Available online at http://ijim.srbiau.ac.ir/

Int. J. Industrial Mathematics (ISSN 2008-5621)

Vol. 5, No. 1, 2013 Article ID IJIM-00329, 11 pages

Research Article

Science and Research Branch (IAU)

The use of iterative methods for solving Black-Scholes equation

T. Allahviranloo [*] [†] , Sh. S. Behzadi [‡]

Abstract

In this paper, the Black-Scholes equation is solved by using the Adomian's decomposition method , modified Adomian's decomposition method , variational iteration method , modified variational iteration method, homotopy perturbation method, modified homotopy perturbation method and homotopy analysis method. The existence and uniqueness of the solution and convergence of the proposed methods are proved in details. A numerical example is studied to demonstrate the accuracy of the presented methods.

Keywords : Black-Scholes equation; Adomian decomposition method (ADM); Modified Adomian decomposition method (MADM); Variational iteration method (VIM); Modified variational iteration method (MVIM); Homotopy perturbation method (HPM); Modified homotopy perturbation method (MHPM); Homotopy analysis method (HAM).

1 Introduction

THe pricing of options is a central problem in financial investment. It is of both theoretical and practical importance since the use of options thrives in the financial market. In option pricing theory, the Black-Scholes equation is one of the most effective models for pricing options. The equation assumes the existence of perfect capital markets and the security prices are log normally distributed or, equivalently, the log-returns are normally distributed. To these, one adds the assumptions that trading in all securities is continuous and that the distribution of the rates of return is stationary. In recent years, some works have been done in order to find the numerical solution of the Black-Scholes equation. For example [5, 6, 7, 8, 9, 10]. In this work, we develope the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve this equation as follows [1, 2, 3, 4]:

$$u_t + ax^2 u_{xx} + bxu_x - ru = 0, \tag{1.1}$$

where,

$$a = \frac{\sigma}{2}, \quad b = r - \delta,$$

r is the risk-free rate, σ is the volatility, and δ is the dividend yield. With the initial condition:

$$u(x, 0) = g(x). \tag{1.2}$$

The paper is organized as follows. In Section 2, the mentioned iterative methods are introduced for solving Eq. 1.1. In Section 3 we prove the existence , uniqueness of the solution and convergence of the proposed methods. Finally, the numerical example is shown in Section 4. In order to obtain an approximate solution of Eq. 1.1, let us integrate one time Eq. 1.1 with respect to t using the initial condition we obtain,

$$u(x,t) = g(x) - \tag{1.3}$$

$$a \int_0^t F_1(u(x,\tau)d\tau - b \int_0^t F_2(u(x,\tau))d\tau + r \int_0^t u(x,\tau)d\tau,$$

where,

$$F_1(u(x,t)) = x^2 u_{xx}(x,t),$$
$$F_2(u(x,t)) = xu_x(x,t).$$

In Eq. 1.3, we assume $g(x)$ is bounded for all x in $J = [0, T](T \in \mathbb{R})$. The terms $F_1(u(x,t))$ and $F_2(u(x,t))$ are Lipschitz continuous with $| F_1(u) - F_1(u^*) | \leq L_1 | u - u^* |$ and $| F_2(u) - F_2(u^*) | \leq L_2 | u - u^* |$.

[*]Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

[†]Corresponding author. allahviranloo@yahoo.com

[‡]Department of Mathematics, Qazvin Branch, Islamic Azad University, Qazvin, Iran.

2 The iterative methods

2.1 Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = f, \tag{2.4}$$

where $u(x,t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and f is the source term. Applying the inverse operator L^{-1} to both sides of Eq. 2.4, and using the given conditions we obtain

$$u(x,t) = z(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.5}$$

where the function $z(x)$ represents the terms arising from integrating the source term f. The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \tag{2.6}$$

where $A_n,\ n \geq 0$ are the Adomian polynomials determined formally as follows :

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i u_i)]]_{\lambda=0}. \tag{2.7}$$

The first Adomian polynomials (introduced in [11, 12, 13]) are:

$$\begin{aligned}
A_0 &= G_1(u_0),\\
A_1 &= u_1 G_1'(u_0),\\
A_2 &= u_2 G_1'(u_0) + \frac{1}{2!}u_1^2 G_1''(u_0),\\
A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!}u_1^3 G_1'''(u_0), \ldots
\end{aligned} \tag{2.8}$$

2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of $u(x,t)$ in 2.4 as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{2.9}$$

where, the components $u_0, u_1, \ldots$ which can be determined recursively

$$\begin{aligned}
u_0(x,t) &= g(x), \\
u_1(x,t) &= -a\int_0^t A_0(x,t)dt - b\int_0^t B_0(x,t)dt \\
&\quad + r\int_0^t u_0(x,t)dt, \\
&\vdots \\
u_{n+1}(x,t) &= -a\int_0^t A_n(x,t)\ dt - b\int_0^t B_n(x,t)\ dt \\
&\quad + r\int_0^t u_n(x,t)\ dt, \quad n \geq 0.
\end{aligned} \tag{2.10}$$

Substituting 2.8 into 2.10 leads to the determination of the components of u.

