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Research Article

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A Four-Step Approach to Ranking Z-numbers

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Abstract
As the concept of Z-numbers is relatively new in fuzzy sets, its basic theoretical aspects have not yet been paid attention particularly in the field of ranking, which plays a substantial role in data analysis, decision optimization, forecasting, etc. In this matter, a few methods are developed to rank them but by including some shortcomings. It caused that, illogically providing the same ranks for many Z-numbers. Algorithms of some of the presented ranking methods also have computational complexity and as a result, they have a high computational cost. Moreover, the proposed algorithms included computational complications and as a result, they have a high computational cost. To fulfill this shortcoming, this paper develops a novel approach for ranking Z-number. Finally, the correctness of the subject is shown in several examples.

Keywords: Z-Numbers; Ranking; New approach; Data analysis; Decision, Forecasting

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1. Introduction

People can take rational decisions in several conditions based on inconclusive, imprecise, and incomplete information. In this regard, as formulating this ability is somewhat of a challenge that is difficult to accomplish, new numbers are required for such data to show the reliability of ambiguous information. For this purpose, Zadeh (2011) proposed new numbers so-called Z-numbers [1].

Compared to fuzzy numbers, they contain another type of uncertainty from distribution functions in order to increase the human judgment level for decision making and predictions; so, they are more useful than fuzzy numbers. To study about fuzzy numbers and ranking of fuzzy numbers, you can refer to references [2-4].

Here, it is worthwhile to mention that as this concept was developed in the last decade, various studies have been conducted in terms of Z-numbers [5-7], and their ranking as follows:

Kang et al. [8], Bakar and Gegov [9], Jiang et al. [10], Alive et al. [11] proposed a method for ranking Z-numbers. Besides, the method for ranking Z-number was presented by Ezadi and Allahviranloo [12]. Hadayegh et al. [13] described a method for ranking Z-numbers based on the center of gravity. For further reading, see the references [14-16].

Nevertheless, some of these methods contain gaps that have not been fulfilled in recent years. Accordingly, this paper aims to survey them as well as to propose a new method for ranking Z-numbers, so that it does not include previous disadvantages. It means that the proposed method has a simpler algorithm than the existing methods.

It is shown that we do not necessarily need to use the center of gravity method in ranking. There are some Z- numbers that are intuitively clear that they do not have the same rank, but some ranking methods such as [8,9,12,17] show the same rank for them, and our proposed method ranks these numbers.

This paper is organized as follows: In the next section, the basic definitions and concepts are provided. Afterwards, the fundamental concepts are delivered in the next section. In section 3, the shortcomings of some existing methods are first distinguished to rank Z-numbers, and then a novel approach is proposed. In section 4, the obtained results are presented. Ultimately, the conclusion and some further research bring in Section 5.

2. Fundamental concepts

Definition 1.2. Consider X be a global space. The fuzzy set A in X is a space of points defined as the set $A=\left\{x,\mu_A(x)\middle| x\in X\right\}$, which is characterized by the membership function $\mu_A(x)$ where a general element x in X is a real number in the closed interval [0,1]. Note that the values of x represent the degree of membership in set A [4].

Definition 2.2. Fuzzy numbers

The fuzzy number $\tilde{A}=(a,b,c,d;w)$ is defined as a fuzzy subspace of the real set R via the membership function $f_{\tilde{A}}(x)$ including the following characteristics:

a) $f_{\tilde{A}}$ is a continuous function from R to [0, w] where $0 \le w \le 1$.

b) For each member $x\left(-\infty, a\right]$, $f_{\tilde{A}}\left(x\right) = 0$.

c) $f_{\tilde{A}}$ is strictly ascending in the interval $[a, b]$.

d) For each member x of [b,c], $f_{\tilde{A}}\left(x\right) = w$ where w is a constant number and $0 \le w \le 1$.

e) $f_{\tilde{A}}$ is strictly descending in the interval $[c, d]$.

g) For each member x, $\left[d, +\infty\right)$, $f_{\tilde{A}}\left(x\right) = 0$ [4].

Definition 2.3. Fuzzy numbers representation

In the fuzzy number $\tilde{A} = (a, b, c, d; w)$ if w=1, the fuzzy number $\tilde{A}$ is called a normal trapezoidal fuzzy number, represented as $\tilde{A} = (a, b, c, d)$ [4].

If $b = c$, the number is called a triangular fuzzy number.

If $a = b$ and $c = d$, then the fuzzy number $\tilde{A}$ is an interval.

If $a = b = c = d$ then it is called a real number.

