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Addition of Two Generalized Fuzzy Numbers

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Abstract

The objective of this paper is to develop arithmetic operations between generalized trapezoidal (triangular) fuzzy numbers so that the drawbacks of the existing works are overcome. In this respect, the extension principle has been used to calculate different arithmetic operations.

Keywords : Generalized fuzzy number, Extension principle, Function principle

1 Introduction

Present-day science and technology are featured with a complex process and phenomena for which complete information is not always available. For such situations, mathematical models have to be set up using the available data which is only approximately known. To make this possible Zadeh [25] introduced fuzzy set theory. In recent years, this theory has emerged as an interesting branch of pure and applied sciences [1, 18, 20]. In 1985 and 1999, Chen [2, 3] further developed the theory and possible applications of generalized fuzzy numbers. In the paper [2, 3] different arithmetic operations on generalized fuzzy numbers were formulated by proposing the function principle. In 1996, Chen et al. [11] mentioned that the function principle was proposed in order to overcome the complications arising due to the use of extension principle [14, 26]. It was also mentioned in [8] that "the method known as the function principle is more useful than the extension principle for the fuzzy numbers with trapezoidal membership functions". There are few literatures [4–13, 15–17, 19, 21–24] involving generalized fuzzy numbers theory and applications based on Chen's arithmetic operations.

Though the function principle was used to develop arithmetic operations on generalized fuzzy numbers, in practice it has been realized that there are some shortcomings of Chen's

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method. From mathematical point of view it can be said that, computing different arithmetic operations using function principle is basically a point wise operation (addition, subtraction, multiplication and division). Due to this reason, it has been observed that arithmetic operations of generalized trapezoidal (triangular) fuzzy numbers with function principle cause the loss of information and do not give exact results. This motivates us to correct the results of arithmetic operations of generalized trapezoidal (triangular) fuzzy numbers.

The structure of this paper has been organized as follows. In section 2, the basic arithmetic operations between generalized trapezoidal fuzzy numbers have been reviewed. This section also give the examples to show that the existing operations make some approximation, thereby causes loss of information. In section 3 the extension principle has been used to correct the result. Numerical examples have been given in section 4. The paper has been concluded in Section 5.

2 The fuzzy arithmetic operations with function principle

In this section the arithmetic operations between generalized trapezoidal fuzzy numbers are reviewed from [2, 3, 15].

Let us consider $\tilde{A}_1 = (a_1, b_2, c_1, d_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers.

(i) The addition of $\tilde{A}_1$ and $\tilde{A}_2$:

- $\tilde{A}_1(+)\tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min(w_1, w_2))$.

(ii) The subtraction of $\tilde{A}_1$ and $\tilde{A}_2$:

- $\tilde{A}_1(-)\tilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \min(w_1, w_2))$.

(iii) The multiplication of $\tilde{A}_1$ and $\tilde{A}_2$:

- $\tilde{A}_1(\times)\tilde{A}_2 = (a_1 \times a_2, b_1 \times b_2, c_1 \times c_2, d_1 \times d_2; \min(w_1, w_2))$, if $a_1, b_2, c_1, d_1, a_2, b_2, c_2, d_2$ are all positive real numbers.

(iv) The division, $\tilde{A}_1$ is divided by $\tilde{A}_2$:

- $\tilde{A}_1(/)\tilde{A}_2 = (a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2;\ \min(w_1, w_2))$, if $a_1, b_2, c_1, d_1, a_2, b_2, c_2, d_2$ are all nonzero positive real numbers.

It has been found that most of the work [4–13, 15–17, 19, 21–24] done using generalized fuzzy numbers is based on Chen's arithmetic operations. Therefore, the importance of these operations and the high number of citations of Chen's [2, 3] work related to this, warrants a thorough study of their work. In course of this study some shortcomings of Chen's method have been observed which have been illustrated with the help of following example:

2.1 Example

Let us consider two generalized triangular fuzzy numbers of the following form: $\tilde{A}_1 = (0.7, 0.8, 0.9; 0.5)$ and $\tilde{A}_2 = (0.8, 0.9, 1.0; 1.0)$. After performing Chen's addition operation (defined above) between $\tilde{A}_1$ and $\tilde{A}_2$, the result has been obtained as $\tilde{A}_1(+)\tilde{A}_2 = \tilde{B} = (1.5, 1.7, 1.9; 0.5)$, which is a generalized triangular fuzzy number. In our opinion this result does not give the exact value. The result after performing Chen's addition operation has been illustrated with the help of Figure 2.1 given below.