2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [14]. The modified forms was established on the assumption that the function $g(x)$ can be divided into two parts, namely $g_1(x)$ and $g_2(x)$. Under this assumption we set

$$g(x) = g_1(x) + g_2(x). \tag{2.11}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part g_1 be assigned to the zeroth component u_0, whereas the remaining part g_2 be combined with the other terms given in 2.11 to define u_1. Consequently, the modified recursive relation

$$\begin{aligned}
u_0 &= g_1(x), \\
u_1 &= g_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0), \\
&\vdots \\
u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1,
\end{aligned} \tag{2.12}$$

was developed. To obtain the approximation solution of Eq. 1.1, according to the MADM, we can write the iterative formula 2.12 as follows:

$$\begin{aligned}
u_0 &= g_1(x), \\
u_1 &= g_2(x) - a\int_0^t A_0(x,t)\ dt - b\int_0^t B_0(x,t)\ dt \\
&\quad + r\int_0^t u_0(x,t)\ dt, \\
&\vdots \\
u_{n+1} &= -a\int_0^t A_n(x,t)\ dt - b\int_0^t B_n(x,t)\ dt \\
&\quad + r\int_0^t u_n(x,t)\ dt, \quad n \geq 1.
\end{aligned} \tag{2.13}$$

The operators $F_i(u(x,t))$ $(i = 1,2)$ are usually represented by the infinite series of the Adomian polynomials as follows:

$$F_1(u) = \sum_{i=0}^{\infty} A_i,$$

$$F_2(u) = \sum_{i=0}^{\infty} B_i.$$

where A_i and B_i are the Adomian polynomials. Also, we can use the following formula for the Adomian polynomials [15]:

$$\begin{aligned} A_n &= F_1(s_n) - \textstyle\sum_{i=0}^{n-1} A_i, \\ B_n &= F_2(s_n) - \textstyle\sum_{i=0}^{n-1} B_i. \end{aligned} \tag{2.14}$$

Where $s_n = \sum_{i=0}^{n} u_i(x,t)$ is the partial sum.

2.2 Description of the VIM and MVIM

In the VIM [16, 17, 18, 19, 20, 35, 41, 42], it has been considered the following nonlinear differential equation:

$$Lu + Nu = g, \tag{2.15}$$

where L is a linear operator, N is a nonlinear operator and g is a known analytical function. In this case, the functions u_n may be determined recursively by

$$u_{n+1}(x,t) = u_n(x,t)+ \tag{2.16}$$

$$\int_0^t \lambda(x,\tau)\{L(u_n(x,\tau)) + N(u_n(x,\tau)) - g(x,\tau)\}d\tau,$$

$$n \geq 0,$$

where λ is a general Lagrange multiplier which can be computed using the variational theory. Here the function $u_n(x,\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $u_n(x,t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{2.17}$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions. To obtain the approximation solution of Eq. 1.1, according to the VIM, we can write iteration formula 2.16 as follows:

$$u_{n+1}(x,t) = \tag{2.18}$$

$$u_n(x,t) + L_t^{-1}(\lambda[u_n(x,t) - g(x) + a\int_0^t F_1(u_n(x,t))\, dt$$

$$+b\int_0^t F_2(u_n(x,t))\, dt - r\int_0^t u_n(x,t)\, dt]), n \geq 0.$$

Where,

$$L_t^{-1}(.) = \int_0^t (.)\, d\tau.$$

To find the optimal λ, we proceed as

$$\delta u_{n+1}(x,t) = \tag{2.19}$$

$$\delta u_n(x,t)+\delta L_t^{-1}(\lambda[u_n(x,t)-g(x)+a\int_0^t F_1(u_n(x,t))\, dt$$

$$b\int_0^t F_2(u_n(x,t))\, dt - r\int_0^t u_n(x,t)\, dt]).$$

From Eq. 2.19, the stationary conditions can be obtained as follows: $\lambda' = 0$ and $1+\lambda = 0$. Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in 2.18, the following iteration formula is obtained.

$$u_0(x,t) = g(x), \tag{2.20}$$

$$u_{n+1}(x,t) =$$

$$u_n(x,t) - L_t^{-1}(u_n(x,t) - g(x) + a\int_0^t F_1(u_n(x,t))\, dt$$

$$+b\int_0^t F_2(u_n(x,t))\, dt - r\int_0^t u_n(x,t)\, dt), n \geq 0.$$

To obtain the approximation solution of Eq. 1.1, based on the MVIM [21, 22, 23], we can write the following iteration formula:

$$u_0(x,t) = g(x), \tag{2.21}$$

$$u_{n+1}(x,t) =$$

$$u_n(x,t) - L_t^{-1}(a\int_0^t F_1(u_n(x,t) - u_{n-1}(x,t))\, dt$$

$$+b\int_0^t F_2(u_n(x,t) - u_{n-1}(x,t))\, dt$$

$$-r\int_0^t (u_n(x,t) - u_{n-1}(x,t))\, dt), n \geq 0.$$

Relations 2.20 and 2.21 will enable us to determine the components $u_n(x,t)$ recursively for $n \geq 0$.

2.3 Description of the HAM

Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function and x is an independent variable. let $u_0(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H_1(x,t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x,t)] = 0$ when $s(x,t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] - qhH_1(x,t)N[\phi(x,t;q)] = \tag{2.22}$$

$\hat{H}[\phi(x,t;q);u_0(x,t),H_1(x,t),h,q]$.
It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H_1(x,t)$. Enforcing the homotopy 2.22 to be zero, i.e.,

$$\hat{H}_1[\phi(x,t;q);u_0(x,t),H_1(x,t),h,q]=0, \tag{2.23}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q)-u_0(x,t)]=qhH_1(x,t)N[\phi(x,t;q)]. \tag{2.24}$$

When $q=0$, the zero-order deformation Eq. 2.24 becomes

$$\phi(x;0)=u_0(x,t), \tag{2.25}$$

and when $q=1$, since $h\neq 0$ and $H_1(x,t)\neq 0$, the zero-order deformation Eq. 2.24 is equivalent to

$$\phi(x,t;1)=u(x,t). \tag{2.26}$$

Thus, according to 2.25 and 2.26, as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [23, 24, 25, 26, 40]. Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{2.27}$$

where,

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}\ |_{q=0}\ .$$

Let the initial guess $u_0(x,t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series 2.27 of $\phi(x,t;q)$ converges at $q=1$, then, we have under these assumptions the solution series

$$u(x,t)=\phi(x,t;1)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t). \tag{2.28}$$