Furthermore, a real number is a special form of a fuzzy number where $a = b = c = d$, and every interval is either a fuzzy interval or fuzzy number where $a = b$ and $c = d$. Here, it should be mentioned that although generating such numbers is very simple, determining the relevant parameters (i.e. points a, b, c, d) strongly depends on the related fields, which is required to be optimized.

Definition 2.4. Definition of Z-numbers

Zadeh (2011) developed Z-numbers with an unknown variable X in the form of (A, B) [1]. The first component A represents the limitation whereas the second component B, represents the certainty of the first component. It should be noted that the triple (x, A, B) is introduced as a Z-value, indicating that this triple is correspondence to that x is equal to (A, B). Here, $Z = (A, B)$ provides information about the variable x.

To better understand it, suppose that 50 minutes be very sure while 40 minutes be sure. Thus, regarding the valuation proposed by Zadeh [17], variable x is interpreted as follows:

$$prob\left(x\, is\, A\right) is\, B$$

In other words, the probability that variable x is around 50 minutes is very certain whereas it is certain if it is around 40 minutes. Indeed:

$$R\left(x\right): x\, is\, A \rightarrow \mathrm{P}oss\left(x = u\right) = \mu_A\left(u\right)$$

$$p\left(x\, is\, A\right) = \int_R \mu_A\left(u\right) p_x\left(u\right) du \;\; is \;\; B$$

where μ_A is the membership function of the fuzzy set A and u is a value of x. $p_x\left(u\right)$ is the probability density function of x and $p\left(x = u\right)$ is the probability function of x. It is worth mentioning that if the basic probability distribution is unknown, it is obvious that the probability distribution is a fuzzy number.

3. The problem and proposed approach

Now, two simple examples are provided to survey the weakness of the methods of Mohammad et al. [17], Bakr [9], Kang [8] and the hyperbolic method of Ezadi and Allahviranloo [12].

For instance, suppose $Z_i\left(A, B_i\right), i=1,2$ are Z-numbers such that $A=\left(0.1, 0.3, 0.3, 0.5; 1\right)$

Let $B_1=\left(0.1, 0.3, 0.3, 0.3\right)$ be fuzzy numbers while $B_2=\left(0.3, 0.3, 0.3, 0.3\right)$ be real numbers. Obviously, Z_1 and Z_2 do not have the same rank, but the methods of [8,9,12,17] indicate the same rank.

Meanwhile, consider $Z_1=\left(A_1, B_1\right)$ and $Z_2=\left(A_2, B_2\right)$ are two Z-numbers such that $A_1=\left(0.1, 0.3, 0.3, 0.5\right)$, $A_2=\left(0.1, 0.2, 0.4, 0.5\right)$, $B_1=\left(0.1, 0.2, 0.4, 0.5\right)$ and $B_2=\left(0.1, 0.3, 0.3, 0.5\right)$. Hence, these two Z-numbers do not have the same rank intuitively, but the methods [8,9,12,17] reveal the same rank.

3.1. The proposed method

As mentioned before, a novel method id developed to rank Z-numbers.

Let $Z_i=\left(A_i, B_i\right)$ be a Z-number such that $A_i=\left(a_{i1}, a_{i2}, a_{i3}, a_{i4}; \omega_{A_i}\right)$ and $B_i=\left(b_{i1}, b_{i2}, b_{i3}, b_{i4}; \omega_{B_i}\right)$. Then:

Step 1: First, A_i is converted into a standard number $A_i^*=\left(a_{i1}^*, a_{i2}^*, a_{i3}^*, a_{i4}^*; \omega_{A_i}\right)$ so that for $j=1,2,3,4$, $i=1,\ldots,n$, $a_{ij}^*=\dfrac{a_{ij}}{r}, \omega_{A_i}\in\left[0,1\right]$ and $r=\max\limits_{i,j}\left(a_{ij},1\right)$ Also, the fuzzy number B_i is converted into a standard number $B_i^*=\left(b_{i1}^*, b_{i2}^*, b_{i3}^*, b_{i4}^*; \omega_{B_i}\right)$, so that for $j=1,2,3,4$, $i=1,\ldots,n$, $b_{ij}^*=\dfrac{b_{ij}}{k}, \omega_{B_i}\in\left[0,1\right]$ and $k=\max\limits_{i,j}\left(b_{ij},1\right)$

Step 2: Calculating $\overline{A}_i^*$ and $\overline{B}_i^*$ as follows

$$\overline{A}_i^*=\frac{a_{i1}^*, a_{i2}^*, a_{i3}^*, a_{i4}^*}{4} \qquad (3.1)$$

$$\overline{B}_i^*=\frac{b_{i1}^*, b_{i2}^*, b_{i3}^*, b_{i4}^*}{4} \qquad (3.2)$$

