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Gauss Elimination Algorithm for Interval Matrix

M. Adabitabar firozja *, F. babakordi, M. Shahhosseini

Department of mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran.

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Abstract

Let $[A] = [[a_{ij}]]_{n\times n}$be an interval for $[a_{ij}] = [\underline{a}_{ij}, \overline{a}_{ij}]$ (i,j=1,2,...,n). In transformation a matrix into upper triangular matrix by Gauss method, entries under the main diameter should be zero by elementary row operations. In this paper we consider zero as an interval; and, we would apply Gauss elimination method on interval matrix by arithmetic operations on intervals and the definition of comparison method interval dates, we would use Gauss elimination method on interval matrix.

Keywords : Interval linear equation; Gauss elimination algorithm.

1 Introduction

The problems of interval equations solution are of perennial interest, because of their direct relevance to practical modeling and optimization of real-world processes including finance [2,4], economy [1,3,8,11], and mechanics [5]. Gauss elimination method is used for transformation matrix into upper triangular form. This upper triangular form can be used for solving system[9].Gunter worked on feasibility result for interval Gaussian elimination [6].Jurgen presented a method by which the breakdown of interval Gaussian elimination caused by division of an interval containing zero can be avoided for some classes of matrices [7]. Dymova and Sevastjanov proposed "interval extended zero"method for solving interval and fuzzy equations and they applied it for Gauss elimination algorithm [10]. We simply see if we use the method, coefficient matrix will not be transformed into upper triangular form.Therefor, we proposed a new method for solving the problem. The organization of the paper is as follows: In Section 2, we will remind some basic definitions of interval mathematics, interval extended zero method,and Gauss elimination algorithm. "Interval extended zero" method will be used for Gauss elimination algorithm in section 3. Examples will be provided in Section 4 and finally, Section 5 is allocated to conclusion.

*Corresponding author. Email address: mohamadsadega@yahoo.com.

2 Preliminaries

Definition 2.1. *We call* $[a] = [\underline{a}, \overline{a}]$ *an interval number if* $\underline{a} \leq \overline{a}$*, and we consider arithmetic operations on these two intervals of* $[a], [b]$ *as follows:*

$$[a] + [b] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \tag{2.1}$$

$$k[a] = \begin{cases} [k\underline{a}, k\overline{a}] & k \geq 0, \\ [k\overline{a}, k\underline{a}] & k < 0 \end{cases} \tag{2.2}$$

$$[a] - [b] = [a] + (-[b]) = [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \tag{2.3}$$

$$[a] \times [b] = [min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}], \tag{2.4}$$

$$[a] \diagup [b] = [a] \times [\frac{1}{\overline{b}}, \frac{1}{\underline{b}}], \tag{2.5}$$

And, we define the comparison of these two intervals such as:

$$[a] = [b] \longleftrightarrow \underline{a} = \underline{b} \ , \ \overline{a} = \overline{b} \tag{2.6}$$

$$[a] < [b] \longleftrightarrow \overline{a} < \underline{b} \tag{2.7}$$

$$[a] \leq [b] \longleftrightarrow \overline{a} \leq \underline{b} \tag{2.8}$$

$$[a] \approx [b] \longleftrightarrow \underline{a} + \overline{a} = \underline{b} + \overline{b} \tag{2.9}$$

$$[a] \preceq [b] \longleftrightarrow \underline{a} + \overline{a} < \underline{b} + \overline{b} \tag{2.10}$$

2.1 interval Extended zero

In problems with interval entries, it is better to consider zero as a symmetric interval around zero in the form of $[-y, y]$. In which $y \geq 0$ and we show it as $[0_y] = [-y, y]$.