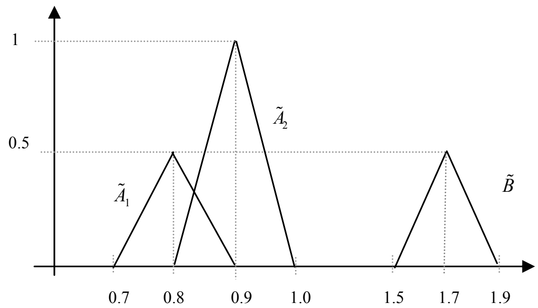

Fig. 1. Sum of two generalized triangular fuzzy numbers

It may be observed from the above figure that, $\min(1, 0.5) = 0.5$. If the both fuzzy numbers are taken to the same level by "truncating the higher one" i.e. if we take 0.5(since $0.5 < 1.0$) cut of $\tilde{A}_2$ then $\tilde{A}_2$ is transformed into a generalized trapezoidal (flat) fuzzy number. Therefore, it is necessary to conserve this flatness into the resultant generalized fuzzy number. In this respect Chen's approach is incomplete and hence loses its significance. In view of this, there is a need to develop arithmetic operations for generalized fuzzy numbers, which calculates the result more appropriately.

Therefore, in the next sections the aim is to develop the arithmetic operations between generalized fuzzy numbers based on the extension principle to improve the method.

3 Operations on generalized fuzzy numbers

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2; w_2)$ be two generalized flat fuzzy numbers with membership functions $\mu_{\tilde{A}_1}$ and $\mu_{\tilde{A}_2}$ respectively, which can be written in the following form:

$$\mu_{\tilde{A}_1}(x) = \begin{cases} 0 & \text{if } -\infty < x \leq a_1 \\ w_1 f_a(x) & \text{if } \ a_1 \leq x \leq b_1 \\ w_1 & \text{if } \ b_1 \leq x \leq c_1 \\ w_1 h_a(x) & \text{if } \ c_1 \leq x \leq d_1 \\ 0 & \text{if } \ \ d_1 < x \leq \infty \end{cases}$$

$$\mu_{\tilde{A}_2}(y) = \begin{cases} 0 & \text{if } -\infty < y \le a_2 \\ w_2 f_b(y) & \text{if } \ a_2 \le y \le b_2 \\ w_2 & \text{if } \ b_2 \le y \le c_2 \\ w_2 h_b(y) & \text{if } \ c_2 \le y \le d_2 \\ 0 & \text{if } \ \ d_2 < y \le \infty \end{cases}$$

Here, $f_a(x)$ is strictly increasing from the interval $[a_1, b_1]$ to the interval $[0, 1]$ and $h_a(x)$ is strictly decreasing from the interval $[c_1, d_1]$ to the interval $[0, 1]$. Similarly $f_b(y)$ is strictly increasing from $[a_2, b_2]$to the interval $[0, 1]$ and $h_b(y)$ is strictly decreasing from $[c_2, d_2]$ to the interval $[0, 1]$.

The membership function $\mu_{\tilde{C}}$ of their composition $\tilde{A}_1(*)\tilde{A}_2$, (where $(*)$ is an extended binary operation to combine two generalized fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$) is defined using Zadeh's extension principle, [14, 26] as follows:

$$\mu_{\tilde{C}}(z) = \sup_{z=x*y} \min\{\mu_{\tilde{A}_1}(x), \mu_{\tilde{A}_2}(y)\}$$

If $\tilde{A}_1(*)\tilde{A}_2$ is denoted by $\tilde{C}$ from the above definition it can be said that $\tilde{C}$ will be a generalized fuzzy number with confidence level $w_c = \min(w_1, w_2)$. Different arithmetic operations between $\tilde{A}_1$ and $\tilde{A}_2$ have been established with the following theorems.