Step 3: B_i^* is converted into a real number β_i using E_q. (3.3)

$$B_i = \left| M_{B_i^*}(2)^2 - \left| M_{B_i^*}(3) \right| \right| * \bar{B}_i^* \quad (3.3)$$

where

$$M_{B_i^*}(t) = \frac{\omega_{B_i}}{t}\left((e-1)\left(b_{i2}^*\right)^t - (1-e)\left(b_{i3}^*\right)^t - \left(b_{i1}^*\right)^t + \left(b_{i4}^*\right)^t\right) \quad (3.4)$$

where $B_i \in [0,1]$ and $e = 3.14$.

Step 4: Calculating the ranks of Z-number:

$$rank(Z_i) = \left| M_{Z_i^*}(2)^2 - \left| M_{Z_i^*}(3) \right| \right| * \bar{A}_i^* \quad (3.5)$$

$$M_{B_i^*}(t) = \frac{B_i \omega_{A_i}}{t}\left((e-1)\left(a_{i2}^*\right)^t - (1-e)\left(a_{i3}^*\right)^t - \left(a_{i1}^*\right)^t + \left(a_{i4}^*\right)^t\right) \quad (3.6)$$

The higher the value of $rank(Z_i)$, the higher the preference of $rank(Z_i)$.

Property 1: Let $Z_i(A_i, B_i)$ be a Z-number. For $A_i = (0,0,0,0;1)$ and $B_i = (0,0,0,0;1)$, it is clear that $rank(Z_i) = 0$.

Proof. According to Eq. (3.3), $rank(A_i) = 0$ and $rank(B_i) = 0$. Besides, according to E_q. (3.6) $M_{B_i^*}(t) = 0$ and as a result $rank(Z_i) = 0$.

Property 2: Suppose $Z_i = (A_i, B_i)$ be a Z-number such that $A_i = (-r-, r, -r, -r; 1)$ and $B_i = (r, r, r, r; 1)$ and $Rank\ (Z_i) = k$ in this case when $A_i = (-r-, r, -r, -r; 1)$ and $B_i = (r, r, r, r; 1)$ then $Rank\ (Z_i) = -k$.

In the following, $Rank\ (Z_i) \in R$ is in the form of ranking function of Z-numbers such that for each $i, j \geq 1$:

1. $Rank\ (Z_i) > Rank\ (Z_j)$ if and only if $Z_i \succ Z_j$.

2. $Rank\ (Z_i) < Rank\ (Z_j)$ if and only if $Z_i \prec Z_j$.

3. $Rank\ (Z_i) = Rank\ (Z_j)$ if and only if $Z_i \sim Z_j$.

4. Results

Regarding the provided examples for assessing the originality of the proposed method:

Example 4.1. Suppose $Z_i = (A, B_i), i = 1,2,3,4,5,6$ are Z-numbers such that $A = (0.1, 0.3, 0.3, 0.5; 1)$ depicted in Fig. 4.1 and B_i in Fig 4.2.

Table 4.1 reports the results of some existing methods with the proposed method. Table 4.2 lists the comparison of the results obtained from the proposed method with some other methods. Table 4.2 provides the obtained results of comparing our proposed method others.