*2.1.*1 Properties of interval extended zero

Zero satisfies some of the properties in real number set,for the present we show in the following equalities that the properties satisfy in interval extended zero, too:

1. $[0_y] + [0_{y'}] = [-y, y] + [-y', y'] = [-(y + y'), (y + y')] = [0_{y+y'}]$
2. $[0_y] \times [0_{y'}] = [-y, y] \times [-y', y'] = [-yy', yy'] = [0_{yy'}]$
3. $[a] + [0_y] = [\underline{a}, \overline{a}] + [-y, y] = [\underline{a} - y, \overline{a} + y] \approx [a]$
4.

$$[a] \times [0_y] = [\underline{a}, \overline{a}] \times [-y, y]$$
$$= \begin{cases} [-\overline{a}y, \overline{a}y] \approx [0_{\overline{a}y}], & [a] > 0 \\ [\underline{a}y, -\underline{a}y] \approx [0_{-\underline{a}y}], & [a] < 0 \\ [-max\{-\underline{a}y, \overline{a}y\}, max\{-\underline{a}y, \overline{a}y\}] \approx [0_{max\{-\underline{a}y, \overline{a}y\}}], & 0 \in [a] \end{cases} \tag{2.11}$$

*2.1.*2 Solving interval equation

Dymova and Sevastjanov proposed interval extended zero method for solving a linear equation as $[a][x]-[b]=[0]$ where $[a],[b],[x]$ are interval numbers [10], and $0 \notin [a]$. In the method,they get the interval solution of this equation as$[x]=[\underline{x},\overline{x}]$by using arithmetic operations, we obtain:

1 If $[a]>0$, $[b]>0$, then $\underline{x}=\frac{\underline{b}}{\overline{a}}$, $\quad \overline{x}=\frac{\underline{b}+\overline{b}}{\overline{a}}-\frac{\underline{a}\underline{b}}{\overline{a}^2}$

2 If $[a]>0$, $[b]<0$, then $\underline{x}=\frac{\underline{b}+\overline{b}}{\overline{a}}-\frac{\underline{a}\overline{b}}{\overline{a}^2}$, $\quad \overline{x}=\frac{\overline{b}}{\overline{a}}$

3 If $[a]<0$, $[b]<0$, then $\underline{x}=\frac{\overline{b}}{\underline{a}}$, $\quad \overline{x}=\frac{\underline{b}+\overline{b}}{\underline{a}}-\frac{\overline{a}\overline{b}}{\underline{a}^2}$

4 If $[a]<0$, $[b]>0$, then $\underline{x}=\frac{\underline{b}+\overline{b}}{\underline{a}}-\frac{\overline{a}\underline{b}}{\underline{a}^2}$, $\quad \overline{x}=\frac{\underline{b}}{\underline{a}}$

5 If $[a]>0$, $0\in b$, then $\underline{x}=\frac{\underline{b}}{\underline{a}}$, $\quad \overline{x}=\frac{\underline{b}+\overline{b}}{\overline{a}}-\frac{\underline{b}}{\underline{a}}$

6 If $[a]<0$, $0\in b$, then $\underline{x}=\frac{\overline{b}}{\overline{a}}$, $\quad \overline{x}=\frac{\underline{b}+\overline{b}}{\underline{a}}-\frac{\overline{b}}{\overline{a}}$

*2.1.*3 Gauss elimination algorithm

Elementary row operations:

1. R_{ij}: we move i-th row and j-th row for each other.
2. $R_i(\alpha)$: we multiply i-th row in α.
3. $R_{ij}(\alpha)$: we multiply i-th row in α,and add the result to j-th row.

