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A Method to Approximate Solution of the First Kind Abel Integral Equation Using Navot's Quadrature and Simpson's Rule

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Abstract

In this paper, we present a method for solving the first kind Abel integral equation. In this method, the first kind Abel integral equation is transformed to the second kind Volterra integral equation with a continuous kernel and a smooth deriving term expressed by weakly singular integrals. By using Sidi's $\sin^m$ - transformation and modified Navot-Simpson's integration rule, an algorithm for solving this kind of integral equation is proposed, then the convergence of algorithm is derived. Some numerical results show the efficiency of the mentioned method.

Keywords : The first kind Abel integral equation, Simpson's rule, Sidi's $\sin^m$- transformation, Zeta function, Navot's quadrature.

1 Introduction

The first kind Abel integral equation

$$\int_0^x \frac{H(x,y)}{(x-y)^\alpha} f(y)dy = g(x) \quad (0 \le x \le 1, 0 < \alpha < 1), \tag{1.1}$$

frequently appears in many physical and engineering problems, e.g., semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions and population dynamics, etc.

There are many classes of numerical methods for the approximate solution of Eq. (1.1) such as product-integration methods, collocation methods, fractional multistep methods, etc.

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Ya-Ping Liu and Lu Tao in [1] proposed quadrature methods and their extrapolation for solving Eq. (1.1). In [2], a method in order to solve the second kind singular Volterra integral equations by modified Navot-Simpson's quadrature rule was proposed. In this paper, we propose a similar approach for solving (1.1) based on the method presented in [1].

Since solving the first kind Abel integral equation is an ill-posed problem we transform that into the second kind. For this purpose, in (1.1) we replace x by s, multiply both sides by $\frac{1}{(x-s)^{1-\alpha}}$ and integrate both sides in $[0,x]$ with respect to s then the following relation is obtained:

$$\int_0^x \int_0^s \frac{H(s,y)}{(x-s)^{1-\alpha}(s-y)^{\alpha}} f(y)dyds = \int_0^x \frac{g(s)}{(x-s)^{1-\alpha}} ds.$$

The double integral in above relation can be written as $\int_0^x (\int_y^x \frac{H(s,y)}{(x-s)^{1-\alpha}(s-y)^{\alpha}} ds) f(y)dy$. Let

$$L(x,y) = \int_y^x \frac{H(s,y)}{(x-s)^{1-\alpha}(s-y)^{\alpha}} ds \quad , \quad G(x) = \int_0^x \frac{g(s)}{(x-s)^{1-\alpha}} ds.$$

If we apply the change of variables $s = y + \tau(x-y)$ for $L(x,y)$ and $s = \tau x$ for $G(x)$ then,

$$L(x,y) = \int_0^1 \frac{H(y+\tau(x-y),y)}{(1-\tau)^{1-\alpha}\tau^{\alpha}} d\tau, \tag{1.2}$$

and

$$G(x) = x^{\alpha} \int_0^1 \frac{g(x\tau)}{(1-\tau)^{1-\alpha}} d\tau. \tag{1.3}$$

Therefore, (1.1) can be written as

$$\int_0^x L(x,y) f(y)dy = G(x). \tag{1.4}$$

By differentiating (1.4) with respect to x, we get

$$\frac{d}{dx} \int_0^x L(x,y)f(y)dy = G'(x) \Longrightarrow L(x,x)f(x) + \int_0^x \frac{\partial}{\partial x} L(x,y)f(y)dy = G'(x).$$

Since $L(x,x) = \frac{H(x,x)}{\sin(\pi\alpha)} \neq 0$ for $0 \leq x \leq 1$ and $G(0) = 0$, then we can write,

$$f(x) + \int_0^x \tilde{L}(x,y)f(y)dy = V(x), \quad 0 \leq x \leq 1, \tag{1.5}$$

where $\tilde{L}(x,y) = L_x(x,y)/L(x,x)$ and $V(x) = G'(x)/L(x,x)$. The Eq. (1.5) is the second kind Volterra integral equation whose kernel and deriving term are expressed by weakly singular integrals.