From Eq. 2.27, we can write Eq. 2.24 as follows

$$(1-q)L[\phi(x,t,q)-u_0(x,t)] \tag{2.29}$$

$$=(1-q)L[\sum_{m=1}^{\infty}u_m(x,t)\ q^m]=q\ h\ H_1(x,t)N[\phi(x,t,q)]$$

$$\Rightarrow L[\sum_{m=1}^{\infty}u_m(x,t)\ q^m]-q\ L[\sum_{m=1}^{\infty}u_m(x,t)q^m]$$

$$=q\ h\ H_1(x,t)N[\phi(x,t,q)]$$

By differentiating 2.29 m times with respect to q, we obtain

$$\{L[\sum_{m=1}^{\infty}u_m(x,t)\ q^m]-q\ L[\sum_{m=1}^{\infty}u_m(x,t)q^m]\}^{(m)}=$$

$$\{q\ h\ H_1(x,t)N[\phi(x,t,q)]\}^{(m)}=$$

$$m!\ \ L[u_m(x,t)-u_{m-1}(x,t)]=$$

$$h\ H_1(x,t)\ m\ \frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}\ .$$

Therefore,

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]= \tag{2.30}$$

$$hH_1(x,t)\Re_m(u_{m-1}(x,t)),$$

where,

$$\Re_m(u_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{2.31}$$

and

$$\chi_m=\begin{cases}0, & m\leq 1\\ 1, & m>1\end{cases}$$

Note that the high-order deformation Eq. 2.30 is governing the linear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by 2.31 for any nonlinear operator N. To obtain the approximation solution of Eq. 1.1, according to HAM, let

$$N[u(x,t)]=u(x,t)-g(x)+$$

$$a\int_0^t F_1(u(x,t))dt+b\int_0^t F_2(u(x,t))dt-r\int_0^t u(x,t)dt,$$

so,

$$\Re_m(u_{m-1}(x,t))=u_{m-1}(x,t)-g(x)+$$

$$a\int_0^t F_1(u_{m-1}(x,t))\ dt+b\int_0^t F_2(u_{m-1}(x,t))\ dt \tag{2.32}$$

$$-r\int_0^t u_{m-1}(x,t)\ dt.$$

Substituting 2.32 into 2.30

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hH_1(x,t)[u_{m-1}(x,t) \tag{2.33}$$

$$+a\int_0^t F_1(u_{m-1}(x,t))\ dt+b\int_0^t F_2(u_{m-1}(x,t))\ dt$$

$$-r\int_0^t u_{m-1}(x,t)\ dt+(1-\chi_m)g(x)(x)].$$

We take an initial guess $u_0(x,t)=g(x)$, an auxiliary linear operator $Lu=u$, a nonzero auxiliary parameter $h=-1$, and auxiliary function $H_1(x,t)=1$. This is substituted into 2.33 to give the recurrence relation

$$u_0(x,t)=g(x),$$

$$u_{n+1}(x,t)=-a\textstyle\int_0^t F_1(u_n(x,t))\ dt$$

$$-\ b\int_0^t F_2(u_n(x,t))\ dt+r\int_0^t u_n(x,t)\ dt,\quad n\geq 0. \tag{2.34}$$

Therefore, the solution $u(x,t)$ becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = g(x) + \tag{2.35}$$

$$\sum_{n=1}^{\infty} \Big(-a \int_0^t F_1(u_n(x,t))\, dt - b \int_0^t F_2(u_n(x,t))\, dt$$

$$+ r \int_0^t u_n(x,t)\, dt.$$

Which is the method of successive approximations. If

$$\mid u_n(x,t) \mid < 1,$$

then the series solution 2.35 convergence uniformly.

2.4 Description of the HPM and MHPM

To explain HPM [27, 28, 34, 36, 37, 38, 39], we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u), \tag{2.36}$$

with initial conditions

$$u(x,0) = f(x).$$

According to HPM, we construct a homotopy which satisfies the following relation

$$H(u,p) = Lu - Lv_0 + p\, Lv_0 + p\, [Nu - f(u)] = 0, \tag{2.37}$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions. In HPM, the solution of Eq. 2.37 is expressed as

$$u(x,t) = u_0(x,t) + p\, u_1(x,t) + p^2\, u_2(x,t) + \ldots \tag{2.38}$$

Hence the approximate solution of Eq. 2.36 can be expressed as a series of the power of p, i.e.

$$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \ldots$$

where,

$$u_0(x,t) = g(x),$$

$$\vdots$$

$$u_m(x,t) = \sum_{k=0}^{m-1} -a \int_0^t F_1(u_{m-k-1}(x,t))\, dt - \tag{2.39}$$

$$b \int_0^t F_2(u_{m-k-1}(x,t))\, dt + r \int_0^t u_{m-k-1}(x,t)\, dt,$$

$$m \geq 1.$$

To explain MHPM [30, 31, 32], we consider Eq. 1.1 as

$$L(u) = u(x,t) - g(x) + a \int_0^t F_1(u(x,t))\, dt +$$

$$b \int_0^t F_2(u(x,t))\, dt - r \int_0^t u(x,t)\, dt.$$

Where $F_1(u(x,t)) = g_1(x)h_1(t)$ and $F_2(u(x,t)) = g_2(x)h_2(t)$. We can define homotopy $H(u,p,m)$ by

