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On the Laplace Transform Decomposition Algorithm for Solving Nonlinear Differential Equations

A. R. Vahidi *, Gh. Asadi Cordshooli

Department of Mathematics, Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran.

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Abstract

In this paper, we present a comparative study between the Adomian decomposition method (ADM) and the Laplace transform decomposition algorithm (LTDA) for solving nonlinear differential equations. For the Bratu's boundary value problem and the Duffing's equation, we show that the LTDA is equivalent to the ADM.

Keywords : Adomian decomposition method; Laplace transform decomposition algorithm; Bratu's problem; Duffing's equation

1 Introduction

Sometimes using several numerical methods to solve a nonlinear problem may give similar results. It is noticeable that applying different numerical methods to solve a problem may provide just the same results. For example it is shown that using the ADM and successive approximation method for linear integral equations [8], give just the same results and also it will be hold for the ADM and the power series method for differential equations [9], and the ADM and the Jacobi iterative method for system of linear equation [6].

From the beginning of 1980's that George Adomian introduced his decomposition method, this method has been applied to a wide class of functional equations [1, 2] and it is demonstrated that the method provides accurate and computable solutions for a wide class of linear or nonlinear functional equations.

The LTDA is an approach based on the ADM, which is considered as an effective method in solving many problems because it provides, in general, a rapidly convergent series solution. Since the Laplace transform converts the differentiation to simple algebraic operations and as the algebraic equations are solvable by the ADM, we can combine the

*Corresponding author. Email address: alrevahidi@yahoo.com.

Laplace transform and the ADM to solve differential equations. The LTDA approximates the exact solution with a high degree of accuracy using only few terms of the iterative scheme [5]. Many authors have used this method to solve the Bratu's equation [7], the Duffing equation [11], and integro-differential equation [10].

In this paper, we use both introduced methods to solve two famous and nonlinear problems, namely Bratu and Duffing equations. We will show that these methods are equivalent.

2 Bratu's boundary value problem

In this section, we apply the ADM and the LTDA for the Bratu's boundary value problem and show that the results are exactly the same.

2.1 ADM for solving the Bratu's boundary value problem

Consider the Bratu's boundary value problem as follow

$$-u''(x) = \lambda e^{u(x)} \quad ; \quad u(0) = 0, \quad u(1) = 0, \quad \lambda > 0. \tag{2.1}$$

Denoting $\frac{d^2}{dx^2}$ by G, we have G^{-1} as two-fold integration from 0 to x, through which the differential equation in (2.1), can be written as

$$Gu(x) = -\lambda e^{u(x)}. \tag{2.2}$$

After operating with the inverse operator G^{-1}, substituting the initial condition $u(0) = 0$ and considering $u'(0) = k$, one gets

$$u(x) = kx + G^{-1}(-\lambda e^{u(x)}). \tag{2.3}$$

In the ADM, the solution $u(x)$ is considered as an infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{2.4}$$

and the nonlinear part of the equation (2.3) is replaced by

$$N(u(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{2.5}$$

where $A_n's$ are the Adomian's polynomials that can be calculated by the following formula

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty} \lambda^i u_i(x))]_{\lambda=0}, \quad n = 0, 1, 2, \ldots. \tag{2.6}$$

Substituting (2.4) and (2.5) into (2.3), we obtain

$$\sum_{n=0}^{\infty} u_n(x) = u(0) + u'(0)x - \lambda L^{-1}(\sum_{n=0}^{\infty} A_n). \tag{2.7}$$

Considering the initial conditions given in the problem (2.1), we can choose the first term of the series (2.4) as follows:

$$u_0(x) = kx, \tag{2.8}$$

and its other terms will be obtained by the recursion relation

$$u_{i+1}(x) = -\lambda \int_0^x \int_0^x A_i dx dx. \tag{2.9}$$

The components of $u_i(x)$ for $n = 1, 2, 3$ can be obtained using (2.12) as

$$u_1(x) = \lambda(\frac{1}{k^2} + \frac{x}{k} - \frac{e^{kx}}{k^2}), \tag{2.10}$$

$$u_2(x) = -\lambda^2(\frac{5}{4k^2} + \frac{x}{2k^3} - \frac{e^{kx}}{k^4} + \frac{xe^{kx}}{k^3} - \frac{e^{2kx}}{4k^4}), \tag{2.11}$$

$$u_3(x) = 2\lambda^3(-\frac{e^{3kx}}{24k^6} - \frac{e^{2kx}}{4k^6} + \frac{xe^{2kx}}{4k^5} - \frac{5e^{kx}}{8k^6} + \frac{3xe^{kx}}{4k^5} - \frac{x^2e^{kx}}{4k^4} + \frac{11}{12k^6} + \frac{x}{4k^5}). \tag{2.12}$$

Similarly, the components $u_n(x)$ are calculated for $n = 4, 5, \ldots$ but they are not listed here for brevity.