Since the solution $f(x)$ of (1.5) or its derivative $f'(x)$ may be unbounded at the origin, Baratella and Orsi [3] proposed to take the change of variable $x = \gamma(t) = t^q$, $(0 \leq t \leq 1)$ in (1.5), where q is an undetermined positive constant. Then (1.5) is written as

$$f(\gamma(t)) + \int_0^{\gamma(t)} \tilde{L}(\gamma(t),y)f(y)dy = V(\gamma(t)), \quad (0 \leq t \leq 1).$$

Letting $y = \gamma(s)$, we have

$$f(\gamma(t)) + \int_0^t \tilde{L}(\gamma(t), \gamma(s)) f(\gamma(s)) \gamma'(s) ds = V(\gamma(t)) \quad (0 \leq t \leq 1). \tag{1.6}$$

Multiply (1.6) by $\gamma'(t)$ and set

$$u(t) = \gamma'(t) f(\gamma(t)), \eta(t) = \gamma'(t) V(\gamma(t)), \bar{L}(t,s) = \gamma'(t) \tilde{L}(\gamma(t), \gamma(s)).$$

Consequently the Eq. (1.6) is simplified as

$$u(t) + \int_0^t \bar{L}(t,s) u(s) ds = \eta(t), \quad (0 \leq t \leq 1). \tag{1.7}$$

With a suitable choice of the parameter q we can ensure that the solution $u(t)$ and $\eta(t)$ of (1.7) are sufficiently smooth [1].

3 The numerical method

Since the kernel and the deriving term of the integral equation (1.7) are expressed by weakly singular integrals, we must use a numerical method which is able to compute these integrals with weak singularity at the end points. For this purpose, Navot's quadrature rule is used. This special quadrature is applied for functions having a singularity of any type on or near the integration interval.

We recall that, in the interval [0,1], the trapezoidal rule is defined as, $Tf = \frac{1}{N}\sum_{j=1}^{N-1} f(\frac{j}{N}) + \frac{1}{2N}[f(0) + f(1)]$, and for the midpoint rule, $Mf = \frac{1}{N}\sum_{j=1}^{N} f(\frac{2j-1}{2N})$. So, if N is even then the Simpson's integration rule can be defined as [4,5], $Sf = \frac{2}{3}Mf + \frac{1}{3}Tf$. Now, we can imply the following lemma in the interval [a,b] for introducing the modified Simpson's rule by Navot's Quadrature. This lemma is an improvement of the asymptotic expansion of the trapezoidal rule which has been presented in [1].

Lemma 3.1. *Let $g(x) \in C^{2l+1}[a,b] (l \in Z^+)$, $G(x) = (b-x)^\alpha g(x)$, $h = (b-a)/N$, N is even and $x_i = a + ih$, $i = 0, 1, \ldots, N$, then the modified Simpson's integration rule $S_N(G)$ has an asymptotic expansion as follows,*

$$\begin{aligned} S_N(G) = & \frac{h}{3} G(x_0) + \frac{4}{3} h \sum_{i=1}^{\frac{N}{2}} G(x_{2i-1}) + \frac{2}{3} h \sum_{i=1}^{\lfloor \frac{N-1}{2} \rfloor} G(x_{2i}) \\ & - g(b) \left(\frac{2}{3} \xi(-\alpha, \frac{1}{2}) + \frac{1}{3} \xi(-\alpha) \right) h^{1+\alpha} \\ = & \int_a^b (b-x)^\alpha g(x) dx + \sum_{j=1}^{l} P_j G^{(2j-1)}(a) h^{2j} \\ & + \sum_{j=1}^{2l} (-1)^j \frac{g^{(j)}(b) h^{j+\alpha+1}}{j!} \left(\frac{1}{3} + \frac{2}{3}(2^{-\alpha-j} - 1) \right) \xi(-\alpha - j) + O(h^{2l+1}), \end{aligned} \tag{2.1}$$

where $-1 < \alpha < 0$, $\xi(-\alpha, \frac{1}{2}) = (2^{-\alpha} - 1)\xi(-\alpha)$, $\xi(x)$ is the Riemann-Zeta function and P_j $(j = 1, \ldots, l)$ are all constants independent of h.