$$H(u,0,m) = f(u), \quad H(u,1,m) = L(u),$$

where, m is an unknown real number and

$$f(u(x,t)) = u(x,t) - z(x,t).$$

Typically we may choose a convex homotopy by $H(u,p,m) =$

$$(1-p)f(u) + p\, L(u) + p\, (1-p)[m(g_1(x) + g_2(x))] = 0, \tag{2.40}$$

$$0 \leq p \leq 1.$$

Where m is called the accelerating parameters, and for $m = 0$ we define $H(u,p,0) = H(u,p)$, which is the standard HPM. The convex homotopy 2.40 continuously trace an implicity defined curve from a starting point $H(u(x,t) - f(u), 0, m)$ to a solution function $H(u(x,t), 1, m)$. The embedding parameter p monotonically increase from 0 to 1 as trivial problem $f(u) = 0$ is continuously deformed to original problem $L(u) = 0$. The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \tag{2.41}$$

when $p \to 1$, Eq. 2.37 corresponds to the original one and Eq. 2.41 becomes the approximate solution of Eq. 1.1, i.e.,

$$u = \lim_{p \to 1} v = \sum_{m=0}^{\infty} u_m.$$

Where,

$$\begin{aligned}
&u_0(x,t) = g(x),\\
&u_1(x,t) = -a \textstyle\int_0^t F_1(u_0(x,t))\, dt - b \int_0^t F_2(u_0(x,t))\, dt\\
&+ r \textstyle\int_0^t u_0(x,t)\, dt - m(g_1(x) + g_2(x)),\\
&u_2(x,t) = -a \textstyle\int_0^t F_1(u_1(x,t))\, dt - b \int_0^t F_2(u_1(x,t))\, dt\\
&+ r \textstyle\int_0^t u_1(x,t)\, dt + m(g_1(x) + g_2(x)),\\
&\vdots\\
&u_m(x,t) = \textstyle\sum_{k=0}^{m-1} -a \int_0^t F_1(u_{m-k-1}(x,t))\, dt\\
&-b \textstyle\int_0^t F_2(u_{m-k-1}(x,t))\, dt + r \int_0^t u_{m-k-1}(x,t)\, dt,\\
&m \geq 3.
\end{aligned} \tag{2.42}$$

3 Existence and convergency of iterative methods

We set,

$$\alpha_1 := T(\mid a \mid L_1 + \mid b \mid L_2 + \mid r \mid),$$

$$\beta_1 := 1 - T(1-\alpha_1), \quad \gamma_1 := 1 - T\alpha_1.$$

Theorem 3.1 *Let $0 < \alpha_1 < 1$, then Black-Scholes equation 1.1, has a unique solution.*

Proof. Let u and u^* be two different solutions of 1.3 then

$| u - u^* | = | -a \int_0^t [F_1(u(x,t)) - F_1(u^*(x,t))]\ dt$

$-b \int_0^t [F_2(u(x,t)) - F_2(u^*(x,t))]\ dt$

$+r \int_0^t [u(x,t) - u^*(x,t)]\ dt\ |$

$\leq | a | \int_0^t | F_1(u(x,t)) - F_1(u^*(x,t)) |\ dt +$

$| b | \int_0^t | F_2(u(x,t)) - F_2(u^*(x,t)) |\ dt +$

$| r | \int_0^t | u(x,t) - u^*(x,t) |\ dt \leq$

$T(| a | L_1 + | b | L_2 + | r |)\ | u - u^* | = \alpha_1 | u - u^* |.$

From which we get $(1-\alpha_1) \mid u - u^* \mid \leq 0$. Since $0 < \alpha_1 < 1$, then $| u - u^* | = 0$. Implies $u = u^*$ and completes the proof. □

Theorem 3.2 *The series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem 1.1 using MADM convergence when $0 < \alpha_1 < 1$, $| u_1(x,t) | < \infty$.*

Proof. Denote as $(C[J], \| \ . \ \|)$ the Banach space of all continuous functions on J with the norm $\| g(t) \| = max \mid g(t) \mid$, for all t in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\| s_n - s_m \| =$$

$$max_{\forall t \in J} \mid s_n - s_m \mid = max_{\forall t \in J} \mid \sum_{i=m+1}^{n} u_i(x,t) \mid =$$

$$max_{\forall t \in J} \mid -a \int_0^t (\sum_{i=m}^{n-1} A_i)\ dt - b \int_0^t (\sum_{i=m}^{n-1} B_i)\ dt +$$

$$r \int_0^t (\sum_{i=m}^{n-1} u_i)\ dt \mid .$$

From [15], we have

$$\sum_{i=m}^{n-1} A_i = F_1(s_{n-1}) - F_1(s_{m-1}),$$
$$\sum_{i=m}^{n-1} B_i = F_2(s_{n-1}) - F_2(s_{m-1}),$$
$$\sum_{i=m}^{n-1} u_i = (s_{n-1}) - (s_{m-1}).$$

So,

$$\| s_n - s_m \| = max_{\forall t \in J} \mid -a \int_0^t [F_1(s_{n-1}) - F_1(s_{m-1})]\ dt$$

$$-b \int_0^t [F_2(s_{n-1}) - F_2(s_{m-1})] dt$$

$$+r \int_0^t [(s_{n-1}) - (s_{m-1})] dt \mid \leq \mid a \mid \int_0^t \mid F_1(s_{n-1})$$

$$-F_1(s_{m-1}) \mid dt + \mid b \mid \int_0^t \mid F_2(s_{n-1}) - F_2(s_{m-1}) \mid dt$$

$$+ \mid r \mid \int_0^t \mid (s_{n-1}) - (s_{m-1}) \mid \ dt \leq \alpha_1 \| s_n - s_m \| .$$

Let $n = m+1$, then

$$\| s_n - s_m \| \leq \alpha_1 \| s_m - s_{m-1} \| \leq$$

$$\alpha_1^2 \| s_{m-1} - s_{m-2} \| \leq ... \leq \alpha_1^m \| s_1 - s_0 \| .$$