2.2 LTDA for solving Bratu's problem

S. A. Khuri in [7] applied the LTDA to solve the equation (1). We describe this procedure briefly as follow.

Operating both sides of the differential equation given in (2.1) by Laplace transform integral operator (denoted throughout this paper by L), gives

$$L[u''(x)] = -\lambda L[e^{u(x)}]. \tag{2.13}$$

Applying the formulas of the Laplace transform, tends to

$$s^2 L[u(x)] - u(0)s - u'(0) = -\lambda L[e^{u(x)}]. \tag{2.14}$$

Using the initial condition $u(0) = 0$ and considering $u'(0) = k$, yields

$$s^2 L[u(x)] = k - \lambda L[e^{u(x)}], \tag{2.15}$$

that can be solved for $L[u]$ as

$$L[u(x)] = \frac{k}{s^2} - \frac{\lambda}{s^2} L[e^{u(x)}]. \tag{2.16}$$

Now, the Bratu's differential equation is converted to the algebraic equation (2.16) that will be solved using ADM. By substituting (2.4) and (2.5) into (2.16), we obtain

$$L[\sum_{n=0}^{\infty} u_n(x)] = \frac{k}{s^2} - \frac{\lambda}{s^2} L[\sum_{n=0}^{\infty} A_n(x)]. \tag{2.17}$$

We can replace the Laplace transform by the summation and since Laplace transform is a linear operator, the following result will be hold

$$\sum_{n=0}^{\infty} L[u_n(x)] = \frac{k}{s^2} - \frac{\lambda}{s^2}\sum_{n=0}^{\infty} L[A_n(x)]. \tag{2.18}$$

Choosing $L[u_0] = \frac{k}{s^2}$ and $L[u_{i+1}(x)] = -\frac{\lambda}{s^2}L[A_i(x)]$ and calculating the inverse of the Laplace transform, we obtain

$$u_0(x) = kx, \tag{2.19}$$

$$u_1(x) = \lambda(\frac{1}{k^2} + \frac{x}{k} - \frac{e^{kx}}{k^2}), \tag{2.20}$$

$$u_2(x) = -\lambda^2(\frac{5}{4k^2} + \frac{x}{2k^3} - \frac{e^{kx}}{k^4} + \frac{xe^{kx}}{k^3} - \frac{e^{2kx}}{4k^4}), \tag{2.21}$$

$$u_3(x) = 2\lambda^3(-\frac{e^{3kx}}{24k^6} - \frac{e^{2kx}}{4k^6} + \frac{xe^{2kx}}{4k^5} - \frac{5e^{kx}}{8k^6} + \frac{3xe^{kx}}{4k^5} - \frac{x^2e^{kx}}{4k^4} + \frac{11}{12k^6} + \frac{x}{4k^5}), \tag{2.22}$$

The components $u_n(x)$ are calculated for $n = 4, 5, \ldots$ but for brevity they will not be listed. Comparing (2.8)- (2.12) by (2.19)-(2.22), one by one, shows that $u_i(x)$ for $i = 0, 1, ..., 3$ obtained by the LTDA are just as the same terms obtained by the ADM.

3 Duffing problem

In this section, we apply the methods ADM and LTDA for a spicial version of the Duffing equation. The Duffing equation

$$y'' + 3y - 2y^3 = cosxsin2x, \tag{3.23}$$

with the initial conditions $y(0) = y'(0) = 1$ [4] is solved using ADM and LTDA in this section.

3.1 ADM for solving the Duffing equation

Denoting $\frac{d^2}{dt^2}$ by G, we have G^{-1} as a two-fold integration given by $G^{-1}(.) = \int_0^x \int_0^x (.)dxdx$. Using the operator G, the equation (3.23) becomes

$$Gy = f(x) - 3y + 2y^3, \tag{3.24}$$

where $f(x)$ has seven terms of the Taylor expansion of excitation term about $x = 0$ as follow

$$f(x) = cosxsin2x \simeq 1 - x - \frac{3x^2}{2} + \frac{x^3}{6} + \frac{7x^4}{8} - \frac{x^5}{120} - \frac{61x^6}{240}. \tag{3.25}$$