3.1 Sum of two generalized fuzzy numbers

Let $\tilde{A}_1$ and $\tilde{A}_2$ be two generalized flat fuzzy numbers, as defined above. If $\tilde{C} = \tilde{A}_1(+)\tilde{A}_2$, then the following relation can be written

$$\mu_{\tilde{C}}(z) = \sup_{z=x+y} \min\{\mu_{\tilde{A}_1}(x), \mu_{\tilde{A}_2}(y)\}$$

Theorem 3.1. *Let $\mu_{\tilde{C}}$ be the membership function of $\tilde{A}_1(+)\tilde{A}_2$; then*

$$\mu_{\tilde{C}}(z) = \begin{cases} 0 & \forall z \in (-\infty, a_1 + a_2] \cup [d_1 + d_2, \infty) \\ [0, w_c] & \forall z \in [a_1 + a_2, f_a^{-1}(w_c) + f_b^{-1}(w_c)] \cup [h_a^{-1}(w_c) + h_b^{-1}(w_c), d_1 + d_2] \\ w_c & \forall z \in [f_a^{-1}(w_c) + f_b^{-1}(w_c), \ h_a^{-1}(w_c) + h_b^{-1}(w_c)] \end{cases}$$

where $w_c = \min(w_1, w_2)$.

Proof: For

$$\tilde{C} = \tilde{A}_1(+)\tilde{A}_2 = \left\{(z, \mu_{\tilde{C}}(z)) : z = x + y \text{ and } \tilde{A}_1 = (x, \mu_{\tilde{A}_1}(x)), \ \tilde{A}_2 = (y, \mu_{\tilde{A}_2}(y))\right\}$$

depending on the positions of x and y the following three cases may arise:

Case 1: Let us consider $x \in (-\infty, a_1]$ and $y \in (-\infty, a_2] \Rightarrow x + y = z \in (-\infty, a_1 + a_2]$. As $\mu_{\tilde{A}_1}(x) = 0 \ \forall x \in (-\infty, a_1]$ and $\mu_{\tilde{A}_2}(y) = 0 \ \ \forall y \in (-\infty, a_2]$, we need to prove $\mu_{\tilde{C}}(z) = 0 \ \forall \ z \in (-\infty, a_1 + a_2]$.

To evaluate $\mu_{\tilde{C}}(z)$ we must consider every pair (p, r) such that, $z = p + r$. Now for every pair (p, r) (for $p + r = z$) the following two possibilities have been considered:

(i) For $p < x$ and $r > y$, $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = 0$ as $\mu_{\tilde{A}_1}(p) = 0 \ \forall p < x$,

(ii) For $p > x$ and $r < y$, $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = 0$ as $\mu_{\tilde{A}_2}(r) = 0 \ \forall r < y$.

From both (i) and (ii), $\sup_{z=p+r} \min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = 0 \Rightarrow \mu_{\tilde{C}}(z) = 0$.
Since this result holds for any arbitrary z, therefore, $\mu_{\tilde{C}}(z) = 0 \forall z \in (-\infty, a_1 + a_2]$.

In a similar manner it can be proved that $\mu_{\tilde{C}}(z) = 0 \ \forall \ z \in [d_1 + d_2, \infty)$.
Thus, $\mu_{\tilde{C}}(z) = 0 \ \forall \ z \in (-\infty, a_1 + a_2] \cup [d_1 + d_2, \infty)$.

Case 2: Suppose $\min(w_1, w_2) = w_2$

For $x \in [a_1, f_a^{-1}(w_2)]$ and $y \in [a_2, b_2]$, i.e. $x \in [a_1, f_a^{-1}(w_2)]$ and $y \in [a_2, f_b^{-1}(w_2)]$,
$\Rightarrow x + y = z \in [a_1 + a_2, f_a^{-1}(w_2) + f_b^{-1}(w_2)]$.

We need to prove $\mu_{\tilde{C}}(z) \in [0, w_2] \ \forall z \in [a_1 + a_2, f_a^{-1}(w_2) + f_b^{-1}(w_2)]$.