In Gauss elimination algorithm, all entries under the main diameter should be zero by elementary row operations to make matrix,upper triangular.In order to this matrix is considered as $A=[a_{ij}]_{n\times n}$,where $a_{ij}\in\Re$ and $a_{ii}\neq 0$. Let we are in $k-$th iterate. At the beginning of $k-$th iteration, $A^{(k-1)}$ as follows:

$$A^{(k-1)}=\begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & \dots & \dots & \dots & a_{1n}^{(0)} \\ 0 & a_{21}^{(1)} & \dots & \dots & \dots & a_{2n}^{(1)} \\ \vdots & \ddots & \ddots & & & \vdots \\ 0 & \dots & 0 & a_{kk}^{(k-1)} & \dots & a_{kn}^{(k-1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & a_{nk}^{(k-1)} & \dots & a_{nn}^{(k-1)} \end{bmatrix}$$

We put $m_{ik}=-\frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}$,all entries under main diameter will be zero by $R_{ki}(m_{ik})(i=k+1,...,n,k=1,2,...,n-1)$and we have:

$$a_{ij}^{(k)}=a_{ij}^{(k-1)}+m_{ik}a_{kj}^{(k-1)}(i,j=k+1,...,n,k=1,2,...,n) \tag{2.12}$$

3 Interval extension of Gauss elimination algorithm

Let $[A] = [[a_{ij}]]_{n\times n}$ is a matrix of $n \times n$ with the entries $[a_{ij}] = [\underline{a}_{ij}, \overline{a}_{ij}]$ (i,j=1,2,...,n). In interval Gauss elimination method,we have:

$$[a_{ij}]^{(k)} = [a_{ij}]^{(k-1)} + [m_{ik}][a_{kj}] \tag{3.13}$$

In Gauss elimination method the interval matrix should be transformed into upper triangular matrix. We solve the following equation to obtain $[m_{ik}] = -\frac{[a_{ik}]^{(k-1)}}{[a_{kk}]^{(k-1)}}$:

$$[a_{kk}]^{(k-1)}[m_{ik}] + [a_{ik}]^{(k-1)} = 0 \tag{3.14}$$

Therefore, by interval extended zero method we get:

$$[m_{ik}] = [\underline{m}_{ik}, \overline{m}_{ik}] \tag{3.15}$$

Where we will obtain $[m_{ik}]$ from Section (2.1.2),

1. If $[a_{kk}]^{(k-1)} > 0$, $[a_{ik}]^{(k-1)} > 0$, then

$$\underline{m_{ik}} = -\frac{\underline{a_{ik}}^{(k-1)} + \overline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}} + \frac{\underline{a_{kk}}^{(k-1)}\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)2}},$$

$$\overline{m_{ik}} = -\frac{\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}}$$

2. If $[a_{kk}]^{(k-1)} > 0$, $[a_{ik}]^{(k-1)} < 0$, then

$$\underline{m_{ik}} = -\frac{\overline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}},$$

$$\overline{m_{ik}} = -\frac{\overline{a_{ik}}^{(k-1)} + a_{ik}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}} + \frac{a_{kk}^{(k-1)}\overline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)2}}$$

3. If $[a_{kk}]^{(k-1)} < 0$, $[a_{ik}]^{(k-1)} > 0$, then

$$\underline{m_{ik}} = -\frac{a_{ik}^{(k-1)}}{\underline{a_{kk}}^{(k-1)}},$$

$$\overline{m_{ik}} = -\frac{\underline{a_{ik}}^{(k-1)} + \overline{a_{ik}}^{(k-1)}}{\underline{a_{kk}}^{(k-1)}} + \frac{\overline{a_{kk}}^{(k-1)}a_{ik}^{(k-1)}}{\underline{a_{kk}}^{(k-1)2}}$$

4. If $[a_{kk}]^{(k-1)} < 0$, $[a_{ik}]^{(k-1)} < 0$, then

$$\underline{m_{ik}} = -\frac{\underline{a_{ik}}^{(k-1)} + \overline{a_{ik}}^{(k-1)}}{\underline{a_{kk}}^{(k-1)}} + \frac{\overline{a_{kk}}^{(k-1)}\overline{a_{ik}}^{(k-1)}}{\underline{a_{kk}}^{(k-1)2}},$$

$$\overline{m_{ik}} = -\frac{\overline{a_{ik}}^{(k-1)}}{\underline{a_{kk}}^{(k-1)}}$$