Proof. In [1,4] the modified trapezoidal rule $T_{h'}(G)$ has been introduced by using Navot's quadrature as follows:

$$\begin{aligned} T_{h'}(G) &= \frac{h'}{2}G(x'_0) + h'\sum_{j=1}^{M-1} G(x'_j) + \xi(-\alpha)g(b){h'}^{1+\alpha} \\ &= \int_a^b (b-x)^\alpha g(x)dx + \sum_{j=1}^{l} \frac{B_{2j}}{(2j)!}G^{(2j-1)}(a){h'}^{2j} \\ &\quad + \sum_{j=1}^{2l}(-1)^j \frac{g^{(j)}(b){h'}^{j+\alpha+1}}{j!}\xi(-\alpha-j) + O({h'}^{2l+1}), \end{aligned} \tag{2.1}$$

where, $\xi(x)$ is the Riemann-Zeta function and $B_{2j}, j = 1, ..., l$, are the Bernoulli numbers and $x'_j = a + jh', j = 0, ..., M-1,\ h' = \frac{b-a}{M}$. The similar formula can be written for the modified mid-point rule as follows [5]:

$$M_{h'}(G) = h'\sum_{j=1}^{M} G(x'_j - \frac{h'}{2}) - (2^{-\alpha}-1)\xi(-\alpha)g(b){h'}^{1+\alpha}.$$

Since the number of the points when we combine the modified trapezoidal and mid-point rules is $N = 2M$ which is even, hence $\lfloor\frac{N-1}{2}\rfloor = \frac{N}{2} - 1$ and $\lfloor\frac{N}{2}\rfloor = \frac{N}{2}$. If $h = \frac{b-a}{N}$ then, $x_{2j-1} = x'_j - h, j = 1, 2, ..., M$ and $x_{2j} = x'_j, j = 1, 2, ..., M-1$, hence we can compute $S_N(G)$ in the interval $[a, b]$ as follows:

$$S_N(G) = \frac{2}{3}M_{h'}(G) + \frac{1}{3}T_{h'}(G) =$$

$$\frac{2}{3}[h'\sum_{j=1}^{M} G(x'_j - \frac{h'}{2}) - (2^{-\alpha}-1)\xi(-\alpha)g(b){h'}^{1+\alpha}] + \frac{1}{3}[\frac{h'}{2}G(x'_0) + h'\sum_{j=1}^{M-1} G(x'_j) - \xi(-\alpha)g(b){h'}^{1+\alpha}] =$$

$$\frac{h}{3}G(x_0) + \frac{4}{3}h\sum_{j=1}^{\frac{N}{2}} G(x_{2j-1}) + \frac{2}{3}h\sum_{j=1}^{\frac{N}{2}-1} G(x_{2j}) - [\frac{2}{3}\xi(-\alpha, \frac{1}{2}) + \frac{1}{3}\xi(-\alpha)]g(b)h^{1+\alpha}.$$

Also, if $P_j, j = 1, 2, ..., l$, are the constant values independent of h, the following relation can be proved similarly by using (2.1).

$$\begin{aligned} S_N(G) &= \int_a^b (b-x)^\alpha g(x)dx + \sum_{j=1}^{l} P_j G^{(2j-1)}(a)h^{2j} \\ &+ \sum_{j=1}^{2l}(-1)^j \frac{g^{(j)}(b)h^{j+\alpha+1}}{j!}\left(\frac{1}{3} + \frac{2}{3}(2^{-\alpha-j}-1)\right)\xi(-\alpha-j) + O(h^{2l+1}). \end{aligned}$$