Now for $z \in [a_1 + a_2, f_a^{-1}(w_2) + f_b^{-1}(w_2))$, for every pair (p, r)(for $p + r = z$)the following two possibilities have been considered:

(i) For $p < x$ and $r > y$,

- **(a)** For $p > a_1$ and $r < f_b^{-1}(w_2) = b_2$, we have $0 < \min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} < w_2$.
- **(b)** For $p > a_1$ and $r > b_2 = f_b^{-1}(w_2)$, we have $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} < w_2$.
- **(c)** For $p < a_1$ and $r > b_2$, we can obtain $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = 0$.
- **(d)** For $p < a_1$ and $r < b_2$, we have $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = 0$.

(ii) For $p > x$ and $r < y$,

- **(a)** For $p < f_a^{-1}(w_2)$ and $r > a_2$, we have $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = 0$.
- **(b)** For $p < f_a^{-1}(w_2)$ and $r < a_2$, we obtain $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = 0$.
- **(c)** For $p > f_a^{-1}(w_2)$ and $r > a_2$, we have $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} < w_2$.
- **(d)** For $p > f_a^{-1}(w_2)$ and $r < a_2$, we have $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = 0$.

Consequently for both (i) and (ii), $\forall z \in [a_1 + a_2, f_a^{-1}(w_2) + f_b^{-1}(w_2))$ we have, $0 \leq \sup_{z=p+r} \min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} < w_2$.

And for $z = f_a^{-1}(w_2) + f_b^{-1}(w_2)$, i.e. when $x = f_a^{-1}(w_2)$ and $y = f_b^{-1}(w_2)$, the following holds: $\min\{\mu_{\tilde{A}_1}(x), \mu_{\tilde{A}_2}(y)\} = w_2$.

As before, if we consider the following possibilities:

(iii) For $p < x$ and $r > y$, we get $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} < w_2$.

(iv) For $p > x$ and $r < y$, we get $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} < w_2$.

(v) For $p = x$ and $r = y$ only, we get $\min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = w_2$.

Consequently for (iii), (iv) and (v), we have, $\sup\limits_{z=p+r} \min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} = w_2$.

Therefore, from the above five possibilities, we can say that for any arbitrary $z \in [a_1 + a_2, f_a^{-1}(w_2) + f_b^{-1}(w_2)]$, $0 \leq \sup\limits_{z=p+r} \min\{\mu_{\tilde{A}_1}(p), \mu_{\tilde{A}_2}(r)\} \leq w_2 \Rightarrow \ 0 \leq \mu_{\tilde{C}}(z) \leq w_2$

Since this result holds for any arbitrary $z \in [a_1 + a_2, f_a^{-1}(w_2) + f_b^{-1}(w_2)]$, so we can say that this is true for all $z \in [a_1 + a_2, f_a^{-1}(w_2) + f_b^{-1}(w_2)]$.

Therefore, $\mu_{\tilde{C}}(z) \in [0, w_2] \ \forall z \in [a_1 + a_2, f_a^{-1}(w_2) + f_b^{-1}(w_2)]$.

In a similar manner we can prove that $\mu_{\tilde{C}}(z) \in [0, w_2] \ \forall \ z \in [h_a^{-1}(w_2) + h_b^{-1}(w_2), d_1 + d_2]$.

Similar result can be obtained if we consider $\min(w_1, w_2) = w_1$.

Hence, in general, it can be written that if $\min(w_1, w_2) = w_c$ then $\mu_{\tilde{C}}(z) \in [0, w_c]$ for all values of z belongs to $[a_1 + a_2, f_a^{-1}(w_c) + f_b^{-1}(w_c)] \cup [h_a^{-1}(w_c) + h_b^{-1}(w_c), d_1 + d_2]$.

Case 3: Let us consider $x \in [f_a^{-1}(w_c), h_a^{-1}(w_c)]$ and $y \in [f_b^{-1}(w_c), h_b^{-1}(w_c)] \Rightarrow x + y = z \in [f_a^{-1}(w_c) + f_b^{-1}(w_c), \ h_a^{-1}(w_c) + h_b^{-1}(w_c)]$

Now $\mu_{\tilde{A}_1}(x) = w_c \ \forall x \in [f_a^{-1}(w_c), h_a^{-1}(w_c)]$ and $\mu_{\tilde{A}_2}(y) = w_c \ \ \forall y \in [f_b^{-1}(w_c), h_b^{-1}(w_c)]$.