Applying the inverse operator G^{-1} on both sides of (3.24) and considering the initial condition, yield

$$y = t + G^{-1}[f(x)] - 3G^{-1}[y] + 2G^{-1}[y^3]. \tag{3.26}$$

In order to apply the ADM let

$$y = \sum_{n=0}^{\infty} y_n, \tag{3.27}$$

and

$$N(y) = y^3 = \sum_{n=0}^{\infty} A_n, \tag{3.28}$$

where A_n's are the Adomian polynomials depending on $y_0, y_1, \ldots, y_n$. Replacing (3.27) and (3.28) into (3.26), we obtain

$$y_0 = t + G^{-1}f(x) = x + \frac{1}{3} - \frac{7}{60}x^5 + \frac{61}{2520}x^7 - \frac{547}{181440}x^9 \tag{3.29}$$

and the recurrence relation

$$y_{n+1} = -3G^{-1}[y_n] + 2G^{-1}[A_n], \quad n = 0, 1, 2, \ldots \tag{3.30}$$

using (3.30), we can obtain the components of y_i as follows:

$$y_1 = \frac{-1}{2}x^3 + \frac{1}{20}x^5 + \frac{47}{840}x^7 - \frac{89}{60480}x^9 + \ldots \tag{3.31}$$

$$y_2 = \frac{3}{40}x^5 - \frac{3}{40}x^7 - \frac{523}{20160}x^9 + \ldots \tag{3.32}$$

$$y_3 = \frac{-3}{560}x^7 + \frac{29}{960}x^9 + \ldots \tag{3.33}$$

Similarly, the components y_n are calculated for $n = 3, 4, \ldots$ that was skipped to be listed here.

3.2 LTDA for solving the Duffing problem

E. Yusufoglu in [11] used the LTDA to solve the Duffing equation that is explained briefly in this subsection. Operating both sides of differential equation given in (3.23) by the Laplace transform integral operator gives

$$L[y''] + L[3y] - L[2y^3] = L[f(x)]. \tag{3.34}$$

Applying the the Laplace transform formulas and using the initial conditions, one gets

$$s^2L[y] - 1 + 3L[y] - 2L[y^3] = L[f(x)], \tag{3.35}$$

that can be solved for $L[y]$ as

$$L[y] = \frac{1}{s^2} - \frac{3}{s^2}L[y] + \frac{2}{s^2}L[y^3] + \frac{1}{s^2}L[f(x)]. \tag{3.36}$$

Substituting (3.27) and (3.28) into (3.36), we obtain

$$L[\sum_{n=0}^{\infty} y_n] = \frac{1}{s^2} - \frac{3}{s^2}L[\sum_{n=0}^{\infty} y_n] + \frac{2}{s^2}L[\sum_{n=0}^{\infty} A_n] + \frac{1}{s^2}L[f(x)]. \tag{3.37}$$

Matching both sides of (3.37), the following iterative algorithm will be obtained

$$L[y_0] = \frac{1}{s^2} + \frac{1}{s^2}L[f(x)], \tag{3.38}$$

$$L[y_{i+1}] = -\frac{3}{s^2}L[y_i] + \frac{2}{s^2}L[A_i], \quad i = 0, 1, 2, \ldots \tag{3.39}$$

Calculating the inverse of the Laplace transforms in (3.38) and three first terms of (3.39), we obtain

$$y_0 = x + \frac{1}{3} - \frac{7}{60}x^5 + \frac{61}{2520}x^7 - \frac{547}{181440}x^9 \tag{3.40}$$

$$y_1 = \frac{-1}{2}x^3 + \frac{1}{20}x^5 + \frac{47}{840}x^7 - \frac{89}{60480}x^9 + \ldots \tag{3.41}$$

$$y_2 = \frac{3}{40}x^5 - \frac{3}{40}x^7 - \frac{523}{20160}x^9 + \ldots \tag{3.42}$$

$$y_3 = \frac{-3}{560}x^7 + \frac{29}{960}x^9 + \ldots. \tag{3.43}$$

In a similar manner, the components y_i are calculated for $i = 4, 5, \ldots$ but for brevity they will not be listed. Note that y_0, y_1, y_2, y_3, and y_4 obtained by the ADM as given in equations (3.29), (3.31), (3.32), and (3.33) are equal to similar terms obtained by the LTDA as can be seen in (3.40)-(3.43) . This equality for other terms takes place, too.

4 Conclusion

In this paper, the ADM and LTDA were applied for solving the Bratu's boundary value problem and the Duffing's equation. We showed that the ADM is equivalent to the LTDA from the point of view of estimate the solution for the Bratu's boundary value problem and the Duffing's problem.

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