This completes the proof. □

Since the periodization methods play an important roles in increasing accuracy of quadrature rules, we will use a Sidi's $\sin^m$- transformation [6], which is constructed by

$$\psi_m(y) = \frac{\theta_m(y)}{\theta_m(1)} \quad \text{with} \quad \theta_m(y) = \int_0^y (\sin(\pi t))^m dt, \quad m = 1, 2, \ldots.$$

$\psi_m(y)$ has the following asymptotic expansions

$$\begin{cases} \psi_m(y) \sim \varepsilon_m y^{m+1} + \sum_{i=1}^{\infty} \varepsilon_{m,i} y^{m+1+2i} & \text{as} \quad y \longrightarrow 0^+, \\ \psi_m(y) \sim 1 - \hat{\varepsilon}_m (1-y)^{m+1} - \sum_{i=1}^{\infty} \hat{\varepsilon}_{m,i} (1-y)^{m+1+2i} & \text{as} \quad y \longrightarrow 1^-, \end{cases} \tag{2.2}$$

where $\varepsilon_m \neq 0$, $\hat{\varepsilon}_m \neq 0$, $\varepsilon_{m,i}$ and $\hat{\varepsilon}_{m,i}$, $(i = 1, 2, \ldots)$ are all constants.

From (1.2),

$$L_x(x,y) = \frac{\partial}{\partial x} L(x,y) = \int_0^1 H'(y + \tau(x-y), y) \frac{\tau^{1-\alpha}}{(1-\tau)^{1-\alpha}} d\tau,$$

where $H'(x,y) = \frac{\partial}{\partial x} H(x,y)$. By Letting $\tau = \psi_m(t)$, then

$$L_x(x,y) = \int_0^1 \Phi(x,y,t) dt, \tag{2.3}$$

with

$$\Phi(x,y,t) = H'(y + \psi_m(t)(x-y), y) \frac{(\psi_m(t))^{1-\alpha}}{(1-\psi_m(t))^{1-\alpha}} \psi'_m(t). \tag{2.4}$$

By (2.2), there are functions $B(t)$ and $C(t)$ such that

$$\frac{\psi'_m(t)}{(1-\psi_m(t))^{1-\alpha}} = (m+1)(\hat{\varepsilon}_m)^{\alpha} (1-t)^{(m+1)\alpha - 1} \frac{1 - C(t)(1-t)^2}{(1 - B(t)(1-t)^2/(\hat{\varepsilon}_m)^{\alpha})^{1-\alpha}},$$

which has a zero or pole of βth order at $t = 1$, where $\beta = (m+1)\alpha - 1$. Then, we have

$$\Phi(x,y,t) = (1-t)^{\beta} \phi(x,y,t),$$

where,

$$\phi(x,y,t) = (m+1)(\hat{\varepsilon}_m)^{\alpha} H'(y + \psi_m(t)(x-y), y) \frac{(\psi_m(t))^{1-\alpha}(1 - C(t)(1-t)^2)}{(1 - B(t)(1-t)^2/(\hat{\varepsilon}_m)^{\alpha})^{1-\alpha}}.$$

We observe that ϕ is nonsingular at $t = 1$.

From lemma 2.1, since $\psi_m(0) = 0$ and $\psi_m(1) = 1$, we may derive an approximation $L_x^h(x,y)$ of $L_x(x,y)$ as follows:

$$\begin{aligned} L_x^h(x,y) = {} & \frac{4}{3} \sum_{j=1}^{\frac{N}{2}} h\Phi(x,y,t_{2j-1}) + \frac{2}{3} \sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor} h\Phi(x,y,t_{2j}) \\ & - (m+1)(\hat{\varepsilon}_m)^{\alpha} h^{\beta+1} H'(x,y) \left(\frac{2}{3}\xi(-\beta, \frac{1}{2}) + \frac{1}{3}\xi(-\beta) \right). \end{aligned} \tag{2.5}$$

where $t_j = jh$, $j = 0, 1, \ldots, N$, $h = \frac{1}{N}$.