Therefore, it is obvious that $\forall z \in [f_a^{-1}(w_c) + f_b^{-1}(w_c), \ h_a^{-1}(w_c) + h_b^{-1}(w_c)]$, $\mu_{\tilde{C}}(z) = w_c$.

Thus, the proof has been completed.

Therefore, for $\tilde{C} = \tilde{A}_1(+)\tilde{A}_2$, the membership function $\mu_{\tilde{C}}(z)$ can be written as follows:

$$\mu_{\tilde{C}}(z) = \begin{cases} 0 & \text{if } -\infty < z \leq a_1 + a_2 \\ w_c f_c(z) & \text{if } \ a_1 + a_2 \leq z \leq f_a^{-1}(w_c) + f_b^{-1}(w_c) \\ w_c & \text{if } \ f_a^{-1}(w_c) + f_b^{-1}(w_c) \leq z \leq h_a^{-1}(w_c) + h_b^{-1}(w_c) \\ w_c h_c(z) & \text{if } \ h_a^{-1}(w_c) + h_b^{-1}(w_c) \leq z \leq d_1 + d_2 \\ 0 & \text{if } \ d_1 + d_2 < z \leq \infty \end{cases}$$

Where $f_c(z) = \sup\limits_{z=x+y} \min\{f_a(x), f_b(y)\}$ and $h_c(z) = \sup\limits_{z=x+y} \min\{h_a(x), h_b(y)\}$.

*3.1.*1 Particular Case: For generalized triangular fuzzy numbers

Let us consider, two generalized triangular fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$ denoted as $\tilde{A}_1 = (a_1, b_1, c_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2; w_2)$.

Theorem 3.2. *Addition of two triangular fuzzy numbers with different confidence levels generates a trapezoidal fuzzy number as follows:*

$$\begin{aligned}
\tilde{A}_3 &= \tilde{A}_1(+)\tilde{A}_2 = (a_3, b_3, c_3, d_3; w_3) where, \\
a_3 &= a_1 + a_2 \\
b_3 &= a_1 + a_2 + (b_1 - a_1)w_3/w_1 + (b_2 - a_2)w_3/w_2 \\
c_3 &= c_1 + c_2 - (c_1 - b_1)w_3/w_1 - (c_2 - b_2)w_3/w_2 \\
d_3 &= c_1 + c_2
\end{aligned}$$

and $w_3 = \min(w_1, w_2)$*;* $w_1 \neq w_2$*.*

Proof: Generalized triangular fuzzy numbers $\tilde{A}_1$ and $\tilde{A}_2$ have the membership functions of the following form:

$$\mu_{\tilde{A}_1}(x) = \begin{cases} 0 & \text{if } -\infty < x \leq a_1 \\ w_1(x - a_1)/(b_1 - a_1) & \text{if } \quad a_1 \leq x \leq b_1 \\ w_1(c_1 - x)/(c_1 - b_1) & \text{if } \quad b_1 \leq x \leq c_1 \\ 0 & \text{if } \quad c_1 < x \leq \infty \end{cases} \tag{3.7}$$

$$\mu_{\tilde{A}_2}(y) = \begin{cases} 0 & \text{if } -\infty < x \leq a_2 \\ w_2(y - a_2)/(b_2 - a_2) & \text{if } \quad a_2 \leq y \leq b_2 \\ w_2(c_2 - y)/(c_2 - b_2) & \text{if } \quad b_2 \leq y \leq c_2 \\ 0 & \text{if } \quad c_2 < y \leq \infty \end{cases} \tag{3.8}$$

It can be said that for a fixed value of $w \in [0, \min(w_1, w_2)]$, there exists $(x, y) \in \mathbb{R}^2$ such that $\mu_{\tilde{A}_1}(x) = \mu_{\tilde{A}_2}(y) = w = \mu_{\tilde{A}_3}(z)$ holds,where $z = x + y$.
For obtaining $\tilde{A}_3$, z needs to be found with respect to w. For this purpose the increasing part of the membership functions of $\tilde{A}_1$ and $\tilde{A}_2$ has been considered.