From the triangle inquality we have

$$\| s_n - s_m \| \leq \| s_{m+1} - s_m \| + \| s_{m+2} - s_{m+1} \| + ... +$$

$$\| s_n - s_{n-1} \| \leq [\alpha_1^m + \alpha_1^{m+1} + ... + \alpha_1^{n-m-1}] \| s_1 - s_0 \|$$

$$\leq \alpha_1^m [1 + \alpha_1 + \alpha_1^2 + ... + \alpha_1^{n-m-1}] \| s_1 - s_0 \|$$

$$\leq \alpha_1^m [\frac{1-\alpha_1^{n-m}}{1-\alpha_1}] \| u_1(x,t) \| .$$

Since $0 < \alpha_1 < 1$, we have $(1 - \alpha_1^{n-m}) < 1$, then

$$\| s_n - s_m \| \leq \frac{\alpha_1^m}{1-\alpha_1} max_{\forall t \in J} \mid u_1(x,t) \mid . \quad (3.43)$$

But $| u_1(x,t) | < \infty$, so, as $m \to \infty$, then $\| s_n - s_m \| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete. □

Theorem 3.3 *The maximum absolute truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ to problem 1.1 by using MADM is estimated to be*

$$max \mid u(x,t) - \sum_{i=0}^{m} u_i(x,t) \mid \leq \frac{k\alpha_1^m}{1-\alpha_1}. \quad (3.44)$$

Proof. From inequality 3.43, when $n \to \infty$, then $s_n \to u$ and

$$max \mid u_1(x,t) \mid \leq T(\mid a \mid max_{\forall t \in J} \mid F_1(u_0(x,t)) \mid +$$

$$\mid b \mid max_{\forall t} \mid F_2(u_0(x,t)) \mid + \mid r \mid max_{\forall t \in J} \mid u_0(x,t) \mid).$$

Therefore,

$$\| u(x,t) - s_m \| \leq \tfrac{\alpha_1^m}{1-\alpha_1} T(\mid a \mid max_{\forall t \in J} \mid F_1(u_0(x,t)) \mid$$

$$+ \mid b \mid max_{\forall t} \mid F_2(u_0(x,t)) \mid + \mid r \mid max_{\forall t} \mid u_0(x,t) \mid).$$

Finally the maximum absolute truncation error in the interval J is obtained by 3.44.

Theorem 3.4 *The solution $u_n(x,t)$ obtained from the relation 2.20 using VIM converges to the exact solution of the problem 1.1 when $0 < \alpha_1 < 1$ and $0 < \beta_1 < 1$.*

Table 1: Numerical results for Example 4.1

(x,t)	Errors			
	ADM(n=27)	MADM(n=25)	VIM(n=21)	MVIM(n=18)
$(0.1, 0.13)$	0.0092646	0.0083454	0.0063284	0.00522411
$(0.2, 0.18)$	0.0093652	0.0084525	0.0064473	0.0054136
$(0.3, 0.27)$	0.0094713	0.0085407	0.0064603	0.0054718
$(0.4, 0.32)$	0.0094747	0.0085638	0.0065227	0.0055328
$(0.5, 0.38)$	0.0095805	0.0086159	0.0065863	0.0055787
$(0.7, 0.43)$	0.0096129	0.0086326	0.0066178	0.0056251

(Continue Table 1).

(x,t)	Errors		
	HPM(n=22)	MHPM(n=19)	HAM(n=14)
$(0.1, 0.13)$	0.0073609	0.0054283	0.0044536
$(0.2, 0.18)$	0.0074447	0.0055081	0.0045475
$(0.3, 0.27)$	0.0075226	0.0056205	0.0046231
$(0.4, 0.32)$	0.0075656	0.0056738	0.0046815
$(0.5, 0.38)$	0.0076267	0.0057048	0.0047462
$(0.7, 0.43)$	0.0076926	0.0057659	0.0047826

Proof.

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}([u_n(x,t) - g(x) \\ + a\int_0^t F_1(u_n(x,t))\ dt \\ + b\int_0^t F_2(u_n(x,t))\ dt \\ - r\int_0^t u_n(x,t))\ dt]), \end{aligned} \tag{3.45}$$

$$\begin{aligned} u(x,t) &= u(x,t) \\ &- L_t^{-1}([u(x,t) - g(x) + a\int_0^t F_1(u(x,t))\ dt \\ &+ b\int_0^t F_2(u(x,t))\ dt - r\int_0^t u(x,t))\ dt]). \end{aligned} \tag{3.46}$$

By subtracting relation 3.45 from 3.46,

$$\begin{aligned} u_{n+1}(x,t) - u(x,t) = u_n(x,t) - u(x,t) - L_t^{-1}(u_n(x,t) \\ - u(x,t) + a\int_0^t [F_1(u_n(x,t)) \\ - F_1(u(x,t))]\ dt \\ + b\int_0^t [F_2(u_n(x,t)) \\ - F_2(u(x,t))]\ dt \\ - r\int_0^t [u_n(x,t) - u(x,t)]\ dt), \end{aligned}$$

if we set, $e_{n+1}(x,t) = u_{n+1}(x,t) - u_n(x,t)$, $e_n(x,t) = u_n(x,t) - u(x,t)$,$\mid e_n(x,t^*) \mid = max_t \mid e_n(x,t) \mid$ then since e_n is a decreasing function with respect to t from the mean value theorem we can write,

$$\begin{aligned} e_{n+1}(x,t) = e_n(x,t) + L_t^{-1}(-e_n(x,t) \\ - a\int_0^t [F_1(u_n(x,t)) - F_1(u(x,t))]\ dt \\ - b\int_0^t [F_2(u_n(x,t)) - F_2(u(x,t))]\ dt \\ + r\int_0^t [u_n(x,t) - u(x,t)]\ dt) \end{aligned}$$