5. If $[a_{kk}]^{(k-1)} > 0$, $0 \in [a_{ik}]^{(k-1)}$, then

$$\underline{m_{ik}} = -\frac{\overline{a_{ik}}^{(k-1)}}{\underline{a_{kk}}^{(k-1)}},$$

$$\overline{m_{ik}} = -\frac{\underline{a_{ik}}^{(k-1)} + \overline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}} + \frac{\overline{a_{ik}}^{(k-1)}}{\underline{a_{kk}}^{(k-1)}}$$

6. If $[a_{kk}]^{(k-1)} < 0$, $0 \in [a_{ik}]^{(k-1)}$, then

$$\underline{m_{ik}} = -\frac{\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}},$$

$$\overline{m_{ik}} = -\frac{\underline{a_{ik}}^{(k-1)} + \overline{a_{ik}}^{(k-1)}}{\underline{a_{kk}}^{(k-1)}} + \frac{\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}}$$

Proposition 3.1. *By $R_{ki}([m_{ik}])$;is obtained $[a_{ik}]^{(k+1)} = [0_{ik}]$ for $(i = k+1, ..., n, k = 1, ..., n-1)$.*

Proof: In $R_{ki}([m_{ik}])$,i-th row is multiplied in $[m_{ik}]$ and the result is added to k-th row,therefore,we obtain $[m_{ik}]$, if $[a_{kk}]^{(k-1)} > 0$, $[a_{ik}]^{(k-1)} > 0$Then

$$[m_{ik}] = [\underline{m_{ik}}, \overline{m_{ik}}] = [-\frac{\underline{a_{ik}}^{(k-1)} + \overline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}} + \frac{\underline{a_{kk}}^{(k-1)}\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)2}}, -\frac{\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}}] \tag{3.16}$$

Hence,

$$\begin{aligned}
[a_{ik}]^{(k)} &= [a_{ik}]^{(k-1)} + [m_{ik}][a_{kk}]^{(k-1)} \\
&= [\underline{a_{ik}}^{(k-1)}, \overline{a_{ik}}^{(k-1)}] \\
&+[-\frac{\underline{a_{ik}}^{(k-1)} + \overline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}} + \frac{\underline{a_{kk}}^{(k-1)}\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)2}}, -\frac{\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}}][\underline{a_{kk}}^{(k-1)}, \overline{a_{kk}}^{(k-1)}] \\
&= [\underline{a_{ik}}^{(k-1)}, \overline{a_{ik}}^{(k-1)}] + [-\underline{a_{ik}}^{(k-1)} - \overline{a_{ik}}^{(k-1)} + \frac{\underline{a_{kk}}^{(k-1)}\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}}, -\frac{\underline{a_{kk}}^{(k-1)}\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}}] \\
&= [-\overline{a_{ik}}^{(k-1)} + \frac{\underline{a_{kk}}^{(k-1)}\underline{a_{ik}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}}, -\frac{\underline{a_{ik}}^{(k-1)}\underline{a_{kk}}^{(k-1)}}{\overline{a_{kk}}^{(k-1)}} + \overline{a_{ik}}^{(k-1)}] = [0_{ik}]
\end{aligned}$$

Other cases are similar.

In $(n-1)$-th iteration, we get an upper triangular matrix with interval zero as:

$$A^{(n-1)} = \begin{bmatrix} [a_{11}]^{(0)} & [a_{12}]^{(0)} & \dots & [a_{1n}]^{(0)} \\ [0_{21}] & [a_{22}]^{(1)} & \dots & [a_{2n}]^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ [0_{n1}] & \dots & [0_{nn-1}] & [a_{nn}]^{(n-1)} \end{bmatrix}$$

Where

$$[a_{ij}]^{(k)} = [a_{ij}]^{(k-1)} + [m_{ik}][a_{kj}]^{(k-1)}, (k = 1, 2, ..., n-1) \tag{3.17}$$

Remark 3.1. *If we don't have row changing, then the determinant of interval matrix will be as follows:*

$$|[A]| = \prod_{i=1}^{n} [a_{ii}]^{(i-1)} \tag{3.18}$$

This obvious that, $[A]$ is nonsingular if and only if $0 \notin [a_{ii}]^{(i-1)}$ for $i = 1, ..., n$.