For the kernel $\bar{L}(x, y)$ of (1.7) the corresponding approximate expression is

$$\bar{L}^h(t, s) = \gamma'(t)\frac{L_x^h(\gamma(t), \gamma(s))}{L(\gamma(t), \gamma(t))}, \tag{2.6}$$

where $L(\gamma(t), \gamma(t)) = \frac{\pi H(\gamma(t),\gamma(t))}{\sin(\pi\alpha)} \neq 0$.

On the other hand by (1.3),

$$\begin{aligned} G'(x) &= \alpha x^{\alpha-1}\int_0^1 \frac{g(x\tau)}{(1-\tau)^{1-\alpha}}d\tau + x^\alpha \int_0^1 \frac{g'(x\tau)\tau}{(1-\tau)^{1-\alpha}}d\tau \\ &= \alpha x^{\alpha-1}\int_0^1 G_1(x,\tau)d\tau + x^\alpha \int_0^1 G_2(x,\tau)d\tau, \end{aligned}$$

where,

$$G_1(x,t) = \frac{g(x\psi_m(t))}{(1-\psi_m(t))^{1-\alpha}}\psi'_m(t), \quad G_2(x,t) = \frac{g'(x\psi_m(t))}{(1-\psi_m(t))^{1-\alpha}}\psi_m(t)\psi'_m(t).$$

In (1.7),

$$\eta(t) = \frac{\gamma'(t)G'(\gamma(t))}{L(\gamma(t),\gamma(t))} = \frac{\alpha\gamma'(t)(\gamma(t))^{\alpha-1}}{L(\gamma(t),\gamma(t))}\int_0^1 G_1(\gamma(t),\tau)d\tau + \frac{\gamma'(t)(\gamma(t))^{\alpha}}{L(\gamma(t),\gamma(t))}\int_0^1 G_2(\gamma(t),\tau)d\tau.$$

We put

$$I_1(t) = \frac{\alpha\gamma'(t)(\gamma(t))^{\alpha-1}}{L(\gamma(t),\gamma(t))}\int_0^1 G_1(\gamma(t),\tau)d\tau,$$

and

$$I_2(t) = \frac{\gamma'(t)(\gamma(t))^{\alpha}}{L(\gamma(t),\gamma(t))}\int_0^1 G_2(\gamma(t),\tau)d\tau.$$

We have $G'(x) = O(x^{\alpha-1})$, as $x \longrightarrow 0$, and since $\gamma(t) = t^q$ we have that $I_1(t) = O(t^{q\alpha-1})$ and $I_2(t) = O(t^{q\alpha+q-1})$, as $t \longrightarrow 0$. Choosing $q > \frac{1}{\alpha}$, we get $I_1(0) = I_2(0) = 0$. Similar to above discussion, we can obtain approximate expressions for $I_1(t)$ and $I_2(t)$ as follows:

$$\begin{aligned} I_1^h(t) = \frac{\alpha\gamma'(t)(\gamma(t))^{\alpha-1}}{L(\gamma(t),\gamma(t))}&\left[\frac{4}{3}\sum_{j=1}^{\frac{N}{2}} hG_1(\gamma(t),t_{2j-1}) + \frac{2}{3}\sum_{j=1}^{\lfloor\frac{N-1}{2}\rfloor} hG_1(\gamma(t),t_{2j})\right. \\ &\left. -(m+1)h^{\beta+1}(\hat{\varepsilon}_m)^\alpha g(\gamma(t))\left(\frac{2}{3}\xi(-\beta,\frac{1}{2}) + \frac{1}{3}\xi(-\beta)\right)\right] \quad (0 \leq t \leq 1), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} I_2^h(t) = \frac{\gamma'(t)(\gamma(t))^{\alpha}}{L(\gamma(t),\gamma(t))}&\left[\frac{4}{3}\sum_{j=1}^{\frac{N}{2}} hG_2(\gamma(t),t_{2j-1}) + \frac{2}{3}\sum_{j=1}^{\lfloor\frac{N-1}{2}\rfloor} hG_2(\gamma(t),t_{2j})\right. \\ &\left. -(m+1)h^{\beta+1}(\hat{\varepsilon}_m)^\alpha g'(\gamma(t))\left(\frac{2}{3}\xi(-\beta,\frac{1}{2}) + \frac{1}{3}\xi(-\beta)\right)\right] \quad (0 \leq t \leq 1) \end{aligned} \tag{2.8}$$