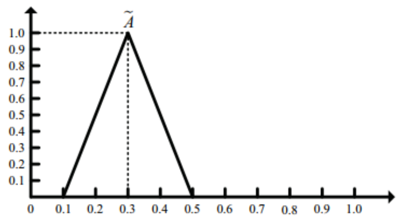

Fig. 4.1. A for all Z-numbers

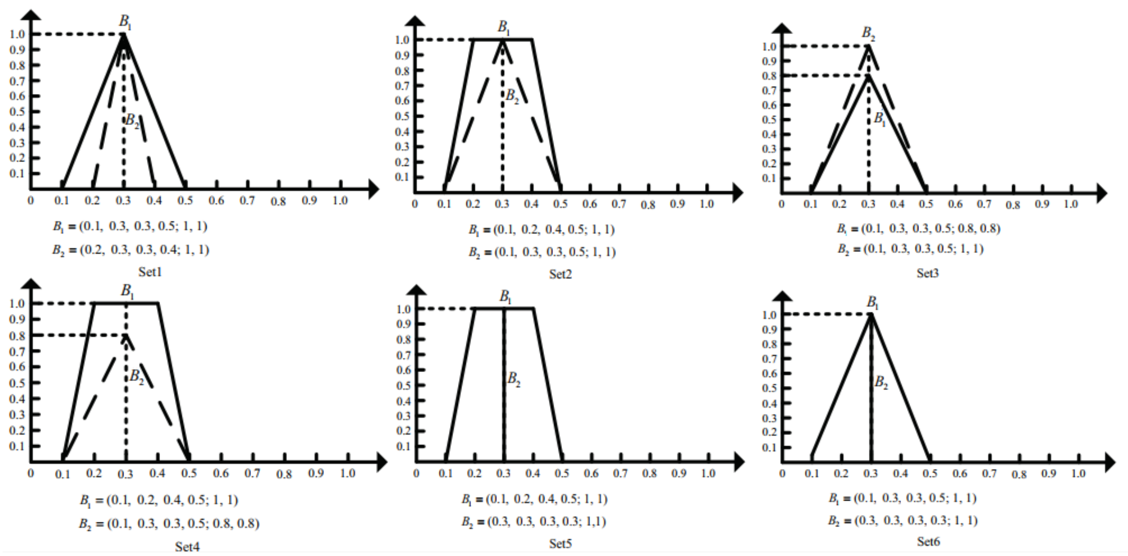

Fig. 4.2. B_i for six sets of Z-numbers.

Table 4.1. The ranks of the proposed method with some existing methods

Method	Set1		Set2		Set3	
	Z_1	Z_2	Z_1	Z_2	Z_1	Z_2
Mohammad et al. [17]	0.0774	0.0774	0.0774	0.0774	0.0774	0.0774
Bakar [9]	0.0288	0.0288	0.0288	0.0288	0.0288	0.0288
Kang et al. [8]	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000
Hyperbolic [12]	0.71	0.71	0.71	0.71	0.71	0.71
Jiang [10]	0.1166	0.1197	0.1217	0.1166	0.1146	0.1166
The proposed method	0.00013	9.1e-05	0.00015	0.00013	8.2e-05	0.00013

Method	Set4		Set5		Set6	
	Z_1	Z_2	Z_1	Z_2	Z_1	Z_2
Mohammad et al. [17]	0.0774	0.0774	0.0774	0.0774	0.0774	0.0774
Bakar [9]	0.0288	0.0288	0.0288	0.0288	0.0288	0.0288
Kang et al. [8]	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000
Hyperbolic [12]	0.71	0.71	0.71	0.71	0.71	0.71
Jiang [10]	0.1217	0.1146	0.1217	0.1512	0.1166	0.1512
The proposed method	0.00015	8.2e-05	0.00015	6.2e-05	0.00013	6.2e-05

Table 4.2. Comparison of the results of the proposed method with some other methods

Method	Set1		Set2		Set3	
	Z_1	Z_2	Z_1	Z_2	Z_1	Z_2
Mohammad et al. [17]	$Z_1 \approx Z_2$		$Z_1 \approx Z_2$		$Z_1 \approx Z_2$	
Bakar [9]	$Z_1 \approx Z_2$		$Z_1 \approx Z_2$		$Z_1 \approx Z_2$	
Kang et al. [8]	$Z_1 \approx Z_2$		$Z_1 \approx Z_2$		$Z_1 \approx Z_2$	
Hyperbolic [12]	$Z_1 \approx Z_2$		$Z_1 \approx Z_2$		$Z_1 \approx Z_2$	
Jiang [10]	$Z_1 \prec Z_2$		$Z_1 \succ Z_2$		$Z_1 \prec Z_2$	
The proposed method	$Z_1 \succ Z_2$		$Z_1 \succ Z_2$		$Z_1 \prec Z_2$	
Method	Set4		Set5		Set6	
	Z_1	Z_2	Z_1	Z_2	Z_1	Z_2
Mohammad et al. [17]	$Z_1 \approx Z_2$		$Z_1 \approx Z_2$		$Z_1 \approx Z_2$	
Bakar [9]	$Z_1 \approx Z_2$		$Z_1 \approx Z_2$		$Z_1 \approx Z_2$	
Kang et al. [8]	$Z_1 \approx Z_2$		$Z_1 \approx Z_2$		$Z_1 \approx Z_2$	
Hyperbolic [12]	$Z_1 \approx Z_2$		$Z_1 \approx Z_2$		$Z_1 \approx Z_2$	
Jiang [10]	$Z_1 \succ Z_2$		$Z_1 \prec Z_2$		$Z_1 \prec Z_2$	
The proposed method	$Z_1 \succ Z_2$		$Z_1 \succ Z_2$		$Z_1 \succ Z_2$	

Example 4.2. Suppose $Z_1 = (A_1, B)$ and $Z_2 = (A_2, B)$ are two Z-numbers such that $A_1 = (-\frac{5}{2}, -\frac{1}{2}, 0, 1)$,$A_2 = (-\frac{9}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4})$,$B = (0.1, 0.3, 0.3, 0.5)$.