From (3.7), the increasing part of $\mu_{\tilde{A}_1}(x)$ gives the following relation between x and w

$$w_1(x - a_1)/(b_1 - a_1) \ = w \Rightarrow x = a_1 + (b_1 - a_1).w/w_1 \tag{3.9}$$

And similarly from (3.8) we get the following

$$w_2(y - a_2)/(b_2 - a_2) = w \Rightarrow y = a_2 + (b_2 - a_2).w/w_2 \tag{3.10}$$

Therefore, to compute z, from (3.9) and (3.10) we get

$$w = \frac{z - (a_1 + a_2)}{(b_1 - a_1)/w_1 + (b_2 - a_2)/w_2} \tag{3.11}$$

On the other hand the increasing part of the membership function of $\tilde{A}_3$ can be written in the following form:

$$\mu_{\tilde{A}_3}(z) = w = w_3(z - a_3)/(b_3 - a_3) \tag{3.12}$$

Now equation (3.11) can be expressed in the following form:

$$w/w_3 = \frac{z - (a_1 + a_2)}{w_3\left((b_1 - a_1)/w_1 + (b_2 - a_2)/w_2\right)} \tag{3.13}$$

Comparing (3.12) and (3.13), the following can be written

$$a_3 = a_1 + a_2 \quad \text{and} \quad b_3 - a_3 = (b_1 - a_1)w_3/w_1 + (b_2 - a_2)w_3/w_2$$

Therefore, $b_3 = a_1 + a_2 + (b_1 - a_1)w_3/w_1 + (b_2 - a_2)w_3/w_2$.

In the same way considering the decreasing part of the membership functions of $\tilde{A}_1$ and $\tilde{A}_2$ the following can be proved

$$c_3 = c_1 + c_2 - (c_1 - b_1)w_3/w_1 - (c_2 - b_2)w_3/w_2, \quad d_3 = c_1 + c_2$$

Thus, finally the following can be obtained

$$\begin{aligned}
\tilde{A}_3 &= \tilde{A}_1(+)\tilde{A}_2 = (a_3, b_3, c_3, d_3; w_3)\text{where,} \\
a_3 &= a_1 + a_2 \\
b_3 &= a_1 + a_2 + (b_1 - a_1)w_3/w_1 + (b_2 - a_2)w_3/w_2 \\
c_3 &= c_1 + c_2 - (c_1 - b_1)w_3/w_1 - (c_2 - b_2)w_3/w_2 \\
d_3 &= c_1 + c_2
\end{aligned}$$

Thus, the required value of a_3, b_3, c_3 and d_3 can be obtained. Hence, the proof has been completed.

3.1.2 More on the example of subsection 2.1

As shown in the example of section 2.1, after performing the addition operation between $\tilde{A}_1$ and $\tilde{A}_2$, we obtained $\tilde{A}_1(+)\tilde{A}_2 = \tilde{B} = (1.5, 1.7, 1.9; 0.5)$. But after the discussion stated above in Theorem (3.2), it can be said that the addition of $\tilde{A}_1$ and $\tilde{A}_2$ generates a generalized trapezoidal fuzzy number and that should be $\tilde{A}_3 = (1.5, 1.65, 1.75, 1.9; 0.5)$ which gives a better result. The difference between the two operations has been shown in Figure 3.1. From the figure it is now clear that Chen's addition causes the loss of information.

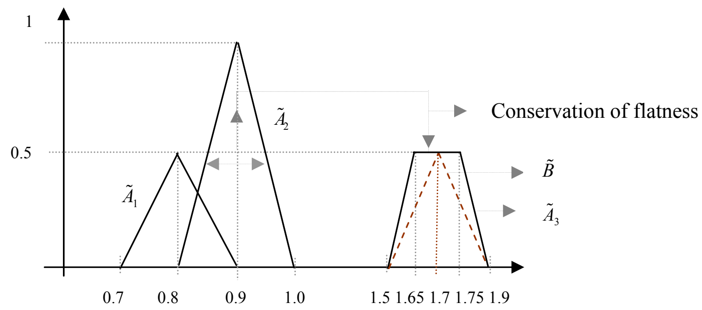

Fig. 2. Comparison between the two additions

Theorem 3.3. *Addition of two generalized trapezoidal fuzzy numbers* $\tilde{A}_1 = (a_1, b_1, c_1, d_1; w_1)$ *and* $\tilde{A}_2 = (a_2, b_2, c_2, d_2; w_2)$*, with different confidence levels generates a trapezoidal fuzzy number as follows:*