$$\leq e_n(x,t) + L_t^{-1}[-e_n(x,t) + L_t^{-1} \mid e_n(x,t) \mid$$

$$(T(\mid a \mid L_1 + \mid b \mid L_2 + \mid r \mid)] \leq e_n(x,t) - Te_n(x,\eta)$$

$$+T(\mid a \mid L_1 + \mid b \mid L_2 + \mid r \mid)L_t^{-1}L_t^{-1} \mid e_n(x,t) \mid$$

$$\leq (1 - T(1-\alpha_1) \mid e_n(x,t^*) \mid,$$

where $0 \leq \eta \leq t$. Hence, $e_{n+1}(x,t) \leq \beta_1 \mid e_n(x,t^*) \mid$. Therefore,

$$\|e_{n+1}\| = max_{\forall t \in J} \mid e_{n+1} \mid \leq \beta_1\ max_{\forall t \in J} \mid e_n \mid$$

$$\leq \beta_1 \|e_n\|.$$

Since $0 < \beta_1 < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete. □

Theorem 3.5 *The solution $u_n(x,t)$ obtained from the relation 2.22 using MVIM for the problem 1.1 converges when $0 < \alpha_1 < 1$, $0 < \gamma_1 < 1$.*

Proof. The Proof is similar to the previous theorem.

Theorem 3.6 *The maximum absolute truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ to problem 1.1 by using VIM is estimated to be*

$$\|e_n\| \leq \frac{\beta_1^n k^{'}}{1-\beta_1}, \quad k^{'} = max \mid u_1(x,t) \mid .$$

Proof.

Theorem 3.7 *If the series solution 2.34 of problem 1.1 using HAM convergent then it converges to the exact solution of the problem 1.1.*

Proof. We assume:

$$\begin{aligned} u(x,t) &= \sum_{m=0}^{\infty} u_m(x,t), \widehat{F}_1(u(x,t)) \\ &= \sum_{m=0}^{\infty} F_1(u_m(x,t)), \widehat{F}_2(u(x,t)) \\ &= \sum_{m=0}^{\infty} F_2(u_m(x,t)). \end{aligned}$$

Where,

$$\lim_{m\to\infty} u_m(x,t) = 0.$$

We can write,

$$\begin{aligned} &\sum_{m=1}^{n} [u_m(x,t) - \chi_m u_{m-1}(x,t)] \\ &= u_1 + (u_2 - u_1) + ... + (u_n - u_{n-1}) \\ &= u_n(x,t). \end{aligned} \tag{3.47}$$

Hence, from 3.47,

$$\lim_{n\to\infty} u_n(x,t) = 0. \tag{3.48}$$

So, using 3.48 and the definition of the linear operator L, we have

$$\begin{aligned} &\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] \\ &= L[\sum_{m=1}^{\infty} [u_m(x,t) - \chi_m u_{m-1}(x,t)]] \\ &= 0. \end{aligned}$$

therefore from 2.30, we can obtain that,

$$\begin{aligned} &\sum_{m=1}^{\infty} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] \\ &= hH_1(x,t) \sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x,t)) \\ &= 0. \end{aligned}$$

Since $h \neq 0$ and $H_1(x,t) \neq 0$, we have

$$\sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x,t)) = 0. \tag{3.49}$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation 3.49 and simplifying it , we have

$$\begin{aligned} \sum_{m=1}^{\infty} \Re_{m-1}(u_{m-1}(x,t)) &= \sum_{m=1}^{\infty} [a \int_0^t F_1(u_{m-1}(x,t))\ dt \\ &\quad + b \int_0^t F_2(u_{m-1}(x,t))\ dt \\ &\quad - r \int_0^t u_{m-1}(x,t)\ dt \\ &\quad + (1-\chi_m) g(x)] \\ &= u(x,t) - g(x) \\ &\quad + a \int_0^t \widehat{F}_1(u(x,t))\ dt \\ &\quad + b \int_0^t \widehat{F}_2(u(x,t))\ dt \\ &\quad - r \int_0^t u(x,t)\ dt. \end{aligned} \tag{3.50}$$

From 3.49 and 3.50, we have

$$\begin{aligned} u(x,t) = g(x) - a \int_0^t \widehat{F}_1(u(x,t))\ dt \\ - b \int_0^t \widehat{F}_2(u(x,t))\ dt + r \int_0^t u(x,t)\ dt. \end{aligned}$$

Therefore, $u(x,t)$ must be the exact solution. □

Theorem 3.8 *The maximum absolute truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$to problem 1.1 by using HAM is estimated to be*

$$\|e_n\| \leq \frac{\alpha_1^n k^{'}}{1-\alpha_1}, \quad k^{'} = max \mid u_1(x,t) \mid .$$

Proof. The Proof is similar to the 3.6 Theorem.

Theorem 3.9 *If $\mid u_m(x,t) \mid \leq 1$, then the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem 1.1 converges to the exact solution by using HPM.*

Proof. We set,

$$\phi_n(x,t) = \sum_{i=1}^{n} u_i(x,t),$$

$$\phi_{n+1}(x,t) = \sum_{i=1}^{n+1} u_i(x,t).$$

$$\begin{aligned} &\mid \phi_{n+1}(x,t) - \phi_n(x,t) \\ &\mid \\ &= D(\phi_{n+1}(x,t), \phi_n(x,t)) \\ &= D(\phi_n + u_n, \phi_n) \\ &= D(u_n, 0) \end{aligned}$$

$$\leq \sum_{k=0}^{m-1} \mid a \mid \int_0^t \mid F_1(u_{m-k-1}(x,t)) \mid\ dt$$

$$+ \mid b \mid \int_0^t \mid F_2(u_{m-k-1}(x,t)) \mid \ dt +$$
$$\mid r \mid \int_0^t \mid u_{m-k-1}(x,t) \mid \ dt. \rightarrow$$
$$\sum_{n=0}^{\infty} \parallel \phi_{n+1}(x,t) - \phi_n(x,t) \parallel \leq m\alpha_1 \mid g(x) \mid \sum_{n=0}^{\infty} (m\alpha_1)^n.$$

Therefore,

$$\lim_{n\to\infty} u_n(x,t) = u(x,t).$$

Theorem 3.10 *If $\mid u_m(x,t) \mid \leq 1$, then the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ of problem 1.1 converges to the exact solution by using MHPM.*

Proof. The Proof is similar to the previous theorem.

