879488000}x^{15} \\ &- \frac{239}{20922789888000}x^{16} - \frac{271}{355687428096000}x^{17} - \frac{61}{1280474741145600}x^{18} \\ &- \frac{31}{11058645491712000}x^{19} - \frac{379}{2432902008176640000}x^{20} + O\left(x^{21}\right). \end{aligned}$$

This gives the solution in a closed form by $(1-x^2)e^x$.

Example 3.3.([10]) Consider Eq. (1.1) with $[a,b] = [-1,1]$, $\phi(x) = 1$ and $\psi(x) = -10(2x\sin(x) - 9\cos(x))$, and the boundary conditions

$$\begin{array}{ll} A_0 = 0, & A_1 = -2\cos(1), \\ A_2 = 2\cos(1) - 4\sin(1), & A_3 = 6\cos(1) + 6\sin(1), \\ A_4 = (-12\cos(1) + 8\sin(1)), & \\ B_0 = 0, & B_1 = 2\cos(1), \\ B_2 = 2\cos(1) - 4\sin(1), & B_3 = -6\cos(1) - 6\sin(1), \\ B_4 = -12\cos(1) + 8\sin(1). & \end{array} \tag{3.7}$$

According to (2.2) we have the following iteration formula

$$u_{n+1}(x) = u_n(x) + \frac{1}{9!}\int_0^x (s-x)^9[u_n^{(10)}(s) + u_n(s) + 10(2s\ \sin(s) - 9\cos(s))]ds. \quad (3.8)$$

Now, we begin with an arbitrary initial approximation:

$$u_0(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + ix^8 + jx^9,$$

where $a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i$ and j are constants to be determined. By the variational iteration formula (3.8), we have

$$\begin{aligned} u_1(x) &= u_0(x) + \frac{1}{9!}\int_0^x (s-x)^9[u_0^{(10)}(s) + u_0(s) + 10(2s\ \sin(s) - 9\cos(s))]ds \\ &= -110 + a + bx + (c+35)\,x^2 + dx^3 + \left(e - \frac{5}{4}\right)x^4 + fx^5 + \left(-\frac{1}{72} + g\right)x^6 \\ &\quad + hx^7 + \left(i + \frac{5}{4032}\right)x^8 + jx^9 - \frac{1}{3628800}\,ax^{10} - \frac{1}{39916800}\,bx^{11} \\ &\quad - \frac{1}{239500800}\,cx^{12} - \frac{1}{1037836800}\,dx^{13} - \frac{1}{3632428800}\,ex^{14} \\ &\quad - \frac{1}{10897286400}\,fx^{15} - \frac{1}{29059430400}\,gx^{16} - \frac{1}{70572902400}\,hx^{17} \\ &\quad - \frac{1}{158789030400}\,ix^{18} - \frac{1}{335221286400}\,jx^{19} + 110\,\cos(x) + 20x\,\sin(x)\,. \end{aligned}$$

Incorporating the boundary conditions, Eq. ((3.7)), into $u_1(x)$, yields a system with 10 equations and 10 variable which solving this system simultaneously, we have $a = -1,\ b = 0,\ c = \frac{3}{2},\ d = 0,\ e = \frac{-13}{24},\ f = 0,\ g = \frac{31}{720},\ h = 0,\ i = \frac{-19}{13440}$ and $j = 0$. thus we obtain the following first order approximate solution

$$u_1(x) = -1 + 3/2\,x^2 - \frac{13}{24}\,x^4 + \frac{31}{720}\,x^6 - \frac{19}{13440}\,x^8$$

similarly we can find $u_2(x)$ as follows

$$\begin{aligned} u_2(x) &= u_1(x) + \frac{1}{9!}\int_0^x (s-x)^9[u_1^{(10)}(s) + u_1(s) + 10(2s\ \sin(s) - 9\cos(s))]ds \\ &= -1 + \frac{3}{2}x^2 - \frac{13}{24}x^4 + \frac{31}{720}x^6 - \frac{19}{13440}x^8 + \frac{13}{518400}x^{10} - \frac{19}{68428800}x^{12} \\ &\quad + \frac{61}{29059430400}x^{14} - \frac{241}{20922789888000}x^{16} + \frac{307}{6402373705728000}x^{18} + O\left(x^{20}\right). \end{aligned}$$

This gives the solution in a closed form by $(x^2 - 1)\ \cos(x)$.

4 Conclusions

In this paper, we have studied some tenth-order boundary value problems with the variational iteration method. The initial approximation was selected as a polynomial with unknown constants, which was determined by considering the boundary conditions. The results reveal that the method is remarkably effective. This method is a very promoting method, which will be certainly found widely applications.

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