$$\begin{aligned}
\tilde{A}_4 &= \tilde{A}_1(+)\tilde{A}_2 = (a_4, b_4, c_4, d_4; w_4)\textit{where,}\\
a_4 &= a_1 + a_2\\
b_4 &= a_1 + a_2 + (b_1 - a_1)w_4/w_1 + (b_2 - a_2)w_4/w_2\\
c_4 &= d_1 + d_2 - (d_1 - c_1)w_4/w_1 - (d_2 - c_2)w_4/w_2\\
d_4 &= d_1 + d_2
\end{aligned}$$

and $w_4 = \min(w_1, w_2)$*;* $w_1 \neq w_2$*.*

Proof: The proof is similar to Theorem 3.2.

3.2 Subtraction of two generalized triangular fuzzy numbers

Let us consider two generalized triangular fuzzy numbers $\tilde{A}_1 = (a_1, b_1, c_1; w_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2; w_2)$. In order to compute the subtraction operation between $\tilde{A}_1$ and $\tilde{A}_2$, the value of $\tilde{A}_1(-)\tilde{A}_2$ can be defined as $\tilde{A}_1(-)\tilde{A}_2 = \tilde{A}_1(+)(-\tilde{A}_2)$. Now the addition operation on $\tilde{A}_1$ and $(-)\tilde{A}_2$ as discussed in the section 3.1.1, can be easily performed. Hence, the following can be written

$$\begin{aligned}
\tilde{A}_5 &= \tilde{A}_1(-)\tilde{A}_2 = (a_5, b_5, c_5, d_5; w_5)\text{where,}\\
a_5 &= a_1 - a_2\\
b_5 &= a_1 - c_2 + (b_1 - a_1)w_5/w_1 + (c_2 - b_2)w_5/w_2\\
c_5 &= c_1 - a_2 - (c_1 - b_1)w_5/w_1 - (b_2 - a_2)w_5/w_2\\
d_5 &= c_1 - a_2
\end{aligned}$$

and $w_5 = \min(w_1, w_2)$; $w_1 \neq w_2$

4 Numerical illustration

Example 4.1. *Let* $\tilde{A}_1 = (1, 2, 4, 5; 0.5)$ *and* $\tilde{A}_2 = (5, 6, 8, 9; 1)$ *be two generalized trapezoidal fuzzy numbers. Find* $\tilde{A}_1(+)\tilde{A}_2$*.*

Result: With the help of Theorem 3.3, the following result has been obtained $\tilde{A}_1(+)\tilde{A}_2 = (6, 6.5, 12.5, 14; 0.5)$.

Example 4.2. *Let* $\tilde{A}_1 = (0.4, 0.6, 0.7; 0.8)$ *and* $\tilde{A}_2 = (-0.1, 0.2, 0.4; 1)$ *be two generalized triangular fuzzy numbers. Find* $\tilde{A}_1(+)\tilde{A}_2$.

Result: With the help of Theorem 3.2, the following result has been obtained $\tilde{A}_1(+)\tilde{A}_2 = (0.3, 0.74, 0.84, 1.1; 0.8)$.

Example 4.3. *Let* $\tilde{A}_1 = (0.5, 0.6, 0.7; 0.5)$ *and* $\tilde{A}_2 = (0.3, 0.4, 0.5; 1)$ *be two generalized triangular fuzzy numbers. Find* $\tilde{A}_1(-)\tilde{A}_2$.

Result: Utilizing the process as stated in section 3.2, the result has been obtained as: $\tilde{A}_1(-)\tilde{A}_2 = (0, 0.15, 0.25, 0.4; 0.5)$.

5 Conclusion

In this paper the shortcomings of the existing operations on generalized fuzzy numbers have been discussed. In order to overcome this problem the extension principle has been used to derive different arithmetic operations (addition, subtraction). In future, the result of production and division of generalized fuzzy numbers may also be analyzed.

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