Theorem 3.11 *The maximum absolute truncation error of the series solution $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$to problem 1.1 by using HPM is estimated to be*

$$\|e_n\| \leq \frac{(n\alpha_1)^n nk'}{1-\alpha_1}, \quad k' = max \mid u_1(x,t) \mid.$$

Proof. The Proof is similar to the 3.6 Theorem.

4 Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm 1:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations 2.10 for ADM, 2.13 for MADM, 2.34 for HAM, 2.39 for HPM and 2.42 for MHPM .
Step 3. If $\mid u_{n+1} - u_n \mid < \varepsilon$ then go to step 4, else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.
Algorithm 2:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations 2.20 for VIM and 2.21 for MVIM.
Step 3. If $\mid u_{n+1} - u_n \mid < \varepsilon$ then go to step 4, else $n \leftarrow n+1$ and go to step 2.
Step 4. Print $u_n(x,t)$ as the approximate of the exact solution.

Example 4.1 *Consider the Black-Scholes equation as follows:*

$$u_t(x,t) + x^2 u_{xx}(x,t) + 0.5xu_x(x,t) - u(x,t) = 0.$$

With initial condition:

$$g(x) = x^3.$$

The exact solution is $u(x,t) = x^3 e^{-6.5t}$. $\epsilon = 10^{-3}$.

Table 1, shows that, approximate solution of the Black-Scholes equation is convergence with 14 iterations by using the HAM . By comparing the results of Table 1, we can observe that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM.

5 Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are convergent are rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the Black-Scholes equation. For this purpose, we showed that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM.

Acknowledgments

The authors would like to express their sincere appreciation to the Department of Mathematics, Islamic Azad University, Central Tehran Branch for their cooperation.

References

[1] E. Ahmed, H. A. Abdusalam, *On modified Black-Scholes equation*, Chaos Solitons Fractals 22 (2004) 583-587.

[2] L. Jdar, P. Sevilla-Peris, JC. Corts, R. Sala, *A new direct method for solving the Black-Scholes equation*, Appl. Math. Lett. 18 (2005) 29-32.

[3] Marianito R. Rodrigo, Rogemar S. Mamon, *An alternative approach to solving the Black-Scholes equation with time-varying parameters*, Appl. Math. Lett. 19 (2006) 398-402.

[4] Joseph Stampfli, Victor Goodman, *The mathematics of finance*, in: Brooks/Cole Series in Advanced Mathematics, Brooks/Cole, Pacific Grove, CA, 2001, Modeling and hedging.

[5] M. Bohner, Y. Zheng, *On analytical solution of the Black-Scholes equation*, Appl. Math. Lett 22 (2009) 309-313.

[6] R. Company, E. Navarro, J. R. Pintos, E. Ponsoda, *Numerical solution of linear and nonlinear Black-Scholes option pricing equations*, Comput. Math. Appl. 56 (2008) 813-821.

[7] Z. Cen, A. Le, *A robust and accurate finite difference method for a generalized Black- Scholes equation*, J. Comput. Appl. Math. 235 (2011) 3728-3733.

[8] R. Company, L. Jodar, J. R. Pintos, *A numerical method for European Option Pricing with transaction costs nonlinear equation*, Math. Comput. Modell. 50 (2009) 910-920.

[9] F. Fabiao, M. R. Grossinho, O. A. Simoes, *Positive solutions of a Dirichlet problem for a stationary nonlinear Black Scholes equation*, Nonlinear Anal. 71 (2009) 4624-4631.

[10] P. Amster, C. G. Averbuj, M. C. Mariani, *Solutions to a stationary nonlinear Black- Scholes type equation*, J. Math. Anal. Appl. 276 (2002) 231238.

[11] S. H. Behriy, H. Hashish, I. L. E-Kalla, A. Elsaid, *A new algorithm for the decomposition solution of nonlinear differential equations*, Appl. Math. Comput. 54 (2007) 459-466.

[12] M. A. Fariborzi Araghi, Sh. S. Behzadi, *Solving nonlinear Volterra-Fredholm integral differential equations using the modified Adomian decomposition method*, Comput. Methods in Appl. Math. 9 (2009) 1-11.

[13] A. M. Wazwaz, *Construction of solitary wave solution and rational solutions for the KdV equation by ADM*, Chaos, Solution and fractals 12 (2001) 2283-2293.

[14] A. M. Wazwaz, *A first course in integral equations*, WSPC, New Jersey; 1997.

[15] I. L. El-Kalla, *Convergence of the Adomian method applied to a class of nonlinear integral equations*, Appl. Math. Comput. 21 (2008) 372-376.

[16] J. H. He, X. H. Wu, *Exp-function method for nonlinear wave equations*, Chaos, Solitons and Fractals 30(2006) 700-708.

[17] J. H. He, *Variational principle for some nonlinear partial differential equations with variable cofficients*, Chaos, Solitons and Fractals 19 (2004) 847-851.