Then, the approximation of $\eta(t)$ is

$$\eta^h(t) = I_1^h(t) + I_2^h(t). \tag{2.9}$$

By (2.6) and (2.9), the following approximate integral equation is derived:

$$u(t) + \int_0^t \bar{L}^h(t,s)u(s)ds = \eta^h(t). \tag{2.10}$$

Now, we can apply the Simpson's rule to derive the numerical solution of (2.10).

Algorithm (Simpson's rule)

$$\begin{cases} u_0^s = \eta^s(t_0) = 0 \\ u_i^s = \eta^h(t_i) - \frac{h}{3}\sum_{j=0}^{i} w_{ij}\bar{L}^h(t_i,t_j)u_j^s, \quad i = 1,\dots,N \end{cases} \tag{2.11}$$

where, $w_{i0} = w_{ii} = 1$, $w_{ij} = 4$ $(j = 2k-1)$, $w_{ij} = 2(j = 2k)$, $k \geq 1, 0 < j < i$, $i = 0,1,\dots,N$.

By this algorithm, $\{u_i^s\}$, $i = 0,\dots,N$ are found and $\{u_i^s/\gamma'(t_i)\}$ will be the approximate of the solution $\{f(\gamma(t_i))\}$ of (1.7).

4 Convergence and error estimate

Assume that the data function is not perturbed by noise, then approximations $\bar{L}^h(t,s)$ and $\eta^h(t)$ satisfy the following lemma.

Lemma 4.1. *Let $H(x,.)$, $g(x) \in C^6[0,1]$, then the errors $\bar{L}^h(t,s) - \bar{L}(t,s)$ and $\eta^h(t) - \eta(t)$ have the estimates*

$$\bar{L}^h(t,s) - \bar{L}(t,s) = O(h^\lambda), \tag{3.1}$$

and

$$\eta^h(t) - \eta(t) = O(h^\lambda), \tag{3.2}$$

where $\lambda = \min\{\beta + 3, 4\}$.

Proof. Since $H(x,.), g(x) \in C^6[0,1]$, it follows that $\phi(.,.,t), \in C^5[0,1]$. By lemma 2.1 with $l = 2$,

$$\tilde{L}^h(x,y) = \int_0^1 (1-t)^\beta \frac{\phi(x,y,t)}{L(x,x)}dt + \sum_{j=1}^{2} P_j \frac{\Phi^{(2j-1)}(x,y,0)}{L(x,x)}h^{2j}$$
$$+ \sum_{j=1}^{4}(-1)^j \frac{\phi^{(j)}(x,y,1)h^{j+\beta+1}}{L(x,x)j!}\left(\frac{1}{3} + \frac{2}{3}(2^{-\beta-j} - 1)\right)\xi(-\beta - j) + O(h^5).$$

Then we have,

$$\tilde{L}^h(x,y) - \tilde{L}(x,y) = T_1h^2 + T_2h^4 + T_3h^{2+\beta} + T_4h^{3+\beta} + T_5h^{4+\beta} + T_6h^{5+\beta} + O(h^5),$$