Table 4.3 reports the ranks using the proposed method and its comparison with the hyperbolic method [12].

Table 4.3. The ranks using the proposed method and comparing it with the hyperbolic method [12]

Z_i	Hyperbolic method [12]	proposed method
	$Rank\ (Z_i)$	$Rank\ (Z_i)$
$Z_1 = \left((-\frac{5}{2}, -\frac{1}{2}, 0, 1), (0.1, 0.3, 0.3, 0.5)\right)$	-0.14	-0.14
$Z_2 = \left((-\frac{9}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}), (0.1, 0.3, 0.3, 0.5)\right)$	-0.11	-0.010
Final result	$Z_1 \prec Z_2$	$Z_1 \prec Z_2$

Example 4.3. Suppose Z_1 ،Z_2 , Z_3 are as follows

$$Z_1 = \big((1,3,4,7;0.8),(2,3,4,8;0.8)\big)$$
$$Z_2 = \big((2,4,5,8;0.8),(3,4,5,9;0.8)\big)$$
$$Z_3 = \big((2,3,5,7;0.8),(1,4,5,9;0.8)\big)$$

Step 1. Standardization of the first part of each Z-number. So we have

$$Z_1 = \left(\left(\frac{1}{7},\frac{3}{7},\frac{4}{7},1;0.8\right),\left(\frac{1}{4},\frac{3}{8},\frac{1}{2},1;0.8\right)\right)$$
$$Z_2 = \left(\left(\frac{1}{4},\frac{1}{2},\frac{5}{8},1;0.8\right),\left(\frac{3}{9},\frac{4}{9},\frac{5}{9},1;0.8\right)\right)$$
$$Z_3 = \left(\left(\frac{2}{7},\frac{3}{7},\frac{5}{7},1;0.8\right),\left(\frac{1}{9},\frac{4}{9},\frac{5}{9},1;0.8\right)\right)$$

Step 2. Calculation of β_1 ،β_2 و β_3 using Eq. (3.3)

$$\beta_1 = 0.073$$
$$\beta_2 = 0.126$$
$$\beta_3 = 0.143$$

Step 3. Calculating $Rank\ (Z_1)$ ،$Rank\ (Z_2)$, $Rank\ (Z_3)$

$$Rank\ (Z_1) = 0.011$$
$$Rank\ (Z_2) = 0.023$$
$$Rank\ (Z_3) = 0.028$$

In short, Table 4.4 brings the obtained ranks using the proposed method. Furthermore, Table 4.5 lists the comparison of the proposed method with the hyperbolic method

Table 4.4. The obtained ranks using the proposed method

Z_i	$Rank\ (Z_i)$
$Z_1 = \big((1,3,4,7;0.8),(2,3,4,8;0.8)\big)$	0.011
$Z_2 = \big((2,4,5,8;0.8),(3,4,5,9;0.8)\big)$	0.023
$Z_3 = \big((2,3,5,7;0.8),(1,4,5,9;0.8)\big)$	0.028

Table 4.5. Comparison of the proposed method with the hyperbolic method [12]

methods	Results
Hyperbolic method [6]	$Z_1 \prec Z_2 \prec Z_3$
proposed method	$Z_1 \prec Z_2 \prec Z_3$

Example 4.4. Suppose $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ are two Z-numbers such that

$$A_1 = (0.1, 0.3,0.3,0.5) ،A_2 = (0.1,0.2,0.4,0.5) ،$$
$$B_1 = (0.1, 0.2, 0.4, 0.5)\ B_2 = (0.1, 0.3, 0.3, 0.5)$$

Then, their ranks can be obtained as follows:

$$Rank(Z_1) = 0.00013$$
$$Rank(Z_2) = 0.00015$$

so $Z_1 \prec Z_2$.

5. Conclusion

This paper investigated the shortcomings of some existing methods for ranking Z-numbers, and then proposed a novel technique for ranking Z-numbers. This did not include the shortcomings of some existing methods, which is one of its main novelties. Another advantage of the proposed method was to do not utilize the center of gravity, while most of the existing methods employed it to complete their methods for ranking Z-numbers. We have demonstrated it using several examples.

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