[18] J. H. He, Wang. Shu-Qiang, *Variational iteration method for solving integro-differential equations*, Physics Letters A 367 (2007) 188-191.

[19] J. H. He, *Variational iteration method some recent results and new interpretations*, J. Comp. Appl. Math. 207 (2007) 3-17.

[20] M. A. Fariborzi Araghi, Sh. S. Behzadi, *Solving nonlinear Volterra-Fredholm integro-differential equations using He's variational iteration method*, International Journal of Computer Mathematics, DOI: 10.1007/s12190-010-0417-4, 2010.

[21] T. A. Abassy, El-Tawil, H. El. Zoheiry, *Toward a modified variational iteration method (MVIM)* , J. Comput. Appl. Math. 207 (2007) 137-147.

[22] T. A. Abassy, El-Tawil, H. El. Zoheiry, *Modified variational iteration method for Boussinesq equation*, Comput. Math. Appl. 54 (2007) 955-965.

[23] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman and Hall/CRC Press, Boca Raton, 2003.

[24] S. J. Liao , *Notes on the homotopy analysis method: some definitions and theorems*, Communication in Nonlinear Science and Numerical Simulation 14 (2009) 983-997.

[25] M. A. Fariborzi Araghi, Sh. S. Behzadi, *Numerical solution of nonlinear Volterra-Fredholm integro-differential equations using Homotopy analysis method*, Journal of Applied Mathematics and Computing http://dx.doi.org/10.1080/00207161003770394/.

[26] E. Babolian, J. Saeidian, *Analytic approximate solutions to Burger, Fisher, Huxley equations and two combined forms of these equations*, Commun Nonlinear Sci Numer Simulat. 14 (2009) 1984-1992.

[27] J. Biazar, H. Ghazvini, *Convergence of the homotopy perturbation method for partial differential equations*, Nonlinear Analysis: Real World Application 10 (2009) 2633-2640.

[28] M. Ghasemi , M. Tavasoli , E. Babolian, *Application of He's homotopy perturbation method of nonlinear integro-differential equation*, Appl. Math. Comput. 188 (2007) 538-548.

[29] A. Golbabai , B. Keramati, *Solution of non-linear Fredholm integral equations of the first kind using modified homotopy perturbation method*, Chaos Solitons and Fractals 5 (2009) 2316-2321.

[30] M. Javidi, *Modified homotopy perturbation method for solving linear Fredholm integral equations*, Chaos Solitons and Fractals 50 (2009) 159-165.

[31] M. A. Fariborzi Araghi, S. Sh. Behzadi, *Numerical solution for solving Burger's-Fisher equation by using Iterative Methods*, Mathematical and Computational Applications 16 (2011) 443-455.

[32] S. Abbasbandy, *Modified homotopy perturbation method for nonlinear equations and comparsion with Adomian decomposition method*, Appl. Math. Comput. 172 (2006) 431-438.

[33] A. A.Yildirim, S. T. Mohyud-Din, D. H. Zhang, *Analytical solutions to the pulsed Klein-Gordon equation using Modified Variational Iteration Method (MVIM) and Boubaker Polynomials Expansion Scheme (BPES)*, Computers and Mathematics with Applications 59 (2010) 2473-2477.

[34] S. A. Sezer, A. A.Yildirim, S. T. Mohyud-Din, *Hes homotopy perturbation method for solving the fractional KdV-Burgers-Kuramoto equation*, International Journal of Numerical Methods for Heat and Fluid Flow 21 (2011) 448-458.

[35] A. A.Yildirim, *Variational iteration method for modified Camassa-Holm and Degasperis-Procesi equations*, International Journal for Numerical Methods in Biomedical Engineering 26 (2010) 266-272.

[36] A. A.Yildirim, *Solution of BVPs for Fourth-Order Integro-Differential Equations by using Homotopy Perturbation Method*, Computers and Mathematics with Applications 56 (2008) 3175-3180.

[37] A. A.Yildirim, *The Homotopy Perturbation Method for Approximate Solution of the Modified KdV Equation*, Zeitschrift fr Naturforschung A,A Journal of Physical Sciences 63 (2008) 621-626.

[38] A. A.Yildirim, *Application of the Homotopy perturbation method for the Fokker-Planck equation*, International Journal for Numerical Methods in Biomedical Engineering 26 (2010) 1144-1154.

[39] Sh. S. Behzadi, *The convergence of homotopy methods for nonlinear Klein-Gordon equation*, J. Appl. Math. Informatics 28 (2010) 1227-1237.

[40] Sh. S. Behzadi, MA. Fariborzi Araghi, *The use of iterative methods for solving Naveir-Stokes equation*, J. Appl. Math. Informatics 29 (2011) 1-15.

[41] S. Abbasbandy, *Numerical method for non-linear wave and diffusion equations by the variational iteration method*, Q1 Int. J. Numer. Methods Eng. 73 (2008) 1836-1843.

[42] S. Abbasbandy, A. Shirzadi, *The variational iteration method for a class of eight-order boundary value differential equations*, Z. Naturforsch 63 (2008) 745-751.

Tofigh Allahviranloo was born in the west Azarbayejan, khoy. He has got Phd degree in 2001 now he is a professor of applied mathematics. He has published more than 130 papers in international journals and he is editing more than 5 journals as an editor in chief and associated editor.

Shadan Sadigh Behzadi she has born in Tehran- Iran in 1983. Got B.Sc and M.Sc degrees in applied mathematics, numerical analysis field from Islamic Azad University, Central Tehran Branch and PHD degree in applied mathematics, here numerical analysis field from Science and Research Branch, Islamic Azad University. Main research interest includ numerical solution of nonlinear Fredholm- Volterra integro-differential equation systems, ordinary differential equation, partial differential equation, fuzzy system and some equations in fluid dynamics.