with

$$T_1 = P_1 \frac{\Phi^{(1)}(x,y,0)}{L(x,x)},$$
$$T_2 = P_2 \frac{\Phi^{(3)}(x,y,0)}{L(x,x)},$$
$$T_3 = -\frac{\phi^{(1)}(x,y,1)}{L(x,x)} \left(\frac{1}{3} + \frac{2}{3}(2^{-\beta-1}-1)\right) \xi(-\beta-1),$$
$$T_4 = \frac{\phi^{(2)}(x,y,1)}{2L(x,x)} \left(\frac{1}{3} + \frac{2}{3}(2^{-\beta-2}-1)\right) \xi(-\beta-2),$$
$$T_5 = -\frac{\phi^{(3)}(x,y,1)}{6L(x,x)} \left(\frac{1}{3} + \frac{2}{3}(2^{-\beta-3}-1)\right) \xi(-\beta-3),$$
$$T_6 = \frac{\phi^{(4)}(x,y,1)}{4!L(x,x)} \left(\frac{1}{3} + \frac{2}{3}(2^{-\beta-4}-1)\right) \xi(-\beta-4).$$

By letting

$$R(t) = \frac{1 - C(t)(1-t)^2}{(1 - B(t)(1-t)^2/(\hat{\varepsilon}_m)^\alpha)^{1-\alpha}},$$

then,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x,y,t) &= (m+1)(\hat{\varepsilon}_m)^\alpha (H^{(2)}(y+\psi_m(t)(x-y),y) \\ &\times (x-y)\psi'_m(t)R(t)(\psi_m(t))^{1-\alpha} + H'(y+\psi_m(t)(x-y),y)(\psi_m(t))^{1-\alpha}R'(t) \\ &+ H'(y+\psi_m(t)(x-y),y)R(t)(\psi_m(t))^{-\alpha}\psi'_m(t)).\end{aligned}$$

Since $\psi'_m(1) = 0$, $R'(1) = 0$, $\psi_m(0) = 0$, $\psi'_m(0) = 0$ we get $\frac{\partial}{\partial t}\phi(x,y,1) = 0$ and $\frac{\partial}{\partial t}\phi(x,y,0) = 0$. Therefore, $T_1(x,y) = 0$, and $T_3(x,y) = 0$. From $\bar{L}(t,s) = \gamma'(t)\tilde{L}(\gamma(t),\gamma(s))$, (3.1) is proved.

Using the same technique we can also prove (3.2). This completes the proof of lemma 3.1. □

In order to obtain an error estimate of algorithm, we need the following discrete Gronwall inequality.

Lemma 4.2. *[1].If a nonnegative sequence $\{y_n, n = 0,\ldots,N\}$ satisfies $y_0 = 0$, $y_n \leq A + Bh\sum_{j=0}^{n-1} y_j$, $1 \leq n \leq N$, $h = \frac{1}{N}$, then*

$$\max_{0\leq i\leq N} y_i \leq Ae^B$$

where A and B are positive constants independent of h.

The error of algorithm is estimated as follows.

Theorem 4.3. *Assume that $H(x,.), g(x) \in C^6[0,1]$, $H(.,y) \in C^5[0,1]$ and the step size h is sufficiently small, then the error $e_i^s = u(t_i) - u_i^s$, $i = 0,1,\ldots,N$ of the above Algorithm is obtained by*

$$\max_{1\leq i\leq N} |e_i^s| \leq d_M h^\gamma \tag{3.3}$$

where d_M is a constant independent of h and $2 < \gamma \leq 4$.

Proof. By Euler-Maclaurian formula of the Simpson's integration rule, (1.7) becomes the following equality,

$$\begin{cases} \eta(t_0) = u(t_0) = 0 \\ \eta(t_i) = u(t_i) + \frac{h}{3}\sum_{j=0}^{i} w_{ij}\bar{L}(t_i,t_j)u(t_j) + Q_1(t_i)h^4 + O(h^5), \quad i = 1,2,\ldots,N, \end{cases}$$

with

$$w_{i0} = w_{ii} = 1, \quad w_{ij} = 2 \quad (j = 2k), \quad w_{ij} = 4 \quad (j = 2k-1), \quad k \geq 1,$$

and,

$$Q_1(t_i) = c\frac{d^3}{ds^3}(\bar{L}(t_i,s)u(s))\big|_{s=0}^{s=t_i}, \quad c = -\frac{1}{180}$$

By lemma 3.2 we have,

$$\begin{aligned} \eta^h(t_i) &= \eta(t_i) + O(h^\lambda) \\ &= u(t_i) + \frac{h}{3}\sum_{j=0}^{i} w_{ij}\bar{L}^h(t_i,t_j)u(t_j) + Q_1(t_i)h^4 + O(h^5) + O(h^\lambda), \end{aligned} \tag{3.4}$$

where, $\lambda = \min\{\beta+3, 4\}$.