4 Numerical examples

Here,we illustrate our method.

Example 4.1. *Let us consider*

$$[A] = \begin{bmatrix} [2,3] & [5,6] \\ [3,4] & [1,2] \end{bmatrix}$$

If we use the proposed method for Gauss elimination algorithm we will get:

$$[m_{21}] = [\underline{m}_{21}, \overline{m}_{21}] = [-\frac{5}{3}, -1] \tag{4.19}$$

Then with $R_{12}([m_{21}])$

$$[A^{(1)}] = \begin{bmatrix} [2,3] & [5,6] \\ [-2,2] & [-9,-3] \end{bmatrix} = \begin{bmatrix} [2,3] & [5,6] \\ [0_{21}] & [-9,-3] \end{bmatrix}$$

Example 4.2. *Let us consider* $[A]$ *matrix as follows [8]:*

$$\begin{bmatrix} [0.1389,0.1396] & [0.0804,0.0806] & [0.0033,0.0036] & [0.0001,0.0001] & [0.0321,0.0327] & [0.0052,0.0054] \\ [0.1565,0.1571] & [0.5043,0.5047] & [0.5634,0.5643] & [0.3401,0.3421] & [0.2405,0.2411] & [0.2642,0.2654] \\ [0.0001,0.0002] & [0.0004,0.0005] & [0.0067,0.0069] & [0.0013,0.0015] & [0.0090,0.0090] & [0.0145,0.0149] \\ [0.0110,0.01130] & [0.0178,0.0178] & [0.0296,0.0298] & [0.0140,0.0143] & [0.1103,0.0111] & [0.0413,0.0417] \\ [0.0214,0.0216] & [0.0749,0.0750] & [0.0917,0.0917] & [0.00490,0.0490] & [0.0400,0.0402] & [0.0454,0.0458] \\ [0.0268,0.0271] & [0.0284,0.0407] & [0.0142,0.0144] & [0.0358,0.0363] & [0.1086,0.1090] & [0.0981,0.0987] \end{bmatrix}$$

We will get $[A^{(5)}]$ *as follows:*

$$\begin{bmatrix} [0.1389,0.1396] & [0.0804,0.0806] & [0.0033,0.0036] & [0.0001,0.0001] & [0.0321,0.0327] & [0.0052,0.0054] \\ [-0.0014,0.0014] & [0.4131,0.4146] & [0.5593,0.5606] & [0.3400,0.342] & [[0.2035,0.2051] & [0.2581,0.2596] \\ [-0.0001,0.0001] & [-0.0001,0.0001] & [0.0061,0.0065] & [0.001,0.0013] & [0.0088,0.0089] & [0.0142,0.0147] \\ [-0.0003,0.0003] & [-0.0003,0.0003] & [-0.0015,0.0015] & [0.0015,0.0033] & [0.081,0.0845] & [-0.0008,0.004] \\ [-0.0003,0.0003] & [-0.0005,0.0005] & [-0.0013,0.0013] & [-0.003,0.003] & [0.0698,0.1524] & [-0.013,0.0010] \\ [-0.0004,0.0004] & [-0.0127,0.0127] & [0.0177,0.00177] & [-0.0231,0.0231] & [-0.776,0.776] & [-0.051,0.153] \end{bmatrix}$$

5 Conclusion

In Gauss elimination method all of entries under main diameter with elementary row operations should be zero in order to transform matrix into upper triangular form; but, we simply see if we use the proposed method in[10] for Gauss elimination method, the matrix will not be transformed into upper triangular form. Therefore, in order to solve this problem we proposed to use interval extended zero method for obtaining $[m_{ik}]$ in Gauss elimination algorithm.

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