Subtracting (3.4) from (2.11), we get

$$\begin{cases} e_0^s = 0 \\ e_i^s = -\frac{h}{3}\sum_{j=0}^{i} w_{ij}\bar{L}^h(t_i,t_j)e_j^s + E_{i,t}(t_i,t,u(t)), \end{cases}$$

where, $E_{i,t}(t_i,t,u(t)) = Q_1(t_i)h^4 + O(h^\lambda)$. Let

$$A = \max_{1\leq i\leq N}\max_{0\leq t\leq 1}|E_{i,t}(t_i,t,u(t))|,$$

since $-1 < \beta < m$, we can derive $A = O(h^\gamma)$ where $2 < \gamma \leq 4$.

Now let

$$B = \sup_{h>0}\max_{1\leq i\leq N}\max_{0\leq j\leq i}|\bar{L}^h(t_i,t_j)|,$$

then we have

$$|e_i^s| \leq \frac{h}{3}B\sum_{j=0}^{i}|e_i^s| + A$$

By lemma 3.3, there is a constant d_M independent of h satisfying

$$\max_{1\leq i\leq N}|e_i^s| \leq Ae^B \leq d_M h^\gamma, \quad 2 < \gamma \leq 4.$$

This completes the proof. □

5 Numerical examples

In this section, we apply the above algorithm to solve the following examples [1]. We use $m = 2$ or let $\tau = \psi_2(t) = (\frac{1}{2\pi})\ (2\pi t - \sin 2\pi t)$. The programs have been provided with maple 10.

Example 5.1. In this example, The integral equation (1.1) is considered with

$$\alpha = \frac{1}{2}, \quad H(x,y) = x^2y + e, \quad g(x) = 3x^4\pi + 4e\pi x.$$

The exact solution of the integral equation is $f(x) = 8\sqrt{x}$. We use a smoothing transformation $x = \gamma(t) = t^2$. The relative errors are shown in Table 1.

Table 1. The errors of the example 1 at $x = 1.0$

N	Numerical solution	error
2	8.7770128	9.71266E-2
4	8.6105208	7.63151E-2
8	8.3686739	4.60842E-2
16	8.03111328	3.88916E-3
32	8.01611856	2.01482E-3
64	8.000245921	3.07401E-5

Example 5.2. In this example, we consider the integral equation

$$\frac{1}{\Gamma(1/2)} \int_0^x \frac{1}{(x-y)^{1/2}} f(y)dy = \sqrt{\pi}. \quad (0 \leq x \leq 1)$$

The exact solution is $f(x) = \frac{1}{\sqrt{x}}$. Since the solution is unbounded at the origin. We applying a smoothing transformation $x = \gamma(t) = t^4$. Then, the relative errors are shown in Table 2.

Table 2. The errors of the example 2 at $x = 1.0$

N	Numerical solution	error
2	1.399561	3.99561E-1
4	1.135846	1.35846E-1
8	1.0425398	4.25398E-2
16	1.00773894	7.73894E-3
32	1.0000833910	8.33910E-5
64	1.0000198145	1.98145E-5

6 Conclusions

Many of important mechanical and physical problems are converted to a type of the first kind Abel integral equations. In this work, for solving these kinds of integral equations, we presented a numerical method to approximate the solution by using Navot's quadrature and Simpson's rule. We apply the integral equation which has a singularity at one of the endpoints. One can improve this technique to use the Navot's quadrature and modify it for the case that there are singularity at both of the endpoints of the integration interval.

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